Iyama's finiteness theorem via strongly quasi-hereditary algebras.

Claus Michael Ringel

Abstract. Let Λ be an artin algebra and X a finitely generated Λ-module. Iyama has shown that there exists a module Y such that the endomorphism ring Γ of $X \oplus Y$ is quasi-hereditary, with a heredity chain of length n, and that the global dimension of Γ is bounded by this n. In general, one only knows that a quasi-hereditary algebra with a heredity chain of length n must have global dimension at most $2n - 2$. We want to show that Iyama's better bound is related to the fact that the ring Γ he constructs is not only quasi-hereditary, but even left strongly quasi-hereditary: By definition, the left strongly quasi-hereditary algebras are the quasi-hereditary algebras with all standard left modules of projective dimension at most 1.

The aim of this note is to present a concise proof of Iyama's finiteness theorem. For the benefit of the reader, it is essentially self-contained. Let us stress that all the main arguments used are known: those of sections 1 and 2 are due to Iyama [I1,I2], section 3 follows the ideas of Auslander [A], whereas section 4 is based on our joint work with Dlab [DR1, DR2, DR3]. Quasi-hereditary algebras with all standard left modules of projective dimension at most 1 have been considered in various papers, see for example [DR3], and it seems worthwhile to give them a name: we propose to call them left strongly quasi-hereditary. In our setting, the main advantage of working with left strongly quasi-hereditary, and not just quasi-hereditary algebras lies in the fact that one avoids to deal with factor algebras of endomorphism rings.

1. Preliminaries. Let Λ be an artin algebra. We denote by $\operatorname{mod}\Lambda$ the category of (finitely generated left) Λ-modules. Morphisms will be written on the opposite side of the scalars, thus if $f\colon X \to Y$ and $g\colon Y \to Z$ are Λ-homomorphisms between Λ-modules, then the composition is denoted by fg.

Recall that the radical rad of $\operatorname{mod}\Lambda$ is defined as follows: If X, Y are Λ-modules and $f\colon X \to Y$, then f belongs to $\operatorname{rad}(X, Y)$ provided for any indecomposable direct summand X' of X with inclusion map $u\colon X' \to X$ and any indecomposable direct summand Y' of Y with projection map $p\colon Y \to Y'$, the composition $ufp\colon X' \to Y'$ is non-invertible.

Of course, for any Λ-module X, the set $\operatorname{rad}(X, X)$ is just the radical of the endomorphism ring of X, thus

$$\gamma X = X \operatorname{rad}(X, X)$$

is the radical of X when considered as a right module over its endomorphism ring, and this is a Λ-submodule of X.

Proposition. *Let X be a Λ-module. Then*
(1) *X generates γX.*
(2) *Any radical map $X \to X$ factors through γX,*
(3) *If X is non-zero, then γX is a proper submodule of X.*
(4) *If $X = \bigoplus_i X_i$ with Λ-modules X_i, then $\gamma X = \bigoplus_i (X_i \cap \gamma X)$, and $X_i \cap \gamma X = X \operatorname{rad}(X, X_i)$.*

Proof: (1) Let $\phi_1, \ldots, \phi_m$ be a generating set of $\operatorname{rad}(X, X)$, say as a k-module, where k is the center of Λ. Then $\gamma X = \sum_i X \phi_i$, thus the map $\phi = (\phi_i)_i \colon X^m \to \gamma X$ is surjective.

(2) is obvious.

(3) The ring $\Gamma = \operatorname{End}(X)$ is again an artin algebra and the radical of a non-zero Γ-module is a proper submodule (it is enough to know that Γ is semi-primary).

(4) Clearly $X \operatorname{rad}(X, X_i) \subseteq X_i \cap \gamma X \subseteq \gamma X$. Thus, we only have to show that for $x \in X$ and $\phi \in \operatorname{rad} \operatorname{End}(X)$, the element $x\phi$ belongs to $\bigoplus_i X \operatorname{rad}(X, X_i)$. Let $\pi_i \colon X \to X_i$ be the canonical projection, so that $y = \sum_i y\pi_i$ for all $y \in X$. Then $x\phi = \sum_i x\phi\pi_i$. But with ϕ also $\phi\pi_i$ belongs to rad, thus $x\phi\pi_i \in X \operatorname{rad}(X, X_i)$.

Warning. One may be tempted to say that X generates γX by radical maps, but this is not true! For example, let Λ be the path algebra of the quiver of Dynkin type A_2 and X the minimal projective generator (i.e. the direct sum of the two indecomposable projective modules). Then γX is simple projective and the non-zero maps $X \to \gamma X$ are not radical maps. (What is true, is the following: γX is generated by X using maps which have the property that when we compose them with the inclusion map $\gamma X \subseteq X$, then they become radical maps.)

2. Iteration. We consider a fixed Λ-module X. We define inductively $M_1 = X$ and $M_{t+1} = \gamma M_t$, for $t \geq 1$. According to (3), there is some n such that $M_{n+1} = 0$. The smallest such n will be denoted by $\operatorname{d}(X)$, and we have $\operatorname{d}(X) \leq |X|$, where $|X|$ denotes the length of X. We define $M = \bigoplus_{i=1}^n M_i$ and $M_{>t} = \bigoplus_{i>t} M_i$.

Warning. Note that M_2 usually is different from $X \operatorname{rad}(X, X)^2$, a typical example is a serial module with composition factors $1, 1, 2, 1, 1$ (in this order) such that the submodule of length 2 and the factor module of length 2 are isomorphic. Here, $X \operatorname{rad}(X, X)^2 = 0$, whereas M_2 is simple.

Proposition. *Let $i \geq 1$. Let N be an indecomposable direct summand of M_i which is not a direct summand of M_{i+1}. Let*

$$\alpha N = M_i \operatorname{rad}(M_i, N).$$

Then αN is a proper submodule of N and the inclusion map $\alpha N \to N$ is a right $M_{>i}$-approximation (and of course right minimal).

Proof: First, we show that αN is a direct summand of M_{i+1}. Namely, since N is a direct summand of M_i, it follows that $\alpha N = M_i \operatorname{rad}(M_i, N)$ is a direct summand of M_{i+1}, using (4) for the module M_i. Since we assume that N is not a direct summand of M_i, we see that αN has to be a proper submodule of N.

In order to see that the inclusion map $u\colon \alpha N \to N$ is a right $M_{>i}$-approximation, we have to show that any map $g\colon M_j \to N$ with $j > i$ factors through u, thus that the image of g is contained in αN. Using inductively (1), there are natural numbers t, t' and surjective maps

$$(M_i)^{t'} \xrightarrow{\ \eta'\ } (M_{i+1})^t \xrightarrow{\ \eta\ } M_j.$$

We claim that the composition $\eta' \eta g$ is a radical map. Otherwise, there is an indecomposable direct summand U of $(M_i)^{t'}$ such that the composition

$$U \to (M_i)^{t'} \xrightarrow{\ \eta'\ } (M_{i+1})^t \xrightarrow{\ \eta g\ } N$$

is an isomorphism, but then N is a direct summand of M_{i+1}, which is not the case.

It follows that the image of $\eta' \eta g$ is contained in $M_i \operatorname{rad}(M_i, N) = \alpha N$. Since $\eta' \eta$ is surjective, we see that the image of g itself is contained in αN.

Corollary. *Let N be an indecomposable summand of M_i and of M_j where $i < j$. Then N is a direct summand of M_r for all $i \le r \le j$.*

Proof: Assume that N is not a direct summand of M_{i+1}. Since N is a direct summand of M_j and $j \ge i+1$, we can factor the identity map $N \to N$ through the inclusion map $\alpha N \to N$. But then $\alpha N = N$ and N is a direct summand of M_{i+1}, a contradiction.

Given an indecomposable direct summand N of M, there is a unique index $i \ge 1$ such that N is a direct summand of M_i but not of $M_{>i}$. We call i the *layer* of N.

3. The indecomposable projective Γ-modules.

We are interested in $\Gamma = \operatorname{End}(M)$. Recall that the indecomposable projective Γ-modules are of the form $\operatorname{Hom}(M, N)$ with N an indecomposable direct summand of M and we denote by $S(N)$ the top of the Γ-module $\operatorname{Hom}(M, N)$. If M' is a Λ-module, we denote by $\operatorname{Hom}(M, N)/\langle M' \rangle$ the factor of $\operatorname{Hom}(M, N)$ modulo all maps which factor through $\operatorname{add} M'$.

Proposition. *Let N be an indecomposable direct summand of M with layer i. Then the minimal right $M_{>i}$-approximation $u\colon \alpha N \to N$ yields an exact sequence*

$$0 \to \operatorname{Hom}(M, \alpha N) \xrightarrow{\ \operatorname{Hom}(M,u)\ } \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N)/\langle M_{>i} \rangle \to 0$$

of Γ-modules.

(a) *The Γ-module $R(N) = \operatorname{Hom}(M, \alpha N)$ is a direct sum of modules of the form $\operatorname{Hom}(M, N'')$ with N'' an indecomposable direct summand of M with layer greater than i.*

(b) *Considering the Γ-module $\Delta(N) = \operatorname{Hom}(M, N)/\langle M_{>i}\rangle$, any composition factor of $\operatorname{rad}\Delta(N)$ is of the form $S(N')$ where N' is an indecomposable Λ-module with layer smaller than i.*

Proof: Since u is injective, also $\operatorname{Hom}(u, -)$ is injective. Now αN belongs to M_{i+1}, thus $\operatorname{Hom}(M, \alpha N)$ is mapped under u to a set of maps $f\colon M \to N$ which factor through a module in $\operatorname{add} M_{>i}$. But since u is a right $M_{>i}$-approximation, we see that the converse also is true: any map $M \to N$ which factors through a module in $\operatorname{add} M_{>i}$ factors through u. This shows that the cokernel of $\operatorname{Hom}(M, u)$ is $\operatorname{Hom}(M, N)/\langle M_{>i}\rangle$.

Of course, $R(N)$ is projective. If we decompose αN as a direct sum of inde-composable modules N'', then $\operatorname{Hom}(M, \alpha N)$ is a direct sum of the corresponding projective Γ-modules $\operatorname{Hom}(M, N'')$ with N'' indecomposable and in $\operatorname{add} M_{>i}$. The layer of any indecomposable module in $\operatorname{add} M_{>i}$ is greater than i.

Now we consider $\Delta(N)$. Let N' be an indecomposable direct summand of M such that $S(N')$ is a composition factor of $\Delta(N)$. This means that there is a map $f\colon N' \to N$ which does not factor through $\operatorname{add} M_{>i}$. In particular, N' itself does not belong to $\operatorname{add} M_{>i}$. Assume that N' belongs to $\operatorname{add} M_i$. Also N is in $\operatorname{add} M_i$ and according to (2), any radical map $M_i \to M_i$ factors through M_{i+1}. This shows that f has to be invertible and therefore we deal with the top composition factor of $\Delta(N)$. It follows that the composition factors of $\operatorname{rad}\Delta(N)$ are of the form $S(N')$ with N' indecomposable with layer smaller than i.

4. Left strongly quasi-hereditary algebras. Let Γ be an artin algebra. Let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the set of isomorphism classes of simple Γ-modules. For any module M, let $P(M)$ be the projective cover of M.

We say that Γ is *left strongly quasi-hereditary with n layers* provided there is a function $l\colon \mathcal{S} \to \{1, 2, \ldots, n\}$ (the layer function) such that for any $S \in \mathcal{S}$, there is an exact sequence

$$0 \to R(S) \to P(S) \to \Delta(S) \to 0$$

with the following two properties:
(a) $R(S)$ is a direct sum of projective modules $P(S'')$ with $l(S'') > l(S)$, and
(b) if S' is a composition factor of $\operatorname{rad}\Delta(S)$, then $l(S') < l(S)$.

Recall that Γ is said to be *quasi-hereditary* with respect to a function $l\colon \mathcal{S} \to \{1, 2, \ldots, n\}$ provided for any $S \in \mathcal{S}$, there is an exact sequence

$$R(S) \to P(S) \to \Delta(S) \to 0$$

with properties (a) and (b) mentioned above and the additional property
(c) For any $S \in \mathcal{S}$, the module $P(S)$ has a Δ-filtration (i.e. a filtration with factors of the form $\Delta(S')$ with $S' \in \mathcal{S}$).
(But note that here the map $R(S) \to P(S)$ is not required to be injective.)

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Proposition. *If Γ is left strongly quasi-hereditary with n layers and layer function l, then Γ is quasi-hereditary with respect to l, and the global dimension of Γ is at most n.*

Proof. In order to see that Γ is quasi-hereditary with respect to l, we have to verify property (c). This we show by decreasing induction on $l(S)$. If $l(S) = n$, then $P(S) = \Delta(S)$. Assume we know that all $P(S)$ with $l(S) > i$ have a Δ-filtration. Let $l(S) = i$. Then $R(S)$ is a direct sum of projective modules $P(S')$ with $l(S') > l(S)$, thus it has a Δ-filtration. Then also $P(S)$ has a Δ-filtration. This shows that Γ is quasi-hereditary with respect to l.

Now we have to see that the global dimension of Γ is at most n. We show by induction on $l(S)$ that proj. dim $S \leq l(S)$. We start with $l(S) = 1$. In this case, $\Delta(S) = S$, thus there is the exact sequence $0 \to R(S) \to P(S) \to S \to 0$ with $R(S)$ projective. This shows that proj. dim $S \leq 1$. For the induction step, consider some $i \geq 2$ and assume that proj. dim $S' \leq l(S')$ for all S' with $l(S') < i$. Let S be simple with $l(S) = i$ and consider the exact sequence

$$0 \to R(S) \to \operatorname{rad} P(S) \to \operatorname{rad} \Delta(S) \to 0.$$

All the composition factors S' of $\operatorname{rad} \Delta(S)$ satisfy $l(S') < i$, thus proj. dim $S' < i$. Also, $R(S)$ is projective, thus proj. dim $R(S) = 0 < i$. This shows that $\operatorname{rad} P(S)$ has a filtration whose factors have projective dimension less than i, and therefore proj. dim $\operatorname{rad} P(S) < i$. As a consequence, proj. dim $S \leq i$. This completes the induction. Since all the simple modules have projective dimension at most n, the global dimension of Γ is bounded by n.

The bound for the global dimension cannot be improved in general: *For $n \geq 2$, there are left strongly quasi-hereditary algebras Γ with n layers such that the global dimension of Γ is equal to n.* As an example, take the cyclic quiver with vertices $1, 2, \ldots, n$, arrows $\alpha_i : i \to i-1$ (modulo n) and with relations $\alpha_{i-1}\alpha_i = 0$ for $2 \leq i \leq n$. The indecomposable projective modules $P(i)$ have the following shape:

$$\begin{array}{ccccc} 1 & 2 & 3 & & n \\ n & 1 & 2 & \cdots & n-1 \\ n-1 & & & & \end{array}$$

This is a left strongly quasi-hereditary algebra using the layer function $l(S(i)) = i$. We have $\Delta(1) = S(1)$, whereas $\Delta(i) = P(i)$ for $2 \leq i \leq n$. One easily checks that the projective dimension of $S(i)$ is equal to i, for any i, thus the global dimension is n.

5. Theorem. *Let X be a Λ-module. Then there is a Λ-module Y such that $\Gamma = \operatorname{End}(X \oplus Y)$ is left strongly quasi-hereditary with $\mathrm{d}(X)$ layers. In particular, the global dimension of Γ is at most $\mathrm{d}(X)$.*

In addition, we record that $\mathrm{d}(X) \leq |X|$. Also, the construction of Y shows that we can assume that *any indecomposable direct summand of the module Y is a submodule of an indecomposable direct summand of X.*

Proof: As above, let $M_1 = X$ and $M_{t+1} = \gamma M_t$ for $t \geq 0$. Let $Y = M_{>1}$ and $M = X \oplus Y = \bigoplus_{i=1}^n M_i$ with $n = \mathrm{d}(X)$. According to Proposition 3, Γ is a left strongly quasi-hereditary algebra with n layers, thus we can apply Proposition 4. The additional information comes from (3) and (4).

Several applications should be mentioned.

(1) First, there is the representation dimension of an artin algebra Λ as introduced by Auslander in the Queen Mary Notes. By definition, this is the smallest number which occurs as the global dimension of the endomorphism ring of a Λ-module which is both a generator and a cogenerator. If one takes $X = \Lambda \oplus D\Lambda$, where $D = \mathrm{Hom}_k(-, k)$ is the k-duality functor, then the theorem provides a Λ-module Y such that the global dimension of $\mathrm{End}(X \oplus Y)$ is bounded by $|X| = 2|\Lambda|$. Since $X \oplus Y$ is a generator and a cogenerator, this yields a bound for the representation dimension of Λ. In particular, *the representation dimension is always finite,* this is Iyama's finiteness theorem.

(2) In this way, we obtain for artin algebras also a strenghtening of the main result of [DR2]: *If Λ is an artin algebra, then there is an artin algebra Γ and an idempotent $e \in \Gamma$ with $e\Gamma e = \Lambda$ such that Γ is left strongly quasi-hereditary,* and not only quasi-hereditary. Here, one may start with $X = \Lambda$ (or also with any module which has Λ as a direct summand, as the one in the previous paragraph).

(3) Finally, we see that Auslander algebras are left strongly quasi-hereditary: *If Λ is a representation-finite artin algebra and M is the direct sum of the indecomposable Λ-modules, one from each isomorphism class, then $\Gamma = \mathrm{End}(M)$ is left strongly quasi-hereditary.* Namely, the Theorem asserts that there is a Λ-module Y, such that $\mathrm{End}(M \oplus Y)$ is left strongly quasi-hereditary. But Γ and $\mathrm{End}(M \oplus Y)$ are Morita equivalent, thus also Γ is left strongly quasi-hereditary.

Appendix

For the convenience of the reader, the appendix collects from the literature some further information on left strongly quasi-hereditary algebras. Also, we will add some examples which may be useful.

A1. Characterizations of left strongly quasi-hereditary algebras.

Let us assume that Γ is quasi-hereditary with respect to some layer function l. For any simple module S, let $\Delta(S)$ be the corresponding standard module, $\nabla(S)$ the costandard module. Let T be the characteristic tilting module. Given a set $\mathcal{X}$ of modules, we denote by $\mathcal{F}(\mathcal{X})$ the class of modules which have a filtration with all the factors in $\mathcal{X}$. Finally, recall that a module is said to be *divisible* provided it is generated by an injective module.

Proposition. *For the quasi-hereditary algebra* Γ *the following conditions are equivalent:*

(1) *Any* Δ-*module has projective dimension at most* 1.

(2) *Any module in* $\mathcal{F}(\Delta)$ *has projective dimension at most* 1.

(3) T *has projective dimension at most* 1.

(4) *The modules in* $\mathcal{F}(\nabla)$ *are the modules generated by* T.

(5) *Any module generated by* T *belongs to* $\mathcal{F}(\nabla)$.

(6) $\mathcal{F}(\nabla)$ *is closed under factor modules.*

(7) *Any divisible module belongs to* $\mathcal{F}(\nabla)$.

(8) *For any module* M*, there is an exact sequence* $0 \to M \to D_0 \to D_1 \to 0$ *where* D_0, D_1 *are modules in* $\mathcal{F}(\nabla)$.

(9) *There is an exact sequence* $0 \to \Gamma \to D_0 \to D_1 \to 0$ *where* D_0, D_1 *are modules in* $\mathcal{F}(\nabla)$.

Before we outline the proof, let us stress the following: Condition (3) states that T is what sometimes is called a *classical* tilting module, namely a tilting module of projective dimension at most 1.

Proof. For the equivalence of (1), (2), (6) and (7) we may refer to [DR3], Lemma 4.1 (section 5 of that paper contains also the assertion that (1) implies (4)). Of course, (2) implies (3), and classical tilting theory asserts that (3) implies (4). Trivially, (4) implies (5) and (6), also (6) implies (7), since the injective modules belong to $\mathcal{F}(\nabla)$. In order to see that (7) implies (8), one just takes for D_0 the injective envelope of M. Again (8) implies (9) is trivial. The equivalence of (3) and (9) is part of tilting theory. It remains to see that (5) implies (4), but it is easy to see that any module in $\mathcal{F}(\nabla)$ is generated by T.

A2. The missing left-right symmetry.

An artin algebra Γ is said to be *right strongly quasi-hereditary* provided the opposite algebra Γ^{op} is left strongly quasi-hereditary.

(1) *A left strongly quasi-hereditary algebra need not be right strongly quasi-hereditary.*

As an example, consider the algebra Γ with quiver

$$2 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 1 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3$$

and with relations $\alpha\alpha', \beta\alpha', \beta\beta', \alpha'\alpha\beta'$. The indecomposable projective modules $P(i)$ have the following shape:

$$
\begin{array}{ccc}
\begin{array}{cc} & 1 \\ 2 & 3 \\ 1 & 1 \\ & 2 \end{array}
&
\begin{array}{c} 2 \\ 1 \end{array}
&
\begin{array}{c} 3 \\ 1 \\ 2 \end{array}
\end{array}
$$

It is obvious that the numbering of the simple modules provides a layer function so that $\Delta(2)$ and $\Delta(3)$ are projective, whereas $\Delta(1)$ is simple with an exact sequence

$$0 \to P(2) \oplus P(3) \to P(1) \to \Delta(1) \to 0.$$

Instead of looking at modules over the opposite algebra, we can consider their k-duals. If Γ^{op} would be right strongly quasi-hereditary, we would obtain an exact sequence of Γ-modules of the form

$$0 \to \nabla(1) \to I(1) \to Q(1) \to 0,$$

where $I(1)$ is the injective envelope of 1, where $\nabla(1)$ has only one composition factor of the form 1 and where $Q(1)$ is injective. The indecomposable injective modules have the shape

$$
\begin{array}{ccc}
\begin{array}{cc} 1 & 1 \\ 2 & 3 \\ & 1 \end{array}
&
\begin{array}{c} 1 \\ 3 \\ 1 \\ 2 \end{array}
&
\begin{array}{c} 1 \\ 3 \end{array}
\end{array}
$$

Since $I(1)$ contains three composition factors of the form 1, we see that $Q(1) \neq 0$, thus $\nabla(1)$ has injective dimension equal to 1. But the only submodule of $I(1)$ with injective dimension equal to 1 is of length 3 with two composition factors 1 (and one composition factor 2). This shows that Γ cannot be right strongly quasi-hereditary.

(2) Let Γ be left strongly quasi-hereditary with layer function $l \colon \mathcal{S}(\Gamma) \to \{1, 2, \ldots, n\}$. Again, let $D = \operatorname{Hom}_k(-, k)$. If S is a simple Γ-module, then we define $l(DS) = l(S)$, thus we consider l also as a function $l \colon \mathcal{S}(\Gamma^{\mathrm{op}}) \to \{1, 2, \ldots, n\}$. We have shown above that Γ is quasi-hereditary with respect to l, and it is well-known that then also Γ^{op} is quasi-hereditary with respect to l. In general, Γ^{op} *may not be left strongly quasi-hereditary with respect to l, even if it is left strongly quasi-hereditary with respect to some other layer function.*

As a typical example, consider an algebra Γ such that the quiver of Γ has no oriented cycles. Then there is a layer function l such that the Δ-modules are projective, and then the standard modules for the opposite algebra are the simple modules. In this case, Γ is left strongly quasi-hereditary with respect to l, but it is right strongly quasi-hereditary with respect to this l only in case Γ is hereditary. But of course, always Γ^{op} will be left strongly quasi-hereditary, however we have to use a different layer function.

Also, the example exhibited in section 4 is of this kind: The algebra Γ^{op} is not left strongly quasi-hereditary with respect the the ordering $\{1, 2, \ldots, n\}$, but it is left strongly quasi-hereditary with respect to the ordering $\{n-1, n, 1, \ldots, n-2\}$.

In fact, there is the following general result due to Erdmann-Parker ([EP],2.1):

Proposition. *If Γ is both left strongly quasi-hereditary and right strongly quasi-hereditary with respect to the same function l, then the global dimension of Γ is at most 2.*

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Proof: The implication (1) $\implies$ (7) of Proposition A1 for Γ^{op} shows that if Γ is right strongly quasi-hereditary, then all submodules of projective modules belong to $\mathcal{F}(\Delta)$. If Γ is left strongly quasi-hereditary, then the modules in $\mathcal{F}(\Delta)$ have projective dimension at most 1. But if all submodules of projective modules have projective dimension at most 1, then the global dimension of Γ is at most 2.

(3) If Γ is quasi-hereditary with characteristic tilting module T, then the endomorphism ring Γ' of T is called the R-dual (Ringel-dual) of Γ. It is again quasi-hereditary with respect to a suitable layer function (so that the characteristic tilting module T' for Γ' is given by $\mathrm{Hom}_\Gamma(T, Q)$, where Q is a minimal injective cogenerator for the category of Γ-modules). Again let us mention an observation of Erdmann-Parker ([EP], section 3):

Proposition. *The R-dual of a left strongly quasi-hereditary algebra is right strongly quasi-hereditary.*

Proof: Tilting theory asserts: if T is a tilting module of projective dimension 1, then the injective dimension of $T' = \mathrm{Hom}_\Gamma(T, Q)$ is at most 1.

A3. Historical remarks.

Left strongly quasi-hereditary algebras have been considered in various papers, only the name is new. As we have mentioned, several characterizations of these algebras have been given already in 1992 in our joint survey [DR3] with Dlab.

It is obvious that any hereditary artin algebra is left strongly quasi-hereditary with respect to any total ordering of the simple modules (thus quasi-hereditary with respect to any total ordering [DR1]).

Other important examples of left strongly quasi-hereditary algebras are the Auslander algebras. Several papers by Brüstle, Hille and Röhrle, but also others are devoted to such examples.

Under suitable directedness assumptions, bimodule problems can be described using left strongly quasi-hereditary algebras, see for example Hille and Vossieck [HV].

In the context of preprojective algebras, Geiss, Leclerc and Schröer [GLS] have shown that endomorphism rings of suitable rigid modules are left strongly quasi-hereditary, and this has been generalized by Iyama and Reiten [IR] and by Schröer himself [S].

On the other hand, in the classical realm of the quasi-hereditary arising for semisimple Lie algebras and algebraic groups, one cannot expect that the quasi-hereditary algebras occurring there are left strongly quasi-hereditary. The reason is quite simple: Usually, these quasi-hereditary algebras are R-self-dual, and have quite large global dimension. However, R-self-dual algebras which are left strongly

quasi-hereditary are also right strongly quasi-hereditary, and thus they have global dimension at most 2 (see the results of Erdmann-Parker mentioned in A2).

But note that in this setting, the equivalence of the conditions (1) and (8) mentioned in A1 was already formulated by Friedlander and Parshall ([FP, Proposition 3.4]), before the concept of a quasi-hereditary algebra was introduced. Following [FP] one may say that the ∇-filtration dimension of a module X is at most d provided there exists an exact sequence

$$0 \to X \to D_0 \to D_1 \to \cdots \to D_d \to 0$$

with $D_0, \ldots, D_d \in \mathcal{F}(\nabla)$. Using this terminology, the equivalence of (1) and (8) may be reformulated as the following assertion: *A quasi-hereditary algebras Γ is left strongly quasi-hereditary if and only if the global ∇-filtration dimension of Γ is at most 1.*

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Fakultät für Mathematik, Universität Bielefeld, POBox 100 131, D-33 501 Bielefeld

E-mail address: `ringel@math.uni-bielefeld.de`