Indecomposable representations of the Kronecker quivers

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Abstract. Let k be a field and Λ the n-Kronecker algebra, this is the path algebra of the quiver with 2 vertices, a source and a sink, and n arrows from the source to the sink. It is well-known that the dimension vectors of the indecomposable Λ-modules are the positive roots of the corresponding Kac-Moody algebra. Thorsten Weist has shown that for every positive root there are tree modules with this dimension vector and that for every positive imaginary root there are at least n tree modules. Here, we present a short proof of this result. The considerations used also provide a calculation-free proof that all exceptional modules over the path algebra of a finite quiver are tree modules.

Let k be a field and Q a finite quiver without oriented cycle. Let $\Lambda = kQ$ be the path algebra of Q. The target of the paper is to look for Λ-modules which are tree modules. According to Kac [K], the dimension vectors of the indecomposable Λ-modules are the positive roots of the corresponding Lie algebra: for a real root, there is a unique indecomposable module, for an imaginary root, there are infinitely many provided k is an infinite field. Unfortunately, no effective procedure is known to construct at least one indecomposable module for each positive root. On the other hand, it seems that for each positive root, there exists even a tree module (the definition will be recalled below), and that for any imaginary root, there are several different tree modules (see [R4], Problem 9). Thorsten Weist [W] has shown that this is true for all the Kronecker algebras. Here, we present a short proof of his result by determining the dimension vectors of the "cover-thin" Kronecker modules (Proposition 1.1).

The Kronecker algebras are the path algebras of the Kronecker quivers. The n-Kronecker quiver Q with n arrows looks as follows:

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_n"'] \arrow[r, phantom, "\vdots"] & \underset{2}{\circ} \end{tikzcd}$$

For $n \geq 2$ we obtain in this way representation-infinite algebras, for $n \geq 3$ these algebras are wild. The importance of the Kronecker algebras and their representations is well-known, often they are considered as the basic data in non-commutative geometry.

2000 *Mathematics Subject Classification.* Primary 16G20. Secondary: 05C05, 11B39, 15A22, 16G60, 17B67, 65F50.

Let $M = (M_a, M_\alpha)_{a,\alpha}$ be a finite-dimensional representation of a quiver, thus M attaches to each vertex a of the quiver a vector space M_a and to each arrow α a linear map M_α. The sum of the dimension of these vector spaces is called the *total dimension* $\dim M$ of M. In case M is an indecomposable representation with total dimension d, then M is said to be a *tree module* provided that there is a choice of bases for the vector spaces such that the corresponding matrix presentations of the linear maps M_α involve altogether only $d-1$ non-zero entries (so that the "coefficient quiver" is a tree, see [R3]).

The root system for the n-Kronecker algebra is easy to describe: it consists of the non-zero vectors $(x, y) \in \mathbb{Z}^2$ with $x^2 + y^2 - nxy \leq 1$. The vectors (x, y) with $x^2 + y^2 - nxy = 1$ are called the real roots, the other roots the imaginary ones. The positive real roots are the dimension vectors of the preprojective and the preinjective modules. Now, the preprojective and the preinjective modules are exceptional modules (a module over a hereditary algebra is said to be exceptional if it is indecomposable and has no self-extension) and exceptional modules over the path algebra of a finite quiver are known to be tree modules [R3]. Thus, in order to show that every positive root is the dimension vector of a tree module, we only have to deal with the imaginary roots.

Theorem (Weist [W]). *Let Q be the n-Kronecker quiver. For any positive imaginary root for Q there are at least n tree modules with this dimension vector.*

A proof of the theorem will be given in section 2, it will rely on the use of covering theory. Denote by $\widetilde{Q}$ the universal cover of Q, this is the n-regular tree with bipartite orientation (n-regular means that every vertex has precisely n neighbors, the bipartite orientation is characterized by the property that all vertices are sinks or sources). We denote by $\pi\colon \operatorname{mod} k\widetilde{Q} \to \operatorname{mod} kQ$ the push-down functor.

An indecomposable representation M of a quiver is said to be *thin,* provided the non-zero vector spaces M_a used are 1-dimensional. If M is a thin indecomposable $k\widetilde{Q}$-module, then $\pi(M)$ will be said to be *cover-thin.* Similarly, we say that N is *cover-exceptional* provided there is an exceptional $k\widetilde{Q}$-module M such that $N = \pi(M)$. We are going to show that *given a positive imaginary root for the n-Kronecker quiver Q there are at least n cover-exceptional modules with this dimension vector* (Corollary 2.1). Since cover-exceptional modules are tree modules, the theorem is an immediate consequence.

Here we refer again to the main result of [R3] which asserts that exceptional modules over the path algebra Λ of a quiver are tree modules. In the last section we point out in which way one can invoke the covering theory of the n-Kronecker quiver in order to obtain a conceptual proof of this theorem, avoiding the matrix calculations used in [R3]. And it will turn out that in this way we do not have to worry about higher dimensional Ext^1-groups, but see that there is an inductive construction of all the exceptional Λ-modules using just one-dimensional Ext^1-groups.

Acknowledgment. The author is indebted to Philipp Fahr, Nicolas Poettering and the referee for a careful reading of a first version.

1. Cover-thin Kronecker-modules.

1.1. Proposition. *Consider* $(x, y) \in \mathbb{N}_0^2$ *with* $x \leq y$. *There exists a cover-thin* kQ*-module* N *with dimension vector* $\mathbf{dim}\, N = (x, y)$ *if and only if* $0 < y \leq (n-1)x + 1$.

For $n \geq 3$ *and* $0 < y \leq (n-1)x+1$, *there are at least* n *isomorphism classes of cover-thin modules* N *with* $\mathbf{dim}\, N = (x, y)$, *unless* $(x, y) = (0, 1)$ *or* $(1, n)$. *For* $n = 2$ *and* $0 < x$, *there are precisely* 2 *isomorphism classes of cover-thin modules* N *with* $\mathbf{dim}\, N = (x, x)$.

Proof: Since $\widetilde{Q}$ is a tree, the thin indecomposable $k\widetilde{Q}$-modules are uniquely determined by the corresponding support, this is just a finite subtree of $\widetilde{Q}$. If T is a finite subtree of $\widetilde{Q}$, let $M(T)$ be the $k\widetilde{Q}$-module with support T and $N(T) = \pi(M(T))$. The modules of the form $N(T)$ are the cover-thin kQ-modules.

First, consider the case $n = 2$, thus $\widetilde{Q}$ is the graph $\mathbb{A}_\infty^\infty$ with bipartite orientation. Up to shift, there is a unique subtree of $\widetilde{Q}$ with x sources and $y = x + 1$ sinks, where $x \geq 0$, and there are two subtrees with x sources and $y = x$ sinks, where $x > 0$, and, finally, all subtrees with x sources and y sinks such that $x \leq y$ are obtained in this way. In particular, we see that for $0 < x$, there are precisely two isomorphism classes of cover-thin modules N with $\mathbf{dim}\, N = (x, x)$.

Thus, let $n \geq 3$. If $x = 0$, then there exists an indecomposable kQ-module N with $\mathbf{dim}\, N = (x, y)$ only for $y = 1$. If $x = 1$, then there is a subtree T of $\widetilde{Q}$ with $\mathbf{dim}\, N(T) = (x, y)$ if and only if $0 \leq y \leq n$. If $1 \leq y \leq n-1$, then we obtain in this way at least n isomorphism classes of kQ-modules $N(T)$ with $\mathbf{dim}\, N(T) = (x, y)$.

It follows that we can assume that $x \geq 2$. First, let us assume that T is a finite subtree of $\widetilde{Q}$ such that $\mathbf{dim}\, N(T) = (x, y)$. Since $x \geq 2$, we obviously must have $y > 0$. Let us show that $y \leq (n-1)x + 1$. Recall that an element a of a tree is said to be a *leaf* provided a has at most one neighbor. We choose a source b such that all neighbors of b but one are leaves. Such a vertex exists: namely, let T'' be obtained from T by removing all leaves which are sinks, then T'' is again a tree, thus has leaves, and all the leaves of T'' are sources of $\widetilde{Q}$; any such vertex can be taken as b. Removing from T the vertex b we obtain the disjoint union of a tree T' with at least two vertices and $t \leq n-1$ isolated vertices. Let $\mathbf{dim}\, \pi(M(T')) = (x', y')$, then $(x, y) = (x', y') + (1, t)$. If $x' \leq y'$, then by induction we know that $y' \leq (n-1)x' + 1$. But this inequality $y' \leq (n-1)x' + 1$ obviously also holds if $y' < x'$. It follows from $y' \leq (n-1)x' + 1$ and $t \leq n-1$ that $y = y' + t \leq (n-1)x' + 1 + (n-1) = (n-1)x + 1$.

Conversely, consider (x, y) with $2 \leq x \leq y$ and $y \leq (n-1)x+1$. We try to construct a subtree T of $\widetilde{Q}$ such that $\mathbf{dim}\, N(T) = (x, y)$. Write $y = \sum_{i=1}^{x} y(i)$ with $1 \leq y(i) \leq n-1$ for $1 \leq i \leq x-1$ and $1 \leq y(x) \leq n$ (such a decomposition exists, since $x \leq y \leq (n-1)x+1$).

Fix some sink s_1 of $\widetilde{Q}$ and take the unique path of the form

$$s_1 \xleftarrow{\alpha_1} t_1 \xrightarrow{\alpha_n} s_2 \xleftarrow{\alpha_1} t_2 \xrightarrow{\alpha_n} \quad \cdots \quad \xleftarrow{\alpha_1} t_{x-1} \xrightarrow{\alpha_n} s_x \xleftarrow{\alpha_1} t_x$$

starting at s_1. For $1 \leq i \leq x$, we add the arrows α_j (and their endpoints) starting at t_i, with $2 \leq j \leq y(i)$. We see that we obtain in this way a subtree T of $\widetilde{Q}$, with x sources and $\sum y(i) = y$ sinks, thus $\mathbf{dim}\, N(T) = (x, y)$.

Finally, observe that the module $M(T)$ constructed here for $x \geq 2$ has the property that $\mathrm{Im}(\alpha_1) \cap \mathrm{Im}(\alpha_n) \neq 0$, whereas $\mathrm{Im}(\alpha_i) \cap \mathrm{Im}(\alpha_j) = 0$ for $i < j$ and $(i, j) \neq (1, n)$. Thus, using a permutation of the labels of the arrows, the same construction yields $\binom{n}{2}$ different isomorphism classes and $\binom{n}{2} \geq n$, since $n \geq 3$. This completes the proof.

Duality provides in a similar way cover-thin kQ-modules with dimension vectors (x, y) where $0 \leq y \leq x$ and $0 < x \leq (n-1)y + 1$.

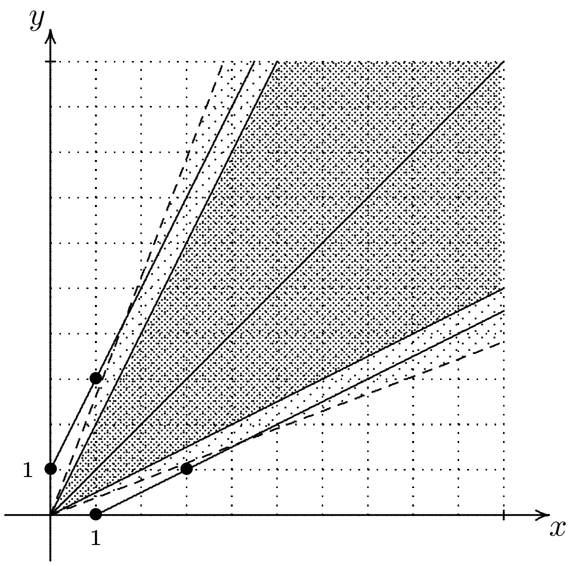
It is well-known (see for example [R1]) that for $n \geq 3$ the region

$$\mathcal{F} = \{(x, y) \in \mathbb{N} \mid \frac{1}{n-1}x < y \leq (n-1)x\}$$

is a fundamental domain for the action of the Coxeter transformation C on the set of positive imaginary roots. Note that this region is contained in the set of dimension vectors of cover-thin kQ-modules and that the vectors $(0, 1),\ (1, 0),\ (1, n),\ (n, 1)$ are real roots. In the case $n = 2$, let $\mathcal{F} = \{(x, x) | x > 0\}$. We see:

1.2. Corollary. *For every $(x, y) \in \mathcal{F}$, there are at least n isomorphism classes of cover-thin kQ-modules N with* $\mathbf{dim}\, N = (x, y)$.

For the benefit of the reader, we provide an illustration for the case $n = 3$:



The union of the shaded areas is the imaginary cone, the dark part being a fundamental domain $\mathcal{F}$ for the action of the Coxeter transformation on the imaginary cone. The bullets indicate the dimension vectors $(0, 1),\ (1, 0),\ (1, 3),\ (3, 1)$, they are outside of the imaginary cone. There are two lines with slope 2 as well as two lines with slope $\frac{1}{2}$: those going through the origin bound the fundamental region $\mathcal{F}$, the parallel ones bound the region of the dimension vectors of cover-thin kQ-modules.

2. Cover-exceptional kQ-modules.

Thin indecomposable modules are exceptional. This implies:

2.1. Corollary. *For every positive imaginary root (x, y) there are at least n isomorphism classes of cover-exceptional kQ-modules with* $\mathbf{dim}\, N = (x, y)$.

Proof. For $n = 2$, this has been shown in Proposition 1.1. Thus let $n \geq 3$. Let τ and $\widetilde{\tau}$ be the Auslander-Reiten translation for kQ and $k\widetilde{Q}$, respectively. Note that $\pi(\widetilde{\tau}M) = \tau(\pi(M))$ for any indecomposable $k\widetilde{Q}$-module M and $\mathbf{dim}\,\tau N = C\,\mathbf{dim}\,N$ for any indecomposable kQ-module N provided N is not projective, or, equivalently, provided $C\,\mathbf{dim}\,N$ has non-negative coordinates. Also note that if M is an exceptional $k\widetilde{Q}$-module, and not projective, then τM is again exceptional.

Let $(x, y) = C^a(x', y')$ with $(x', y') \in \mathcal{F}$ and $a \in \mathbb{Z}$. According to Corollary 1.2, there are pairwise non-isomorphic thin indecomposable $k\widetilde{Q}$-modules $M_1, \ldots, M_n$ with $\mathbf{dim}\,\pi(M_i) = (x', y')$. Consider the kQ-modules $N_i = \pi(\widetilde{\tau}^a M_i)$. Since

$$\mathbf{dim}\,N_i = \mathbf{dim}\,\pi(\widetilde{\tau}^a M_i) = \mathbf{dim}\,\tau^a(\pi(M_i)) = C^a\,\mathbf{dim}\,\pi(M_i) = C^a(x', y') = (x, y),$$

the modules $N_1, \ldots, N_n$ have the required dimension vector (x, y) and they are pairwise non-isomorphic and cover-exceptional.

2.2. Lemma. *Any cover-exceptional kQ-module is a tree module.*

Proof: According to [R3], any exceptional module over a hereditary k-algebra Λ is a tree module. We apply this to the path algebra of (a finite convex subquiver of) $\widetilde{Q}$. Let N be a cover-exceptional kQ module, thus $N = \pi(M)$ for some exceptional $k\widetilde{Q}$-module M. Since M is a tree module, obviously also $\pi(M)$ is a tree module.

The main theorem is a direct consequence of 2.1 and 2.2.

It is known that the only exceptional kQ-modules M are the preprojective and the preinjective modules. Let us consider in more detail a $k\widetilde{Q}$-module N with $M = \pi(N)$ being preprojective.

2.3. Lemma. *Assume that $M = \pi(N)$ is an indecomposable preprojective kQ-module which is not simple. Then there is an exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

of $k\widetilde{Q}$-modules, with N' simple projective, N'' exceptional, and

$$\mathrm{Hom}(N', N'') = \mathrm{Hom}(N'', N') = \mathrm{Ext}^1(N', N'') = 0, \quad \dim \mathrm{Ext}^1(N'', N') = 1.$$

Proof: The preprojective kQ-modules are constructed inductively starting with the simple projective kQ-module and using the Bernstein-Gelfand-Ponomarev reflection functors [BGP] for sources. In the same way, the modules N considered here are constructed inductively starting with a simple projective $k\widetilde{Q}$-module and using the Bernstein-Gelfand-Ponomarev reflection functors for all the sources, simultaneously (see for example [FR]).

To be precise, given a quiver Δ, let $\sigma\Delta$ be obtained from Δ by changing the orientation of all the arrows, thus $\sigma^2\Delta = \Delta$. Let us choose some sink x of $\widetilde{Q}$. Besides $\widetilde{Q}$ also $\sigma\widetilde{Q}$ is a universal cover of Q (since x considered as a vertex of σQ is a source, it has to be sent to the vertex 1 of Q), we denote both push-down functors $\mathrm{mod}\,k\widetilde{Q} \to \mathrm{mod}\,Q$ and

$\operatorname{mod} k\sigma\widetilde{Q} \to \operatorname{mod} Q$ by π. Both for $k\widetilde{Q}$ as well as $k\sigma\widetilde{Q}$, we denote by Φ^- the composition of the Bernstein-Gelfand-Ponomarev reflection functors for all the sources, these are well-defined functors

$$\Phi^-\colon \operatorname{mod} k\widetilde{Q} \longrightarrow \operatorname{mod} k\sigma\widetilde{Q} \qquad \text{and} \qquad \Phi^-\colon \operatorname{mod} k\sigma\widetilde{Q} \longrightarrow \operatorname{mod} k\widetilde{Q}.$$

Since x is a sink of $\widetilde{Q}$, the simple $k\widetilde{Q}$-module with support x is projective and we denote it by $P(0)$. Let $P(t) = (\Phi^-)^t P(0)$, this is a $k\widetilde{Q}$-module for t even and a $k\sigma\widetilde{Q}$-module for t odd. The preprojective kQ-modules are just the modules $\pi(P(t))$ with $t \in \mathbb{N}_0$.

Let us denote by d the distance function for the vertices of $\widetilde{Q}$ and of $\sigma\widetilde{Q}$. We claim: *for any $t \geq 1$ the support of $P(t)$ is the set $B(t) = \{y \mid d(x,y) \leq t\}$ and $\dim P(t)_y = 1$ for all y with $d(x,y) \in \{t-1, t\}$.* This is clear for $t = 1$, since $P(1)$ is indecomposable projective and not simple. Now we use induction. Thus, assume that $t \geq 1$, that $P(t)$ has support $B(t)$ and that $\dim P(t)_y = 1$ for y with $d(x,y) = t$. Obviously, the support of $P(t+1)$ has to be a subset of the set $B(t+1)$. If $d(x,y) = t$, then y is a sink for $\sigma^t Q$, thus $P(t+1)_t = P(t)_t$ is one-dimensional. If $d(x,z) = t+1$, then z has a unique neighbor y with $d(x,y) = t$ and it follows from $\dim P(t)_y = 1$ that $\dim P(t+1)_z = 1$. In this way, we also see that all the vertices z with $d(x,y) = t+1$ belong to the support of $P(t+1)$. Now the support of an indecomposable module has to be connected. But the only connected subquiver with vertices in $B(t+1)$ and which contains all vertices z with $d(x,z) = t+1$, is the full subquiver with vertex set $B(t+1)$. Thus we see that $P(t+1)$ has support $B(t+1)$. This concludes the proof of the claim.

Now, let $N = P(t)$ for some $t \geq 1$. Let z be a vertex with $d(x,z) = t$, and y the unique neighbor of z with $d(x,y) = t-1$. Clearly, z is a sink and we denote by N' the simple module with support z. Since $P(t)_z$ is one-dimensional, we see that N' is a submodule of N, and $\operatorname{Hom}(N', N'') = 0 = \operatorname{Hom}(N'', N')$, where $N'' = N/N'$. Since $\dim N''_y = 1$, we conclude on the one hand that N'' is indecomposable and even exceptional, on the other hand that $\dim \operatorname{Ext}^1(N'', N') = 1$. Since N' is projective, we have $\operatorname{Ext}^1(N', N'') = 0$.

3. Exceptional modules are tree modules.

The proof of Lemma 2.2 is based on the fact that for Λ a finite-dimensional hereditary k-algebra, any exceptional module is a tree module. On the other hand, one can use the considerations of section 2 in order to provide a proof of this result which avoids any calculations. Indeed, the proof given in [R3] required explicit matrix presentation of the preprojective and preinjective Kronecker modules, and, in this way, was quite technical. Here we show that using induction and the covering theory for the Kronecker algebras, one can avoid the matrix calculations.

Using induction on m, we want to present a concise proof of the following result:

3.1. Theorem ([R3]). *Let Λ be the path algebra of a finite quiver. Any exceptional Λ-module is a tree module.*

Proof. Let M be of dimension m. We proceed by induction on m. In the case $m = 1$ nothing has to be shown. Thus let us deal with the induction step, thus let $m > 1$.

First, consider the case where $\Lambda = kQ$ is the n-Kronecker algebra for some $n \geq 1$. In the case $n = 1$, only one module M has to be considered: it has dimension 2 and obviously is a tree module. Thus, assume that $n \geq 2$. The exceptional kQ-modules are the preprojective and the preinjective kQ-modules.

First, consider the case of a preprojective kQ-module M. Let $M = \pi(N)$ where N is a $k\widetilde{Q}$-module. Since $m > 1$, we see that M is not simple, thus Lemma 2.3 provides an exact sequence $0 \to N' \to N \to N'' \to 0$ of $k\widetilde{Q}$-modules, with N' simple and N'' exceptional. Now $\dim N'' = m - 1 < m$, thus by induction we know that N'' is a tree module and then also N is a tree module. But with N also M is a tree module. Thus any preprojective kQ-module is a tree module. By duality, we see that also any preinjective kQ-module is a tree module.

Now assume that we are dealing with an exceptional module M of dimension m such that the support of M has at least three vertices. Schofield induction (see [CB] or also [R2]) asserts that there is an exact sequence

$$0 \to X^a \to M \to Y^b \to 0$$

where X, Y are orthogonal exceptional modules and the pair (a, b) is the dimension vector of a sincere preprojective or preinjective representation Z of an e-Kronecker module, with $e = \dim \operatorname{Ext}^1(Y, X)$. Since $a > 0, b > 0$, it follows that $\dim X < m$, and $\dim Y < m$. Since the support of M has at least three vertices, we see that not both modules X, Y can be simple, thus it follows from $a \dim X + b \dim Y = m$ that $\dim Z = a + b < m$. By induction, all three modules X, Y, Z are tree modules (here, X, Y are Λ-modules, whereas Z is an e-Kronecker module), but then also M is a tree module, see [R3], section 6. This completes the proof.

Remark. The proof presented here seems to explain quite well why exceptional modules are tree modules. The use of Schofield induction shows that inductively one has to look at pairs X, Y of orthogonal exceptional modules with $\operatorname{Ext}^1(Y, X) \neq 0$. If $\dim \operatorname{Ext}^1(Y, X) = 1$, then it is easy to see that the middle term M of a non-split exact sequence $0 \to X \to M \to Y \to 0$ is a tree module, provided both X, Y are known to be tree modules. Thus the interesting cases should be those where $\dim \operatorname{Ext}^1(Y, X) > 1$. However, as we have seen the covering theory allows to reduce the considerations to deal only with orthogonal exceptional modules X', Y' with $\dim \operatorname{Ext}^1(Y', X') = 1$. Altogether we want to stress that in the Schofield construction of exceptional modules the use of the higher dimensional Ext^1-groups can be replaced by looking only at one-dimensional Ext^1-groups.

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