

From submodule categories to preprojective algebras.

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Abstract: Let $S(n)$ be the category of invariant subspaces of nilpotent operators with nilpotency index at most n . Such submodule categories have been studied already in 1934 by Birkhoff, they have attracted a lot of attention in recent years, for example in connection with some weighted projective lines (Kussin, Lenzing, Meltzer). On the other hand, we consider the preprojective algebra Π_n of type \mathbb{A}_n ; the preprojective algebras were introduced by Gelfand and Ponomarev, they are now of great interest, for example they form an important tool to study quantum groups (Lusztig) or cluster algebras (Geiss, Leclerc, Schröer).

We are going to discuss the connection between the submodule category $S(n)$ and the module category $\text{mod } \Pi_{n-1}$ of the preprojective algebra Π_{n-1} . Dense functors $S(n) \rightarrow \text{mod } \Pi_{n-1}$ are known to exist: one has been constructed quite a long time ago by Auslander and Reiten, recently another one by Li and Zhang. We will show that these two functors are full, objective functors with index $2n$, thus $\text{mod } \Pi_{n-1}$ is obtained from $S(n)$ by factoring out an ideal which is generated by $2n$ indecomposable objects.

As a byproduct we also obtain new examples of ideals in triangulated categories, namely ideals \mathcal{I} in a triangulated category \mathcal{T} which are generated by an idempotent such that the factor category \mathcal{T}/\mathcal{I} is an abelian category.

1. Introduction.

Let k be a field. Let $S(n)$ be the category of invariant subspaces of nilpotent operators with nilpotency index at most n . For a detailed analysis of this category we refer to [RS2]. Let $k[x]$ be the polynomial ring in one variable x with coefficients in k and $\Lambda_n = k[x]/\langle x^n \rangle$. The objects of $S(n)$ are the pairs (X, Y) where Y is a Λ_n -module and X is a submodule of Y (or the corresponding inclusion maps $u: X \rightarrow Y$). We denote the indecomposable Λ_n -module of length i by $[i]$, and $[0]$ will denote the zero module.

Let Π_n be the preprojective algebra of type \mathbb{A}_n and $\text{mod } \Pi_n$ the category of the Π_n -modules of finite length. The aim of this note is to show that $\text{mod } \Pi_{n-1}$ is obtained from $S(n)$ by factoring out an ideal \mathcal{I} which is generated by $2n$ indecomposable objects — actually, we will exhibit two possible choices for \mathcal{I} .

Given an additive category \mathcal{A} and an ideal \mathcal{I} in \mathcal{A} , we denote by \mathcal{A}/\mathcal{I} the corresponding factor category (it has the same objects, and the homomorphisms in \mathcal{A}/\mathcal{I} are the residue classes of the homomorphisms in \mathcal{A} modulo \mathcal{I}). If \mathcal{K} is a class of objects of the category \mathcal{A} , we denote by $\langle \mathcal{K} \rangle$ the ideal of \mathcal{A} given by all maps which factor through a direct sum of objects in \mathcal{K} . Instead of writing $\mathcal{A}/\langle \mathcal{K} \rangle$, we just will write \mathcal{A}/\mathcal{K} .

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Theorem 1. *Let \mathcal{U} be the set of objects of $\mathcal{S}(n)$ which are of the form $([i], [j])$ with $i = j$ or $j = n$. Let \mathcal{V} be the set of objects of $\mathcal{S}(n)$ which are of the form $([i], [j])$ with $i = j$ or $i = 0$. Then the categories $\mathcal{S}(n)/\mathcal{U}$ and $\mathcal{S}(n)/\mathcal{V}$ are equivalent to $\text{mod } \Pi_{n-1}$.*

In particular, these categories are abelian categories with enough projective objects. The indecomposable projective objects in $\mathcal{S}(n)/\mathcal{U}$ are the objects of the form $([0], [j])$ with $1 \leq j \leq n-1$, the indecomposable projective objects in $\mathcal{S}(n)/\mathcal{V}$ are the objects of the form $([i], [n])$ with $1 \leq i \leq n-1$.

If \mathcal{A}, \mathcal{B} are Krull-Remak-Schmidt categories, then we say that a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is *objective*, provided its kernel is generated by identity maps of objects (see the appendix). If F is a dense, objective functor, then the number of isomorphism classes of indecomposable objects A in \mathcal{A} with $F(A) = 0$ is called its index. Thus Theorem 1 asserts that there are full, dense, objective functors $\mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$ with index $2n$. These functors are the main target of our considerations. Actually, the two functors $F, G: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$ which we will deal with have been exhibited before: one of them has been constructed quite a long time ago by Auslander and Reiten [AR2], the other one recently by Li and Zhang [LZ], both are based on general considerations by Auslander [A], published already in 1965. What remains to be done is to show that the functors are full and objective.

Both functors F, G are given as compositions of functors which involve some further module categories and also these intermediate categories seem to be of interest.

First of all, let $T_2(\Lambda_n)$ be the ring of upper triangular (2×2) -matrices with coefficients in Λ_n . It is well-known (and easy to see) that the category $\text{mod } T_2(\Lambda_n)$ -modules can be identified with the category of maps between Λ_n -modules. Since the category $\mathcal{S}(n)$ can be considered as the category of monomorphisms of Λ_n -modules, we see that $\mathcal{S}(n)$ is a full subcategory of $\text{mod } T_2(\Lambda_n)$, say with inclusion functor ι . There is a second full embedding $\epsilon: \mathcal{S}(n) \rightarrow \text{mod } T_2(\Lambda_n)$, it sends the pair (X, Y) in $\mathcal{S}(n)$ to the canonical projection $Y \rightarrow Y/X$.

Next, given an algebra Λ of finite representation type, we denote by $A(\Lambda)$ its (basic) Auslander algebra; it is defined as follows: let E be a minimal Auslander generator for Λ , this is the direct sum of all indecomposable Λ -modules, one from each isomorphism class, and $A(\Lambda) = \text{End}(E)^{\text{op}}$. Since the algebras Λ_n are of finite representation type, we may consider $A_n = A(\Lambda_n)$. Of course, in this case $E = \bigoplus_{i=1}^n [i]$. In section 2, we study the Auslander algebra A_n and the category $\mathcal{F}(n)$ of the torsionless A_n -modules. Any Auslander algebra is quasi-hereditary, the Auslander algebras A_n are quasi-hereditary in a unique way and the category $\mathcal{F}(n)$ is just the category of A_n -modules with a Δ -filtration.

The essential tool is the functor

$$\alpha = \text{Cok Hom}_{\Lambda}(E, -): \text{mod } T_2(\Lambda) \longrightarrow \text{mod } A(\Lambda),$$

where Λ is of finite representation type and E is a minimal Auslander generator Λ ; note that α sends a morphism f in $\text{mod } \Lambda$ (thus the object f of the category $\text{mod } T_2(\Lambda)$) to the cokernel of the induced map $\text{Hom}_{\Lambda}(E, f)$; of course, $\text{Hom}_{\Lambda}(E, f)$ is a map of $A(\Lambda)$ -modules. This functor was considered already in 1965 by Auslander [A], and later by Auslander and Reiten in [AR1] and [AR2]. Section 3 and parts of section 6 are devoted to this functor.

Finally, let us note that Π_{n-1} is a factor ring of A_n , namely $\Pi_{n-1} = A_n/\langle e \rangle$, where e is an idempotent of A_n such that $A_n e$ is indecomposable projective-injective. We will consider the functor

$$\delta: \text{mod } A_n \rightarrow \text{mod } \Pi_{n-1}$$

which sends any A_n -module M to its largest factor module which is a Π_{n-1} -module, thus to $M/A_n e M$. Properties of this functor δ will be discussed in section 5. In particular, following [DR], we will look at the restriction F_2 of δ to the subcategory $\mathcal{F}(n)$.

Altogether, we will deal with the following functors

$$\mathcal{S}(n) \begin{array}{c} \xrightarrow{\iota} \\ \xrightarrow{\epsilon} \end{array} \text{mod } T_2(\Lambda_n) \xrightarrow{\alpha} \text{mod } A_n \xrightarrow{\delta} \text{mod } \Pi_{n-1}$$

The upper composition $F = \delta\alpha\iota$ is the functor studied by Li and Zhang [LZ], the lower composition $G = \delta\alpha\epsilon$ is the one considered by Auslander and Reiten [AR2]. We will see that F is a dense functor with kernel the ideal $\langle \mathcal{U} \rangle$, and G is a dense functor with kernel the ideal $\langle \mathcal{V} \rangle$. This yields the first part of Theorem 1.

The image of the functor $\alpha\iota$ is precisely the subcategory $\mathcal{F}(n)$, thus we can write $F = F_2 F_1$, where $F_1: \mathcal{S}(n) \rightarrow \mathcal{F}(n)$ is the functor with $F_1(u) = \alpha(u)$, for $u \in \mathcal{S}(n)$. It is known from the literature that the functors F_1, F_2, G all are dense; our main concern is to show that they are full and objective, with index $n, n, 2n$, respectively. This will be done in sections 4, 5, 6, respectively.

In order to compare the functors F and G , we have to take into account the stable module category $\underline{\text{mod}} \Pi_{n-1}$, it is obtained from the module category $\text{mod } \Pi_{n-1}$ by factoring out the ideal generated by the identity maps of the projective modules. Since the algebra Π_{n-1} is self-injective, the stable module category $\underline{\text{mod}} \Pi_{n-1}$ is a triangulated category. We denote by

$$\pi: \text{mod } \Pi_{n-1} \longrightarrow \underline{\text{mod}} \Pi_{n-1}$$

the canonical projection. Note that this is a full, dense, objective functor of index $n - 1$, its kernel is generated by the indecomposable projective Π_{n-1} -modules.

We denote by $\Omega: \underline{\text{mod}} \Pi_{n-1} \rightarrow \underline{\text{mod}} \Pi_{n-1}$ the syzygy functor, thus $\Omega(M)$ is the kernel of a projective cover of the Π_{n-1} -module M .

Theorem 2. *We have*

$$\pi F = \Omega \pi G.$$

Section 8 draws the attention to the fact that in this setting we also obtain full, dense, objective functors $\mathcal{T} \rightarrow \mathcal{A}$ with finite index, such that \mathcal{T} is a triangulated category, \mathcal{A} an abelian category, thus with ideals in triangulated categories which are generated by an idempotent such that the corresponding factor categories are abelian.

In section 9 we provide illustrations concerning the change of the Auslander-Reiten quiver of $\mathcal{S}(n)$ when we factor out the various ideals mentioned above.

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2. The Auslander algebra $A_n = A(\Lambda_n)$ and the subcategory $\mathcal{F}(n)$ of $\text{mod } A_n$.

We recall that for Λ a ring of finite representation type, $A(\Lambda)$ denotes the basic Auslander algebra of Λ , it is the opposite of the endomorphism ring of a minimal Auslander generator for Λ . We consider here the special case of $\Lambda = \Lambda_n = k[x]/\langle x^n \rangle$ and its Auslander algebra $A_n = (\text{End } E)^{\text{op}}$. Note that $E = \bigoplus_{i=1}^n [i]$, where $[i]$ is the indecomposable Λ_n -module of length i .

Let $P(i) = \text{Hom}_{\Lambda}(E, [i])$. This is an indecomposable projective A_n -module. The inclusions $[i] \rightarrow [i+1]$ in the category $\text{mod } \Lambda_n$ yield a chain of inclusions

$$P(1) \subset P(2) \subset \cdots \subset P(n-1) \subset P(n).$$

Let $\Delta(i) = P(i)/P(i-1)$ (with $P(0) = 0$). Note that A_n is quasi-hereditary (for this ordering and only for this ordering) and the modules $\Delta(i)$ are the standard modules (but observe that the labeling of the simple A_n -modules exhibited here is the opposite of the labeling commonly used (see for example [DR]): in the present paper, it is the module $\Delta(1)$ which is projective and not $\Delta(n)$, and correspondingly, it is $P(n) = I(n)$ which is projective-injective and not $P(1)$).

Let $T(i) = P(n)/P(i-1)$ for $1 \leq i \leq n$. Note that $T(i)$ is also the largest submodule of $P(n-i+1)$ which is generated by $T(n)$. Let $T = \bigoplus_i T(i)$, this is the characteristic tilting module for the quasi-hereditary algebra A_n .

Proposition 1. *The following conditions are equivalent for an A_n -module M .*

- (i) M is torsionless.
- (ii) $\text{Ext}^1(M, T) = 0$.
- (iii) M has a Δ -filtration.
- (iv) The projective dimension of M is at most 1.
- (v) The injective envelope of M is projective.

Remark 1. We have avoided to refer to the labeling of the simple modules. If we use our labels, so that $P(n) = I(n)$ is the unique indecomposable module which is both projective and injective, then (v) can be reformulated as saying that (vi) *the injective envelope IM of M is a direct sum of copies of $I(n)$* , or also that (vii) *the socle of M is a direct sum of copies of $S(n)$* .

Remark 2. It has been shown in [R] that any Auslander algebra is left strongly quasi-hereditary. This means that any module with a Δ -filtration has projective dimension at most 1 (the implication (iii) \implies (iv)). Note that most of the relevant properties of strongly quasi-hereditary algebras have been considered already in [DR] and Proposition 1 is just a reminder.

We denote by $\mathcal{F}(n)$ the full subcategory of $\text{mod } A_n$ given by the A_n -modules which satisfy the equivalent conditions of Proposition 1. It follows directly from (i) or also (v) that $\mathcal{F}(n)$ is closed under submodules.

Proof of Proposition 1. (i) \implies (ii). Let M be torsionless. There is an embedding $M \rightarrow P$ with P projective. Since the injective dimension of T is 1, the canonical map $\text{Ext}^1(P, T) \rightarrow \text{Ext}^1(M, T)$ is surjective. But $\text{Ext}^1(P, T) = 0$, thus $\text{Ext}^1(M, T) = 0$.

(ii) \implies (iii). We have to show that $\text{Ext}^i(M, T) = 0$ for all $i \geq 1$. Since T has injective dimension 1, we only have to look at $i = 1$, but this is assertion (ii).

(iii) \implies (iv). The modules $\Delta(i)$ have projective dimension at most 1.

(iv) \implies (v). Let $0 \rightarrow P_1 \xrightarrow{u} P_0 \rightarrow M \rightarrow 0$ be a projective resolution. We have to show that M embeds into a module which is both injective and projective. Consider the injective envelopes of $v_i: P_i \rightarrow I(P_i)$ for $i = 0, 1$, this yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{u} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & v_1 \downarrow & & \downarrow v_0 & & \downarrow f & & \\ 0 & \longrightarrow & I(P_1) & \xrightarrow{u'} & I(P_0) & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

Since v_0 is injective, the snake lemma yields an embedding of the kernel K of f into the cokernel of v_1 . For any projective module P with injective envelope IP , the cokernel IP/P embeds into a projective-injective module (since the dominant dimension of Λ_n is at least 2).

On the other hand, u' is a monomorphism and $I(P_1)$ is injective, thus u' is a split monomorphism. Thus M' is a direct summand of $I(P_0)$ and this module is projective-injective. The exact sequence $0 \rightarrow K \rightarrow M \rightarrow M'$ shows that M embeds into $IK \oplus M'$ which is projective-injective.

(v) \implies (i). If the injective envelope of M is projective, then M embeds into a projective module, thus M is torsionless. \square

Proposition 2. *If M is an A_n -module of projective dimension at most 1 and generated by $P(n)$, then M belongs to $\text{add } T$.*

Proof: Write $M = P/U$ where P is a direct sum of copies of $P(n)$ and U is a submodule of P . Since the projective dimension of M is at most 1, the submodule U is projective. Now P is injective, thus we may embed an injective envelope IU of U into P , this is a direct summand of P , say $P = IU \oplus C$ for some submodule C . Since P is a direct sum of copies of $P(n)$, also C is a direct sum of copies of $P(n) = T(1)$. On the other hand, the exact sequences $0 \rightarrow P(i) \rightarrow P(n) \rightarrow T(i+1) \rightarrow 0$ for $1 \leq i \leq n$ (with $T(n+1) = 0$) show that for any indecomposable projective module P' , the module $I(P')/P'$ belongs to $\text{add } T$, thus IU/U belongs to $\text{add } T$. Altogether we see that $P/U = IU/U \oplus C$ belongs to $\text{add } T$. \square

Remark. The modules generated by $P(n) = T(1)$ are the modules with a ∇ -filtration (see [DR]), thus proposition 2 just asserts that modules which have both a Δ -filtration and a ∇ -filtration belong to $\text{add } T$.

3. The functor $\alpha: \text{mod } T_2(\Lambda) \rightarrow \text{mod } A(\Lambda)$.

Here we deal with an arbitrary representation-finite algebra Λ with minimal Auslander generator E and consider the corresponding Auslander algebra $A(\Lambda) = (\text{End } E)^{\text{op}}$. We have denoted $T_2(\Lambda)$ the ring of all upper triangular (2×2) -matrices with coefficients in Λ . The category of $T_2(\Lambda)$ -modules may be seen as the category of all morphisms $X \rightarrow Y$ in $\text{mod } \Lambda$. We consider the functor

$$\alpha: \text{mod } T_2(\Lambda) \rightarrow \text{mod } A(\Lambda)$$

defined by $\alpha(f) = \text{Cok Hom}_\Lambda(E, f)$ for a morphism f in $\text{mod } \Lambda$

Proposition 3. *Let \mathcal{X} consist of the two objects $(1: E \rightarrow E)$ and $(E \rightarrow 0)$ in $\text{mod } T_2(\Lambda)$. The functor $\alpha: \text{mod } T_2(\Lambda) \rightarrow \text{mod } A(\Lambda)$ yields an equivalence $(\text{mod } T_2(\Lambda))/\mathcal{X} \rightarrow \text{mod } A(\Lambda)$.*

Thus, if the number of isomorphism classes of indecomposable Λ -modules is m , then $\alpha: \text{mod } T_2(\Lambda) \rightarrow \text{mod } A(\Lambda)$ is a full, dense, objective functor with index $2m$.

For a slightly weaker statement we refer to Theorem 1.1 in [AR2]. Let us recall and complete the proof. Given a morphism $f: X \rightarrow Y$ in $\text{mod } \Lambda$, we obtain an exact sequence

$$\text{Hom}_\Lambda(E, X) \xrightarrow{\text{Hom}_\Lambda(E, f)} \text{Hom}_\Lambda(E, Y) \rightarrow \alpha(f) \rightarrow 0.$$

Since the $A(\Lambda)$ -modules $\text{Hom}_\Lambda(E, X)$ and $\text{Hom}_\Lambda(E, Y)$ are projective, we obtain in this way a projective presentation of $\alpha(f)$. Conversely, given an $A(\Lambda)$ -module M , take a projective resolution $P_1 \xrightarrow{p} P_0 \rightarrow M \rightarrow 0$. Now, the category of projective $A(\Lambda)$ -modules is equivalent to the category $\text{mod } \Lambda$, thus we can assume that there are Λ -modules X and Y such that $P_1 = \text{Hom}_\Lambda(E, X)$ and $P_0 = \text{Hom}_\Lambda(E, Y)$ and a Λ -homomorphism $f: X \rightarrow Y$ such that $\text{Hom}_\Lambda(E, f) = p$. In this way, we see that the functor is dense. Similarly, starting with an $A(\Lambda)$ -homomorphism $M \rightarrow M'$, we can lift it to projective presentations of M and M' and using again the fact that the category of projective $A(\Lambda)$ -modules is equivalent to the category $\text{mod } \Lambda$, we see that $M \rightarrow M'$ is in the image of the functor α . Thus, it remains to calculate the kernel of α .

Of course, under this functor α the two objects in \mathcal{X} are sent to zero. Thus the ideal $\langle \mathcal{X} \rangle$ is contained in the kernel of α . Conversely, assume that there is given a map

$$(g_1, g_0): (f: X_1 \rightarrow X_0) \longrightarrow (f': X'_1 \rightarrow X'_0)$$

(thus $g_0 f = f' g_1$), such that $\alpha(g_1, g_0) = 0$. Thus the following diagram commutes and its rows are projective presentations:

$$\begin{array}{ccccccc} \text{Hom}_\Lambda(E, X_1) & \xrightarrow{\text{Hom}_\Lambda(E, f)} & \text{Hom}_\Lambda(E, X_0) & \xrightarrow{e} & \alpha(f) & \longrightarrow & 0 \\ \text{Hom}_\Lambda(E, g_1) \downarrow & & \downarrow \text{Hom}_\Lambda(E, g_0) & & \downarrow \alpha(g_1, g_0)=0 & & \\ \text{Hom}_\Lambda(E, X'_1) & \xrightarrow{\text{Hom}_\Lambda(E, f')} & \text{Hom}_\Lambda(E, X'_0) & \xrightarrow{e'} & \alpha(f') & \longrightarrow & 0. \end{array}$$

Let us show that the map (g_1, g_0) factors through $[1\ 0]: X_0 \oplus X_1 \rightarrow X_0$. Now $e' \text{Hom}_\Lambda(E, g_0) = 0$, thus there is a map $\tilde{h}: \text{Hom}_\Lambda(E, X_0) \rightarrow \text{Hom}_\Lambda(E, X'_1)$ such that $\text{Hom}_\Lambda(E, f')\tilde{h} = \text{Hom}_\Lambda(E, g_0)$. Again using the equivalence of $\text{mod } \Lambda_n$ and the category of projective A_n -modules, we get a map $h: X_0 \rightarrow X'_1$ with $\tilde{h} = \text{Hom}_\Lambda(E, h)$ and $g_0 = f'h$. Therefore $f'hf = g_0f = f'g_1$, and thus $f'(g_1 - hf) = 0$. As a consequence, the following diagram commutes

$$\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_0 \\
\left[\begin{array}{c} f \\ 1 \end{array} \right] \downarrow & & \downarrow 1 \\
X_0 \oplus X_1 & \xrightarrow{[1\ 0]} & X_0 \\
\left[\begin{array}{c} h \\ g_1 - hf \end{array} \right] \downarrow & & \downarrow g_0 \\
X'_1 & \xrightarrow{f'} & X'_0
\end{array}$$

and the composition of the vertical maps is just (g_1, g_0) . Since E is an Auslander generator, all Λ_n -modules belong to $\text{add } E$, thus $[1\ 0]: X_0 \oplus X_1 \rightarrow X_0$ belongs to $\text{add } \mathcal{X}$. This shows that the map (g_1, g_0) belongs to $\langle \mathcal{X} \rangle$. \square

4. The functor $F_1: \mathcal{S}(n) \rightarrow \mathcal{F}(n)$.

It may be appropriate to focus first the attention to the category $\mathcal{S}(n)$ itself. An object (X, Y) of $\mathcal{S}(n)$ with inclusion map $u: X \rightarrow Y$ will also be denoted by u . This stresses the fact that objects of $\mathcal{S}(n)$ are given by maps in the category $\text{mod } \Lambda_n$, thus we consider $\mathcal{S}(n)$ as a full subcategory of the category of $T_2(\Lambda_n)$ -modules. As we have mentioned, we denote by $\iota: \mathcal{S}(n) \rightarrow \text{mod } T_2(\Lambda_n)$ the inclusion functor. Note that $\mathcal{S}(n)$ turns out to be just the category of all Gorenstein-projective $T_2(\Lambda_n)$ -modules, see for example [Z].

We consider the restriction $\alpha\iota$ of α to the full subcategory $\mathcal{S}(n)$ of $\text{mod } T_2(\Lambda_n)$, thus

$$\alpha\iota(u) = \text{Cok Hom}_\Lambda(E, u).$$

for $(u: X \rightarrow Y)$ in $\mathcal{S}(n)$. Since u is a monomorphism, also $\text{Hom}_\Lambda(E, u)$ is a monomorphism, thus there is the following exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(E, X) \rightarrow \text{Hom}_\Lambda(E, Y) \rightarrow \alpha(u) \rightarrow 0.$$

Since both $\text{Hom}_\Lambda(E, X)$ and $\text{Hom}_\Lambda(E, Y)$ are projective A_n -modules, we see that the projective dimension of $\alpha(u)$ is at most 1, thus $\alpha(u)$ belongs to $\mathcal{F}(n)$. This shows that we can consider $\alpha\iota$ as a functor $\mathcal{S}(n) \rightarrow \mathcal{F}(n)$, we denote it by F_1 (thus $F_1(u) = \alpha(u)$, for $u \in \mathcal{S}(n)$).

Under the functor F_1 , we have

$$\begin{array}{ll}
F_1([i], [i]) = 0 & 1 \leq i \leq n, \\
F_1([i], [n]) = P(n)/P(i) = T(i+1) & \text{for } 0 \leq i \leq n-1, \\
F_1([0], [j]) = P(j) & 1 \leq j \leq n-1.
\end{array}$$

Proposition 4. *Let \mathcal{U}_1 be the set of objects of $\mathcal{S}(n)$ of the form $([i], [i])$ with $1 \leq i \leq n$. The functor F_1 yields an equivalence between the factor category of $\mathcal{S}(n)/\mathcal{U}_1$ and the category $\mathcal{F}(n)$. It maps the pairs $([i], [n])$ with $0 \leq i \leq n-1$ to $T(i+1)$ and the pairs $([0], [j])$ with $1 \leq j \leq n-1$ to $P(j)$.*

Proof. By definition, F_1 is the restriction of the functor α to the subcategory $\mathcal{S}(n)$ of $\text{mod } T_2(\Lambda_n)$. We have noted already that the image of F_1 consists of modules of projective dimension at most 1. Also, conversely, if M is a A_n -module of projective dimension at most 1, say with a projective presentation

$$0 \rightarrow P_1 \xrightarrow{p} P_0 \rightarrow M \rightarrow 0,$$

then we can write $p = \text{Hom}_\Lambda(E, f)$ for some map $f: X_1 \rightarrow X_0$ in $\text{mod } \Lambda_n$. Clearly, f has to be a monomorphism, since E is an Auslander generator. Thus we can assume that f belongs to $\mathcal{S}(n)$ and we have $F_1(f) = M$.

According to Proposition 3 the kernel of the functor F_1 is given by all morphisms in $\mathcal{S}(n)$ which factor through a $T_2(\Lambda_n)$ -module of the form $[1 \ 0]: V \oplus V' \rightarrow V$. Thus assume we have the following commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{u} & Y \\
 \downarrow [g' \] & & \downarrow g'' \\
 V \oplus V' & \xrightarrow{[1 \ 0]} & V \\
 \downarrow [h \ h'] & & \downarrow h'' \\
 X' & \xrightarrow{u'} & Y'.
 \end{array}
 \tag{*}$$

The commutativity of the lower square means that we have

$$[h'', 0] = h''[1, 0] = u'[h, h'] = [u'h, u'h'],$$

thus $u'h' = 0$. Since u' is a monomorphism, it follows that $h' = 0$. But then also the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{u} & Y \\
 g \downarrow & & \downarrow g'' \\
 V & \xrightarrow{1} & V \\
 h \downarrow & & \downarrow h'' \\
 X' & \xrightarrow{u'} & Y'.
 \end{array}$$

commutes and the composition of the vertical maps is the same as the composition of the vertical maps in $(*)$. This shows that the kernel of the functor F_1 consists of all the morphisms in $\mathcal{S}(n)$ which factor through objects of the form $(1: V \rightarrow V)$, but this is just the ideal \mathcal{U}_1 . \square

5. The functor $\delta: \text{mod } A_n \rightarrow \text{mod } \Pi_{n-1}$ and its restriction F_2 to $\mathcal{F}(n)$.

Recall that the indecomposable projective A_n -modules are $P(i) = \text{Hom}_\Lambda(E, [i])$ with $1 \leq i \leq n$, if necessary, we will denote them by $P_A(i)$. The module $P(n)$ is also injective and we may choose an idempotent $e(n)$ in A_n such that $P(n) = A_n e(n)$. Let Π_n be the preprojective algebra of type \mathbb{A}_n ; note that $A_n / \langle e(n) \rangle = \Pi_{n-1}$ (see [DR], Theorem 3, Theorem 4 and Chapter 7).

We consider in $\text{mod } A_n$ the following torsion pair: the torsion modules are the modules generated by $P(n)$, thus the modules with top being a direct sum of copies of $S(n)$, the torsionfree modules are the modules which do not have $S(n)$ as a composition factor (thus the torsionfree modules form a Serre subcategory). Note that the torsionfree modules are just the A_n -modules M with $e(n)M = 0$, thus these are the Π_{n-1} -modules.

Given an A_n -module M , let tM be its torsion submodule, and

$$\delta M = M/tM.$$

The indecomposable projective Π_{n-1} -modules are factor modules of the modules $P_A(i)$ with $1 \leq i \leq n-1$, we denote them by $P_\Pi(i) = P_A(i)/tP_A(i)$.

Since we want to keep track of the selected objects $([i], [j])$ in \mathcal{W} , let us repeat what happens under the functor δ :

$$\begin{aligned} \delta(T(i)) &= 0 & \text{for } & 1 \leq i \leq n, \\ \delta(P_A(j)) &= P_\Pi(j) & & 1 \leq j \leq n-1. \end{aligned}$$

Proposition 5. *The functor δ yields an equivalence between the factor category $\mathcal{F}(n)/T$ and the category $\text{mod } \Pi_{n-1}$. It maps $P_A(j)$ with $1 \leq j \leq n-1$ to $P_\Pi(j)$.*

Proof. First, let us show that the functor δ is dense. Thus, let N be a Π_{n-1} -module, but consider it as an A_n -module. Actually, all the modules to be considered now are A_n -modules; in particular $I(i)$ will denote (as before) the indecomposable injective A_n -module corresponding to the vertex i .

Let $u: N \rightarrow IN$ be an injective envelope of N (as an A_n -module!) and $p: PIN \rightarrow IN$ a projective cover of IN . Let T' be the kernel of p . Now IN is a direct sum of modules of the form $I(i)$ with $1 \leq i \leq n-1$ and the exact sequence $0 \rightarrow T(i+1) \rightarrow P(n) \rightarrow I(i) \rightarrow 0$ shows that the kernel of the projective cover $P(n) \rightarrow I(i)$ is $T(i+1)$. Thus we see that T' is a direct sum of copies of modules of the form $T(j)$ with $2 \leq j \leq n$.

Forming the induced exact sequence with respect to u , we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T' & \xrightarrow{v'} & \tilde{N} & \xrightarrow{p'} & N & \longrightarrow & 0 \\ & & \parallel & & \downarrow u' & & \downarrow u & & \\ 0 & \longrightarrow & T' & \xrightarrow{v} & PIN & \xrightarrow{p} & IN & \longrightarrow & 0 \end{array}$$

The monomorphism u' shows that \tilde{N} is torsionless, thus in $\mathcal{F}(n)$. Clearly, $tT' = T'$ and $tN = 0$, thus $t\tilde{N}$ is the image of v' and therefore $\delta(\tilde{N}) = N$. This shows that δ is dense.

In the same way, we see that δ is also full. Namely, given a morphism $f: N_1 \rightarrow N_2$ of Π_{n-1} -modules, we can extend f to a morphism $f': I(N_1) \rightarrow I(N_2)$ and then lift it to a morphism $f'': PI(N_1) \rightarrow PI(N_2)$. Using the pullback property of the commutative square

$$\begin{array}{ccc} \tilde{N}_2 & \xrightarrow{p'} & N_2 \\ \downarrow u' & & \downarrow u \\ PI(N_2) & \xrightarrow{p} & I(N_2) \end{array}$$

we finally obtain a map $\tilde{f}: \tilde{N}_1 \rightarrow \tilde{N}_2$ such that $\delta(\tilde{f}) = f$.

It remains to determine the kernel of δ . Since the modules $T(i)$ are generated by $T(1)$, they are torsion modules, thus $\delta(T(i)) = 0$ for all $1 \leq i \leq n$. Conversely, let M_1, M_2 belong to $\mathcal{F}(n)$ and let $f: M_1 \rightarrow M_2$ be a homomorphism such that $\delta(f) = 0$. This means that the image of f is contained in the torsion submodule tM_2 . Now tM_2 is a submodule of $M_2 \in \mathcal{F}(n)$ and $\mathcal{F}(n)$ is closed under submodules, thus tM_2 has projective dimension at most 1. On the other hand, tM_2 is generated by $P(n)$. Thus Proposition 2 asserts that tM_2 is in $\text{add } T$. It follows that f belongs to the ideal $\langle T \rangle$.

The last assertion has been mentioned already before. \square

Remark 1. In order to show the density of the functor δ , the paper [DR] started with a Π_{n-1} -module N , considered it as an A_n -module and used a universal extension

$$0 \rightarrow T' \rightarrow \tilde{N} \rightarrow N \rightarrow 0$$

of N by a module T' in $\text{add } T$. Of course, the pullback recipe given in the proof above provides such a universal extension.

Remark 2. Looking at the exact sequence

$$0 \rightarrow T' \rightarrow \tilde{N} \rightarrow N \rightarrow 0$$

constructed in the proof of Proposition 5, one may decompose $T' = \bigoplus_i T(i)^{n(i)}$. Then $n(1) = 0$ and $n(i+1)$ is precisely the multiplicity of $S(i)$ in the socle of N .

Combining Propositions 4 and 5 we obtain the first part of the following proposition:

Proposition 6. *The functor $F: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$ is full, dense and its kernel is $\langle \mathcal{U} \rangle$, thus F induces an equivalence $\mathcal{S}(n)/\mathcal{U} \rightarrow \text{mod } \Pi_{n-1}$.*

Under such an equivalence, the objects of $\mathcal{S}(n)$ of the form $([0], [j])$ with $1 \leq j \leq n-1$ correspond to the indecomposable projective Π_{n-1} -modules.

The last assertion relies on the fact that the set of projective objects of an abelian category is uniquely determined. Since $F([0], [j]) = P_{\Pi}(j)$, for $1 \leq j \leq n-1$, we know that the objects $([0], [1]), \dots, ([0], [n-1])$ in $\mathcal{S}(n)/\mathcal{U}$ are the indecomposable projective objects in $\mathcal{S}(n)/\mathcal{U}$. Thus any equivalence between $\mathcal{S}(n)/\mathcal{U}$ and $\text{mod } \Pi_{n-1}$ sends these objects to the indecomposable projective Π_{n-1} -modules. \square

6. The functor $G: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$.

In this section, we are going to analyze the functor $G: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$. The essential observation is due to Auslander and Reiten, it is valid in the general setting of dealing with a representation-finite algebra Λ as discussed in section 3.

Lemma. *Let Λ be a representation-finite algebra with Auslander generator E and Auslander algebra $A = (\text{End } E)^{\text{op}}$. Let f be a morphism of Λ -modules. Then f is an epimorphism if and only if $\text{Hom}_A(\text{Hom}_\Lambda(E, \Lambda), \alpha(f)) = 0$.*

We may refer to Proposition 4.1 in [AR1], see also the formulation stated in [AR2] just after the proof of Theorem 1.1. But the reader should observe that for the proof one just has to use twice Yoneda isomorphisms. Namely, starting with the map $f: X \rightarrow Y$ of Λ -modules, consider the map $\text{Hom}_\Lambda(E, f): \text{Hom}_\Lambda(E, X) \rightarrow \text{Hom}_\Lambda(E, Y)$ of A -modules; by definition, $\alpha(f)$ is its cokernel, thus there is the exact sequence

$$\text{Hom}_\Lambda(E, X) \xrightarrow{\text{Hom}_\Lambda(E, f)} \text{Hom}_\Lambda(E, Y) \longrightarrow \alpha(f) \longrightarrow 0.$$

If we apply the functor $H = \text{Hom}_A(\text{Hom}_\Lambda(E, \Lambda), -)$, we obtain the sequence

$$H(\text{Hom}_\Lambda(E, X)) \xrightarrow{H(\text{Hom}_\Lambda(E, f))} H(\text{Hom}_\Lambda(E, Y)) \longrightarrow H(\alpha(f)) \longrightarrow 0,$$

it is exact, since $\text{Hom}_\Lambda(E, \Lambda)$ is a projective A -module. In this way, we see that $H(\alpha(f)) = \text{Hom}_A(\text{Hom}_\Lambda(E, \Lambda), \alpha(f))$ is the cokernel of $H(\text{Hom}_\Lambda(E, f))$. There is the following commutative diagram

$$\begin{array}{ccc} H(\text{Hom}_\Lambda(E, X)) & \xrightarrow{H(\text{Hom}_\Lambda(E, f))} & H(\text{Hom}_\Lambda(E, Y)) \\ \downarrow & & \downarrow \\ \text{Hom}_\Lambda(\Lambda, X) & \xrightarrow{\text{Hom}_\Lambda(\Lambda, f)} & \text{Hom}_\Lambda(\Lambda, Y) \end{array}$$

where the vertical maps are Yoneda isomorphisms. The lower map may be identified with the map f (again, this is due to Yoneda isomorphisms). Thus f is an epimorphism if and only if $\text{Hom}_\Lambda(\Lambda, f)$ is an epimorphism if and only if $\text{Hom}_A(\text{Hom}_\Lambda(E, \Lambda), \text{Hom}_\Lambda(E, f))$ is an epimorphism, if and only if the cokernel of $\text{Hom}_A(\text{Hom}_\Lambda(E, \Lambda), \text{Hom}_\Lambda(E, f))$ is zero, thus if and only if $\text{Hom}_A(\text{Hom}_\Lambda(E, \Lambda), \alpha(f)) = 0$. \square

Let us return to the special case of $\Lambda = \Lambda_n$ and $A = A_n$. Recall that we have chosen an idempotent $e(n)$ in A such that $A_n e(n) = P(n) = \text{Hom}_{\Lambda_n}(E, [n]) = \text{Hom}_{\Lambda_n}(E, \Lambda_n)$ and we have identified $A_n / \langle e(n) \rangle = \Pi_{n-1}$. Thus, given an A_n -module M , the condition $\text{Hom}_{A_n}(\text{Hom}_{\Lambda_n}(E, \Lambda_n), M) = 0$ can be rewritten as $\text{Hom}_{A_n}(P(n), M) = 0$, thus as $e(n)M = 0$, and the modules M with this property are just the Π_{n-1} modules. Thus, in our case the Lemma asserts the following: *Let f be a morphism of Λ_n -modules. Then f is an epimorphism if and only if $\alpha(f)$ is a Π_{n-1} -module.*

We consider now the functor $G = \delta\alpha\epsilon$. If $u: X \rightarrow Y$ in $\mathcal{S}(n)$, then $\epsilon(u)$ is an epimorphism, thus $\alpha\epsilon(u)$ is a Π_{n-1} -module and therefore

$$G(u) = \delta\alpha\epsilon(u) = \alpha\epsilon(u).$$

This shows that $G: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$ is defined by

$$G(X, Y) = \text{Cok Hom}_\Lambda(E, Y \rightarrow Y/X),$$

we form the cokernel map $q = \epsilon(u): Y \rightarrow Y/X$ of the inclusion map $u: X \rightarrow Y$, apply the functor $\text{Hom}_\Lambda(E, -)$ so that we obtain a map

$$\text{Hom}_\Lambda(E, q): \text{Hom}_\Lambda(E, Y) \rightarrow \text{Hom}_\Lambda(E, Y/X)$$

and take its cokernel $\text{Cok Hom}_\Lambda(E, q)$.

Remark. We may phrase the definition of G also differently: Let $\mathcal{Q}(n)$ be the category of all epimorphisms $(Y \rightarrow Z)$ in $\text{mod } \Lambda_n$, this is again a subcategory of $\text{mod } T_2(\Lambda_n)$ and we may look at the restriction of the functor α to $\mathcal{Q}(n)$. Of course, there is an obvious categorical equivalence $\mathcal{S}(n) \rightarrow \mathcal{Q}(n)$, it sends an object $(u: X \rightarrow Y)$ to the canonical map $(Y \rightarrow Y/X)$, this is just $\epsilon(u)$ and $G = \alpha\epsilon$.

Instead of looking at the categories $\mathcal{S}(n)$ and $\mathcal{Q}(n)$, we may consider the category $\mathcal{E}(n)$ of all short exact sequences in the category $\text{mod } \Lambda_n$. There are forgetful functors $\mathcal{E}(n) \rightarrow \mathcal{S}(n)$ and $\mathcal{E}(n) \rightarrow \mathcal{Q}(n)$ which send an exact sequence $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{q} Z \rightarrow 0$ to u or q , respectively. Obviously, both functors are categorical equivalences.

Under the functor G , we have

$$\begin{aligned} G([i], [i]) &= 0 & 1 \leq i \leq n, \\ G([i], [n]) &= P_\Pi(n-i) & \text{for } 1 \leq i \leq n-1, \\ G([0], [j]) &= 0 & 1 \leq j \leq n. \end{aligned}$$

(The second assertion is seen as follows: Let $q: [n] \rightarrow [n-i]$ be a cokernel map for the inclusion map $u: [i] \rightarrow [n]$. If we apply $\text{Hom}_\Lambda(E, -)$ to the short exact sequence with maps u and q , we obtain the exact sequence

$$0 \rightarrow P(i) \xrightarrow{\text{Hom}_\Lambda(E, u)} P(n) \xrightarrow{\text{Hom}_\Lambda(E, q)} P(n-i).$$

Now the embedding $\text{Hom}_\Lambda(E, u)$ of $P(i)$ into $P(n)$ has as cokernel the module $T(i+1)$. In this way, $T(i+1)$ is embedded into $P(n-i)$; the image U of this embedding is the largest submodule of $P(n-i)$ generated by $P(n)$. This shows that $P(n-i)/U$ is equal to $P_\Pi(n-i)$. On the other hand, $P(n-i)/U$ is just the cokernel of $\text{Hom}_\Lambda(E, q)$, thus equal to $G([i], [n])$.

Proposition 7. *The functor $G: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$ is full, dense and $\langle V \rangle$ is its kernel. Thus G induces an equivalence $\mathcal{S}(n)/\mathcal{V} \rightarrow \text{mod } \Pi_{n-1}$.*

Under such an equivalence, the objects of $\mathcal{S}(n)$ of the form $([i], [n])$ with $1 \leq i \leq n-1$ correspond to the indecomposable projective Π_{n-1} -modules.

Proof. Proposition 3 asserts that α is a full functor from $\mathcal{Q}(n)$ onto the category $\text{mod } \Pi_{n-1}$ and that its kernel is the set of morphisms which factor through an object of the form $([1, 0]: V \oplus V' \rightarrow V)$. Under the equivalence $\mathcal{S}(n) \rightarrow \mathcal{Q}(n)$, the objects of the form

$([1, 0]: V \oplus V' \rightarrow V)$ in $\mathcal{Q}(n)$ correspond to the objects of the form $\begin{bmatrix} 0 \\ 1 \end{bmatrix}: V' \rightarrow V \oplus V'$ in $\mathcal{S}(n)$, but these are precisely the objects in $\text{add } \mathcal{V}$. This shows that $G = \alpha\epsilon$ yields an equivalence $\mathcal{S}(n)/\mathcal{V} \rightarrow \text{mod } \Pi_{n-1}$.

By proposition 3, we know that the functor $\alpha: \text{mod } T_2(\Lambda_n) \rightarrow \text{mod } A_n$ is full and dense. Let us show that the restriction of α to $\mathcal{Q}(n)$ is a dense functor $\mathcal{Q}(n) \rightarrow \text{mod } \Pi_{n-1}$. If M is a Π_{n-1} -module, the density of α provides a map f of Λ_n -modules such that $\alpha(f)$ is isomorphic to M . But since $\alpha(f)$ is a Π_{n-1} -module, we know that f is an epimorphism (see the reformulation of the Lemma above), thus f belongs to $\mathcal{Q}(n)$.

The last assertion of Proposition 7 relies again on the fact that the set of projective objects of an abelian category is uniquely determined by the categorical structure. \square

This completes the proof of Theorem 1.

7. Comparison of the functors F and G .

We want to compare the functor $F = F_2F_1$ and G . In order to prove the equality $\pi F = \Omega\pi G$, we start with an object $u: X \rightarrow Y$ in $\mathcal{S}(n)$, thus with an exact sequence

$$0 \rightarrow X \xrightarrow{u} Y \xrightarrow{q} Z \rightarrow 0.$$

We apply the functor $\text{Hom}_\Lambda(E, -)$ and obtain the exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(E, X) \xrightarrow{\text{Hom}_\Lambda(E, u)} \text{Hom}_\Lambda(E, Y) \xrightarrow{\text{Hom}_\Lambda(E, q)} \text{Hom}_\Lambda(E, Z)$$

Now, the cokernel of $\text{Hom}_\Lambda(E, q)$ is $G(u)$. The cokernel of $\text{Hom}_\Lambda(E, u)$ and thus the image of $\text{Hom}_\Lambda(E, q)$ is $F_1(u)$, thus there is the following exact sequence

$$0 \rightarrow F_1(u) \rightarrow \text{Hom}_\Lambda(E, Z) \rightarrow G(u) \rightarrow 0$$

and we can assume that the map $F_1(u) \rightarrow \text{Hom}_\Lambda(E, Z)$ is an inclusion map. Since $G(u)$ is a Π_{n-1} -module, we have $tG(u) = 0$, therefore $t\text{Hom}_\Lambda(E, Z) \subseteq F_1(u)$, and therefore $t\text{Hom}_\Lambda(E, Z) = tF_1(u)$. Thus we have the following exact sequence:

$$0 \rightarrow F_1(u)/tF_1(u) \rightarrow \text{Hom}_\Lambda(E, Z)/t\text{Hom}_\Lambda(E, Z) \rightarrow G(u) \rightarrow 0.$$

Note that $F_1(u)/tF_1(u) = F(u)$ and that $\text{Hom}_\Lambda(E, Z)/t\text{Hom}_\Lambda(E, Z)$ is a projective Π_{n-1} -module. It follows that $F(u)$ coincides in $\underline{\text{mod}} \Pi_{n-1}$ with $\Omega G(u)$.

This completes the proof of Theorem 2.

Remark. It should be stressed that for $n \geq 2$, there cannot exist an endofunctor ϕ of $\text{mod } \Pi_{n-1}$ such that $F = \phi G$ or $\phi F = G$. For example, if we would have $F = \phi G$, then the set of objects of $\mathcal{S}(n)$ killed by G would be contained in the set of objects killed by F . However $([0], [1])$ is killed by G , but not by F . Similarly, $([1], [n])$ is killed by F , but not by G .

8. Abelian factor categories of triangulated categories

As a byproduct of our consideration, we see that here we deal with examples of triangulated categories \mathcal{T} with an ideal \mathcal{I} generated by an idempotent such that \mathcal{T}/\mathcal{I} is abelian.

Namely, let $\mathcal{T} = \underline{\mathcal{S}}(n)$ be the stable category of $\mathcal{S}(n)$, it is obtained from $\mathcal{S}(n)$ by factoring out the ideal generated by the objects $([0], [n])$ and $([n], [n])$. Now $\mathcal{S}(n)$ is in a natural way (see [RS2]) a Frobenius category such that $([0], [n])$ and $([n], [n])$ are the only indecomposable objects which are both projective and injective, thus the category \mathcal{T} is a triangulated category.

Let \mathcal{I} be the ideal in \mathcal{T} generated by the objects $([i], [j])$ with $1 \leq i \leq n-1$ and either $i = j$ or $j = n$. Then, as additive categories, we have equivalences

$$\mathcal{T}/\mathcal{I} \simeq \mathcal{S}(n)/\mathcal{U} \simeq \text{mod } \Pi_{n-1},$$

thus \mathcal{T}/\mathcal{I} is an abelian category.

Similarly, let \mathcal{J} be the ideal in \mathcal{T} generated by the objects $([i], [j])$ with $1 \leq j \leq n-1$ and either $i = j$ or $i = 0$. Then, as additive categories, we have equivalences

$$\mathcal{T}/\mathcal{J} \simeq \mathcal{S}(n)/\mathcal{V} \simeq \text{mod } \Pi_{n-1},$$

thus \mathcal{T}/\mathcal{J} is again an abelian category.

Here, the ideals \mathcal{I}, \mathcal{J} of \mathcal{T} each are generated by $2n-2$ indecomposable objects, whereas the rank of the Grothendieck group $K_0(\mathcal{T}/\mathcal{I})$ is $n-1$. One may compare this with examples which involve cluster categories and cluster tilted algebras, see [BMR] and [K]. Let T be a cluster tilting object in a cluster category \mathcal{T} say of type Q , where Q is a directed quiver with n vertices. Then the factor category \mathcal{T}/T is an abelian category and the rank of the Grothendieck group $K_0(\mathcal{T}/T)$ is equal to n , and this is also the number of isomorphism classes of indecomposable direct summands of T .

9. More objective functors $\mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$.

The main part of the paper was devoted to a detailed study of two objective functors $\mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$, but there are also other ones.

As in the previous section, let us denote by \mathcal{T} the stable category of $\mathcal{S}(n)$ by \mathcal{T} . Let \mathcal{I} be again the ideal of \mathcal{T} generated by the objects $([i], [j])$ with $1 \leq i \leq n-1$ and either $i = j$ or $j = n$, thus there is an equivalence $\zeta: \mathcal{T}/\mathcal{I} \rightarrow \text{mod } \Pi_{n-1}$.

Thus, let η be any autoequivalence of \mathcal{T} . Then the composition

$$\mathcal{S}(n) \rightarrow \mathcal{T} \xrightarrow{\eta} \mathcal{T} \rightarrow \mathcal{T}/\mathcal{I} \xrightarrow{\zeta} \text{mod } \Pi_{n-1}$$

(where the first and the third functor are the canonical projections) clearly is a full, dense, objective functor.

We recall from [RS1] that the stable category \mathcal{T} has non-trivial autoequivalences, for example the endofunctor induced by the Auslander-Reiten translation τ is an autoequivalence of order 6.

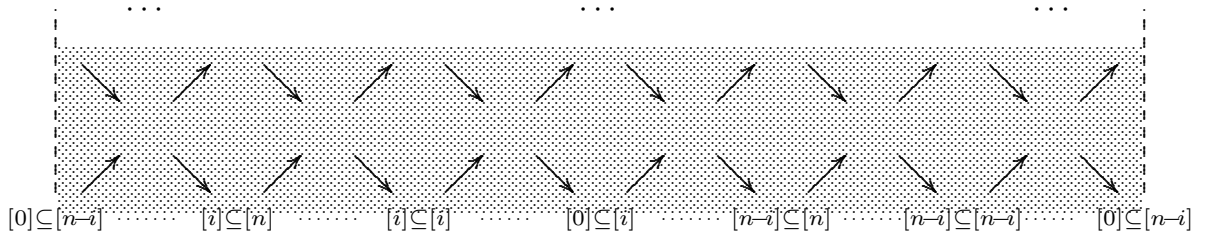
10. Auslander-Reiten orbits.

Some Auslander-Reiten orbits in the category $\mathcal{S}(n)$. Let τ be the Auslander-Reiten translation in $\mathcal{S}(n)$. The paper [RS1] describes in detail how to obtain for the pair (X, Y) in $\mathcal{S}(n)$ the pair $\tau(X, Y)$.

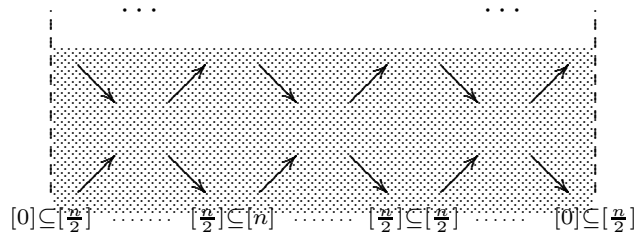
We are interested in some of the objects of the form $([i], [j])$ (with $0 \leq i \leq j \leq n$). The objects $([n], [n])$ and $([0], [n])$ are projective-injective, thus they are sent to zero by τ . The following assertions for $1 \leq i \leq n-1$

$$\begin{aligned}\tau([0], [i]) &= ([i], [i]) \\ \tau([i], [i]) &= ([i], [n]) \\ \tau([i], [n]) &= ([0], [n-i])\end{aligned}$$

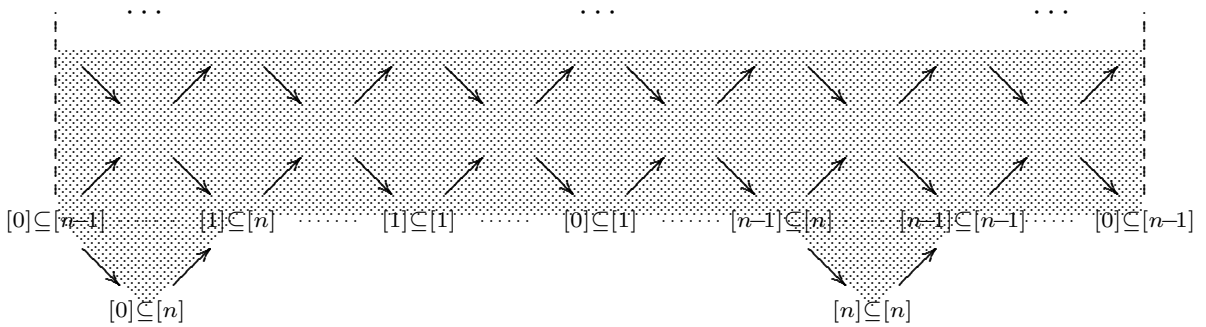
are easily verified. Of course, $1 \leq i \leq n-1$ implies that also $1 \leq n-i \leq n-1$, thus we see that the set of objects of the form $([0], [i]), ([i], [i]), ([i], [n])$ with $1 \leq i \leq n-1$ is closed under τ . Let us present the corresponding parts of Auslander-Reiten components. First of all, for $2 \leq i < \frac{n}{2}$, we deal with a τ -orbit of length 6:



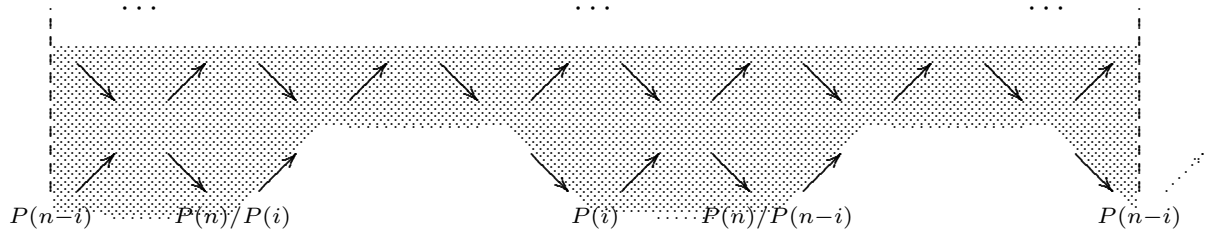
If $n \geq 4$ is even and $i = \frac{n}{2}$, there is the following orbit of length 3:



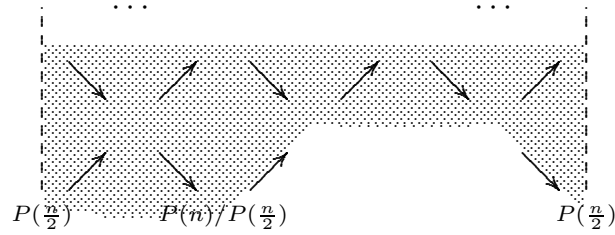
Finally, for $i = 1$ we get:



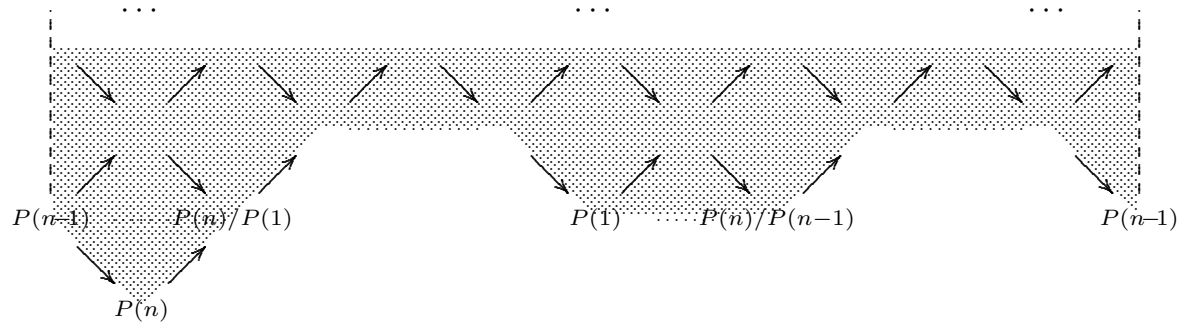
The corresponding τ -orbits in the category $\mathcal{F}(n)$. First, those for $2 \leq i < \frac{n}{2}$:



Second, for $n \geq 4$ even and $i = \frac{n}{2}$:



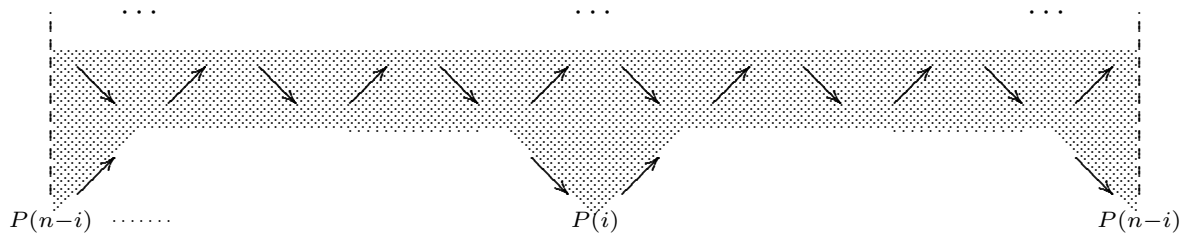
And finally, for $i = 1$ we get:



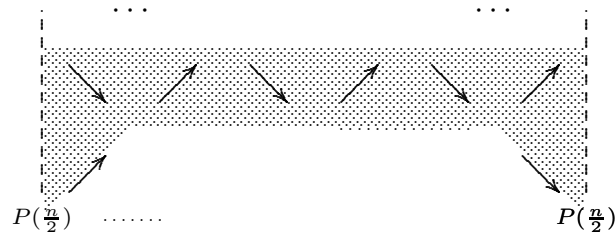
Remark. The paper [RS2] describes in detail the Auslander-Reiten quivers of the categories $\mathcal{S}(n)$ with $1 \leq n \leq 6$. Similarly, in [DR] the Auslander-Reiten quivers of the categories $\mathcal{F}(n)$ with $2 \leq n \leq 5$ are presented. As the functor F_1 shows, the Auslander-Reiten quiver of $\mathcal{F}(n)$ can be obtained from that of $\mathcal{S}(n)$ by just deleting some vertices, thus it is easy to obtain the illustrations presented in [DR] from those in [RS2]. The deletion process explains also some of the features of the shape of the Auslander-Reiten quiver of $\mathcal{F}(n)$: Of course, there is precisely one projective-injective vertex, namely the module $P(n) = I(n) = T(1)$. The remaining transjective orbits contain precisely two vertices, namely $T(n+1-i)$ and $P(i) = \tau_{\mathcal{F}(n)}T(n+1-i)$, here $1 \leq i \leq n-1$. As we now know, this concerns the τ -orbit of $\mathcal{S}(n)$ which contains the pairs $([i], [i])$ and $([n-i], [n-i])$: both pairs are killed by the functor F_1 , but in-between the Auslander-Reiten sequence starting with $([0], [i])$ and ending in $([n-i], [n])$ is not touched and it yields under F_1 the Auslander-Reiten sequence starting with $P(i)$ and ending in $T(n+1-i)$.

The corresponding τ -orbits of the category $\mathcal{S}(n)/\mathcal{U} \simeq \text{mod } \Pi_{n-1}$.

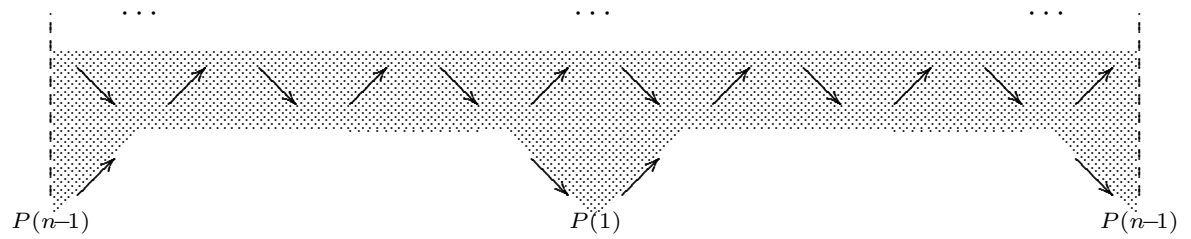
For $2 \leq i < \frac{n}{2}$:



For $n \geq 4$ even and $i = \frac{n}{2}$:



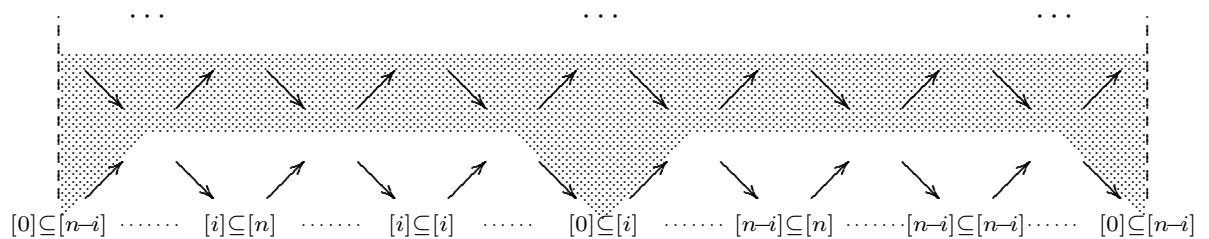
And finally, for $i = 1$ we get:



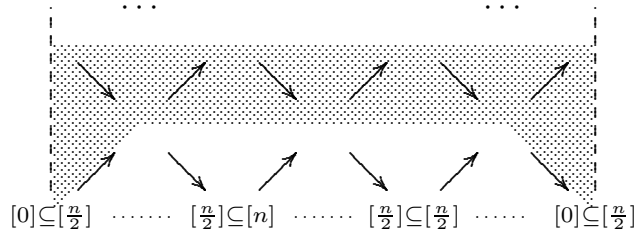
A simultaneous view. Let us draw again the relevant components of $\mathcal{S}(n)$ and use the shading in order to illustrate what remains when we delete the objects in \mathcal{U} . Note that under the functor F , we have

$$\begin{aligned} F([i], [i]) &= 0 & 1 \leq i \leq n, \\ F([i], [n]) &= 0 & \text{for } 0 \leq i \leq n-1, \\ F([0], [j]) &= P_{\Pi}(j) & 1 \leq j \leq n-1. \end{aligned}$$

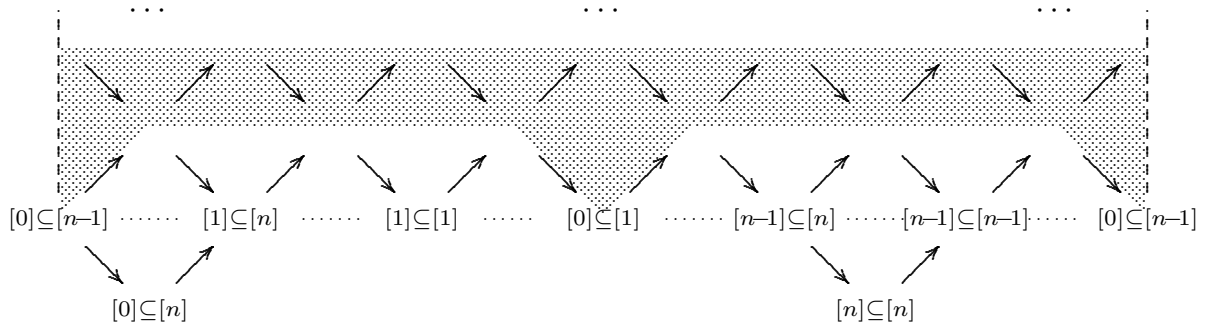
First of all, for $2 \leq i < \frac{n}{2}$, two objects of the $\tau_{\mathcal{S}}$ -orbit of $([i], [i])$ survive:



If $n \geq 4$ is even and $i = \frac{n}{2}$, the pair $([0], [i])$ is the only object in the $\tau_{\mathcal{S}}$ -orbit of $([i], [i])$ which survives:

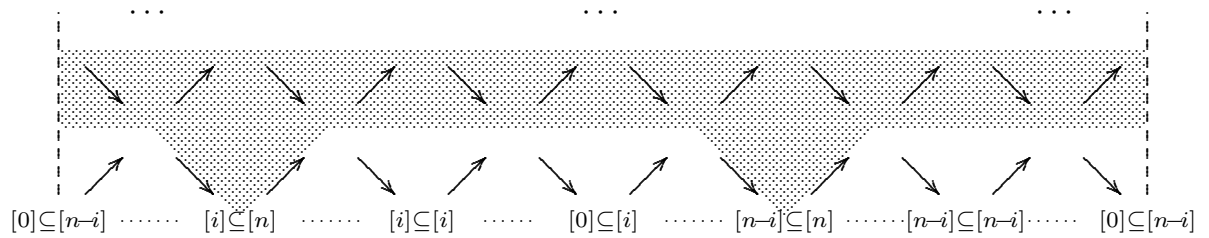


Finally, for $i = 1$, again two objects in the $\tau_{\mathcal{S}}$ -orbit of $([1], [1])$ survive:

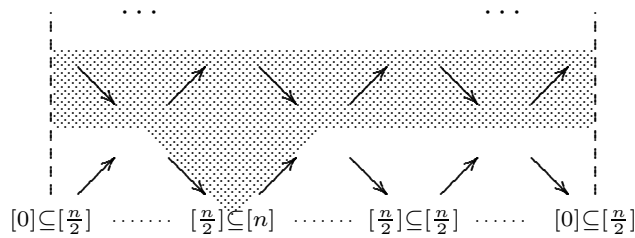


The relevant components of $\mathcal{S}(n)/\mathcal{V}$. In the same way as we have presented components of $\mathcal{S}(n)$ shading the parts which remain after deleting \mathcal{U} , we now show what remains from these components when we remove \mathcal{V} .

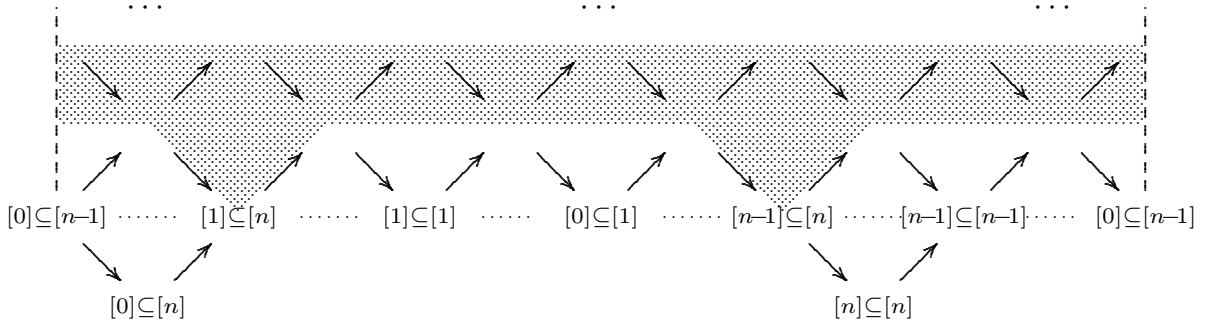
First of all, for $2 \leq i < \frac{n}{2}$, two objects in the $\tau_{\mathcal{S}}$ -orbit of $([i], [i])$ survive:



Next, for $n \geq 4$ even and $i = \frac{n}{2}$, the pair $([i], [n])$ is the only object in the $\tau_{\mathcal{S}}$ -orbit of $([i], [i])$ which survives:



Finally, for $i = 1$, again two objects in the $\tau_{\mathcal{S}}$ -orbit of $([i], [i])$ survive:



Bookkeeping 1. Going from $\mathcal{S}(n)$ to $\mathcal{F}(n)$, or from $\mathcal{F}(n)$ to $\text{mod } \Pi_{n-1}$, the number of indecomposable objects decreases in both step by n . Here is the bookkeeping table. We denote the number of isomorphism classes of indecomposable objects in the category \mathcal{C} by $\# \text{ind } \mathcal{C}$.

n	$\# \text{ind } \mathcal{S}(n)$	$\# \text{ind } \mathcal{F}(n)$	$\# \text{ind mod } \Pi_{n-1}$
1	2	1	0
2	5	3	1
3	10	7	4
4	20	16	12
5	50	45	40
6	∞	∞	∞

Bookkeeping 2. Going from $\underline{\mathcal{S}}(n)$ to $\text{mod } \Pi_{n-1}$ the number of indecomposables decreases by $2(n-1)$, going from $\text{mod } \Pi_{n-1}$ to $\underline{\text{mod}} \Pi_{n-1}$ the number decreases by $n-1$. Here are the actual numbers; for the triangulated categories $\underline{\mathcal{S}}(n)$ and $\underline{\text{mod}} \Pi_{n-1}$ we also list the tree type of the corresponding Auslander-Reiten quivers.

n	$\# \text{ind } \underline{\mathcal{S}}(n)$	$\# \text{ind mod } \Pi_{n-1}$	$\# \text{ind } \underline{\text{mod}} \Pi_{n-1}$
1	0	0	0
2	3 \mathbb{A}_2	1	0
3	8 \mathbb{D}_4	4	2 \mathbb{A}_1
4	18 \mathbb{E}_6	12	9 \mathbb{A}_3
5	48 \mathbb{E}_8	40	36 \mathbb{D}_6
6	∞	∞	∞

11. Appendix: Objective functors.

Let \mathcal{A}, \mathcal{B} be additive categories and let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an (additive) functor. An object A in \mathcal{A} will be called a *kernel object* for F provided $F(A) = 0$. The functor $F: \mathcal{A} \rightarrow \mathcal{B}$ will be said to be *objective* provided any morphism $f: A \rightarrow A'$ in \mathcal{A} with $F(f) = 0$ factors through a kernel object for F . If F is an objective functor, then we will say that the *kernel*

of F is generated by \mathcal{K} , provided \mathcal{K} is a class of objects in \mathcal{A} such that $\text{add } \mathcal{K}$ is the class of all kernel objects for F .

Given an additive category \mathcal{A} and an ideal \mathcal{I} in \mathcal{A} , we denote by \mathcal{A}/\mathcal{I} the corresponding factor category (it has the same objects, and the homomorphisms in \mathcal{A}/\mathcal{I} are the residue classes of the homomorphisms in \mathcal{A} modulo \mathcal{I}). If \mathcal{K} is a class of objects of the category \mathcal{A} , we denote by $\langle \mathcal{K} \rangle$ the ideal of \mathcal{A} given by all maps which factor through a direct sum of objects in \mathcal{K} . Instead of writing $\mathcal{A}/\langle \mathcal{K} \rangle$, we just will write \mathcal{A}/\mathcal{K} .

If $F: \mathcal{A} \rightarrow \mathcal{B}$ is a full, dense, objective functor and the kernel of F is generated by \mathcal{K} , then F induces an equivalence between the category \mathcal{A}/\mathcal{K} and \mathcal{B} . We see that given a full, dense, objective functor $F: \mathcal{A} \rightarrow \mathcal{B}$, the category \mathcal{B} is uniquely determined by \mathcal{A} and a class of indecomposable objects in \mathcal{A} (namely the class of indecomposable kernel objects for F); if F is objective, but not necessarily full or dense, then F induces an equivalence between the category \mathcal{A}/\mathcal{K} and the image category of F .

The composition of objective functors is not necessarily objective. Here is an example: Let \mathcal{B} be the linearization of the chain of cardinality 3, thus \mathcal{B} has three objects b_1, b_2, b_3 with $\text{Hom}(b_i, b_j) = k$ provided $i \leq j$ and zero otherwise, such that the composition $\text{Hom}(b_2, b_3) \otimes \text{Hom}(b_1, b_2) \rightarrow \text{Hom}(b_1, b_3)$ is the multiplication map. Let \mathcal{A} be the full subcategory of \mathcal{B} with objects b_1, b_3 , thus \mathcal{A} is the linearization of a chain of cardinality 2. Let $\mathcal{K} = \{b_2\}$ and $\mathcal{C} = \mathcal{B}/\mathcal{K}$. The inclusion functor $F: \mathcal{A} \rightarrow \mathcal{B}$ and the projection functor $G: \mathcal{B} \rightarrow \mathcal{C}$ both are (full and) objective, however the composition $GF: \mathcal{A} \rightarrow \mathcal{C}$ is not objective (none of the objects b_1, b_3 belongs to the kernel of GF , we have $\text{Hom}_{\mathcal{A}}(b_1, b_3) = k$ and any non-zero map $b_1 \rightarrow b_3$ is mapped to zero under GF). Note that the functor F is not dense.

Let $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$ be objective functors. If F is, in addition, full and dense, then GF is objective (thus, the composition of full, dense, objective functors is full, dense, objective). Proof. Since F, G both are full, also GF is full. Let $a: A_1 \rightarrow A_2$ be a morphism with $GF(a) = 0$. Since G is objective, the morphism $F(a)$ factors through a kernel object B for G , say $F(a) = b_2 b_1$ where $b_1: F(A_1) \rightarrow B$ and $b_2: B \rightarrow F(A_2)$. By assumption, F is dense, thus there is an isomorphism $b: B \rightarrow F(A)$ for some object A in \mathcal{A} . Since F is full, there is a map $a_1: A_1 \rightarrow A$ such that $F(a_1) = b b_1$ and a map $a_2: A \rightarrow A_2$ such that $F(a_2) = b_2 b^{-1}$. We have $F(a) = b_2 b^{-1} b b_1 = F(a_2) F(a_1) = F(a_2 a_1)$, thus $F(a - a_2 a_1) = 0$. Since F is objective, there is a kernel object A' for F such that $a - a_2 a_1$ factors through A' , say $a - a_2 a_1 = a_4 a_3$, with $a_3: A_1 \rightarrow A', a_4: A' \rightarrow A_2$. It follows that $a = a_2 a_1 + a_4 a_3 = \begin{bmatrix} a_2 & a_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix}$, thus this map factors through $A \oplus A'$. But $GF(A \oplus A') = GF(A) \oplus GF(A')$. Now, $F(A)$ is isomorphic to B , thus $GF(A)$ is isomorphic to $G(B) = 0$. Also, $F(A') = 0$, thus $GF(A') = 0$. This shows that $A \oplus A'$ is a kernel object for GF .

We recall that an additive category \mathcal{A} is said to be a Krull-Remak-Schmidt category, provided every object in \mathcal{A} is a (finite) direct sum of objects with local endomorphism rings. Assume now that $F: \mathcal{A} \rightarrow \mathcal{B}$ is an objective functor between Krull-Remak-Schmidt categories \mathcal{A} and \mathcal{B} . Then we are interested in the number $i_0(F)$ of isomorphism classes of indecomposable objects in \mathcal{F} which are kernel objects for F , as well as in the number $i_1(F)$ of isomorphism classes of indecomposable objects B in \mathcal{B} such that B is not isomorphic

to an object of the form $F(A)$ where A is an object in \mathcal{A} . If at least one of the numbers $i_0(F), i_1(F)$ is finite, we call $i(F) = i_0(F) - i_1(F)$ the *index* of F .

The objective functors F considered in the paper are also dense, in this case $i(F) = i_0(F)$ is the number of isomorphism classes of indecomposable kernel objects in \mathcal{A} .

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