Gorenstein-projective modules over short local algebras.

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Abstract: Following the well-established terminology in commutative algebra, any (not necessarily commutative) finite-dimensional local algebra \( A \) with radical \( J \) will be said to be short provided \( J^3 = 0 \). As in the commutative case, we show: if a short local algebra \( A \) has an indecomposable non-projective Gorenstein-projective module \( M \), then either \( A \) is self-injective (so that all modules are Gorenstein-projective) and then \( |J|^2 \leq 1 \), or else \( |J|^2 = |J/J^2| - 1 \) and \( |JM| = |J^2|M/JM| \). More generally, we focus the attention to semi-Gorenstein-projective and \( \infty \)-torsionfree modules, even to \( \infty \)-paths of length 2 or 4. In particular, we show that the existence of a non-projective reflexive module implies that \( |J|^2 < |J/J^2| \) and further restrictions. Also, we consider acyclic minimal complexes of projective modules. Again, as in the commutative case, we see that if such complexes do exist, then \( A \) is self-injective or satisfies the condition \( |J|^2 = |J/J^2| - 1 \). In addition, we draw the attention to the asymptotic behaviour of the Betti numbers of the modules. It may not be surprising that many arguments used in the commutative case actually work in general, but there are some interesting differences. On the other hand, some of our results seem to be new also in the commutative case. In particular, we show that any non-projective semi-Gorenstein-projective module \( M \) satisfies \( \text{Ext}^1(M,M) \neq 0 \). In this way, we see that the Auslander-Reiten conjecture (one of the classical homological conjectures) holds true for any short local algebra.

Key words. Short local algebra, Gorenstein-projective module, semi-Gorenstein-projective module, torsionless module, reflexive module, \( \infty \)-torsionfree module, Betti number, \( \infty \)-quiver, \( \infty \)-path, acyclic complex of projective modules, Auslander-Reiten conjecture.


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1. Introduction.

The modules to be considered are left modules of finite length over a finite-dimensional algebra \( A \) (if not otherwise asserted). We denote by \( |M| \) the length of the module \( M \) and define \( t(M) = t_0(M) = |\text{top } M| \). For \( i \in \mathbb{N} \), let \( t_i(M) = t(\Omega^i M) \), where \( \Omega M = \Omega_A M \) is the first syzygy module of \( M \) (as in commutative algebra [BH,L], one may call these numbers \( t_i(M) \) the \textit{Betti numbers of } \( M \).

A local algebra \( A \) with radical \( J = J(A) \) is said to be short provided \( J^3 = 0 \). All algebras considered here will be local finite-dimensional \( k \)-algebras, where \( k \) is a field, say with radical \( J \), and for simplicity, we will assume that \( A/J = k \). Let \( e = e(A) = |J/J^2| \).
Usually, we will assume that $A$ is short and then we write $a = a(A) = |J^2|$ and call $(e(A), a(A))$ the Hilbert-type of $A$.

We denote by $L(e)$ the local $k$-algebra with $J^2 = 0$ and $|J| = e$ (thus $e(L(e)) = e$ and $a(L(e)) = 0$). If $A$ is a local algebra with $e(A) = e$, then $A/J^2 = L(e)$. We can interpret the $L(e)$-modules as the $A$-modules annihilated by $J^2$, thus as the $A$-modules of Loewy length at most 2. If $M$ is a module with Loewy length at most 2, we call $\dim M = (t(M), |JM|)$ (or its transpose) the dimension vector of $M$ (note that $\dim M$ is only defined for modules $M$ of Loewy length at most 2; we have $\dim S = (1, 0)$ and there is no module with dimension vector $(0, 1)$). Also, let us remark that $|M| = t(M) + |JM|$. We say that a module $M$ is bipartite provided $\text{soc} M = JM$. Non-zero bipartite modules have Loewy length 2. Note that a module has Loewy length at most 2 if and only if it is the direct sum of a bipartite and a semisimple module.

1.1. The aim of the paper is to show that for a short local algebra $A$, the existence of a non-projective Gorenstein-projective module, or of related modules and complexes, forces strong restrictions on the structure of $A$. There will be a second paper [RZ3] devoted to this topic; it will deal with the $\Omega$-growth of modules and, in particular, with the Koszul modules as introduced by Herzog and Iyengar in [HI].

In the last 40 years, short local algebras have found a lot of attention in commutative algebra, since they turned out to provide counter-examples to several conjectures, see [AIS] for a corresponding account. We want to draw the attention to short local algebras which are not necessarily commutative and show in which way results known in the commutative case can be extended to non-commutative algebras. We follow investigations of Yoshino [Y], Christensen-Veliche [CV] and Lescot [L] looking on the one hand at Gorenstein-projective modules and, more generally, at acyclic complexes of projective modules, but also say at reflexive modules, and, on the other hand, at the asymptotic behaviour of Betti numbers, thus of projective resolutions.

It turns out that many arguments used in the commutative case work in general, but there are also some decisive differences. For the convenience of the reader, we are going to provide complete proofs, the only exception will be the use of the appendix of [CV], see 9.1 in the present paper, as well as of some basic observations mentioned in [RZ1].

We often will assume that $A$ is not self-injective (after all, over a self-injective algebra, all the modules are Gorenstein-projective). Note that if $A$ is self-injective, then $e \leq 1$ or $a = 1$ (see 3.1). The representation theory of the self-injective algebras (as well as the representation theory of the radical square zero algebras) is quite well understood. Our Appendix A provides a survey on the shape of the module category of a local algebra with radical square zero and of a self-injective short local algebra.

1.2. Existence of $\mathcal{O}$-paths of length 2 and length 4. Let us recall the notion of the $\mathcal{O}$-quiver of $A$ as introduced in [RZ1]. Given a module $M$, let $\mathcal{O}M$ be the cokernel of a left minimal left $\text{add}(A A)$-approximation $M \to P$. The vertices of the $\mathcal{O}$-quiver of $A$ are the isomorphism classes $[M]$ of the indecomposable non-projective modules $M$, and there is an arrow $[M] \leftrightarrow [M']$ provided that $M$ is torsionless and $M' \simeq \mathcal{O}M$. An $\mathcal{O}$-path is by definition a path in the $\mathcal{O}$-quiver. If $M$ is indecomposable, torsionless and non-projective, then there is an exact sequence $0 \to M \to P \to \mathcal{O}M \to 0$ with $P$ projective; such a sequence is called an $\mathcal{O}$-sequence.
It is well-known that an indecomposable non-projective module $M$ is reflexive if and only if $M$ is the end of an $\mathcal{O}$-path of length 2 (see, for example, [RZ1]). The first theorem concerns the existence of non-projective reflexive modules, thus of $\mathcal{O}$-paths of length 2.

We say that a non-zero module $M$ of Loewy length at most 2 is solid provided any endomorphism of $M$ is a scalar multiplication on $\text{soc} M$ (thus, any non-invertible endomorphism vanishes on the socle). A solid module is of course indecomposable (a characterization of the solid modules will be given in A.2 in Appendix A). In general, $A J$ may be solid, whereas $J A$ is not solid, as the example 5.5 shows.

**Theorem 1.** Let $A$ be a short local algebra. Assume that there exists a reflexive module which is not projective. Then either $a = e = 1$ or else $a \leq e - 1$. Always, $A J$ is a solid module and $J A$ is a solid right module.

Of course, if $A$ has a non-projective reflexive module, then the same is true for $A^{\text{op}}$. Thus, if we show that $A J$ is a solid module, then this shows also that the right module $J A$ has to be solid. The proof of Theorem 1 will be given in section 5.

Since a solid module is indecomposable, it is either simple or bipartite. Thus we see: If $A$ is a short local algebra with a non-projective reflexive module, then $A J$ is either simple, thus $a = 0$ and $e = 1$, or else bipartite, thus $J^2 = \text{soc} A A$ (and also $J^2 = \text{soc} A A$).

**Corollary.** Let $A$ be a short local algebra. Assume that there exists a reflexive module which is not projective. Then either $A$ is self-injective or else $2 \leq a \leq e - 1$.

Namely, if $A$ is a short local algebra with $a \leq 1$ and $A J$ is solid, then $A J$ is uniform and therefore $A$ is self-injective (see 3.2; we recall that a module is said to be uniform provided its socle is simple). Thus, we can assume that $a \geq 2$. Theorem 1 yields the assertion. □

As we will see in section 15, the bound $a \leq e - 1$ in Theorem 1 cannot be improved: for any pair $(e, a)$ with $1 \leq a \leq e - 1$, there exists a short local algebra of Hilbert type $(e, a)$ with non-projective reflexive modules.

Let us jump to $\mathcal{O}$-paths of length 4.

**Theorem 2.** Let $A$ be a short local algebra which is not self-injective. If there exists an $\mathcal{O}$-path of length 4, then $a = e - 1 \geq 2$ (and the modules $A J$ and $J A$ are solid).

### 1.3. Existence of acyclic minimal complexes of projective modules.

A complex $P_\bullet = (P_i, d_i : P_i \to P_{i-1})$, thus

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \to \cdots,$$

with projective modules $P_i$ is said to be minimal provided that any map $d_i : P_i \to P_{i-1}$ maps into the radical of $P_{i-1}$ (thus, is a radical map).

**Theorem 3.** Let $A$ be a short local algebra which is not self-injective. If there exists a non-zero acyclic minimal complex of projective modules, then $a = e - 1 \geq 1$.

Let $P_\bullet = (P_i, d_i)$ be a non-zero acyclic minimal complex of projective modules. Let $M_i$ be the image of $d_i$ and $t_i = t(P_i) = t(M_i)$. Then there are two possibilities:
Type I. We have $t_i = t$ for all $i \in \mathbb{Z}$. Then all the modules $M_i$ are bipartite with $\dim M_i = (t, at)$.

Type II. There is an index $v \in \mathbb{Z}$ such that $t_i = t$ for all $i \leq v$, whereas $t_{i+1} > t_i$ for $i \geq v$. The module $M_{v+1}$ is not bipartite, whereas $M_i$ is bipartite for $i < 0$. Also, $|JM_i| < at_i$ for $i > v$, whereas $\dim M_i = (t, at)$ for all $i \leq v$.

For commutative rings, Theorem 3 is due to Christensen-Veliche [CV]. For commutative rings the case $a = 1$ does not occur, but in general it does, see Example 9.3. Also, for $A$ commutative, and $P_\bullet$, a complex of type II, all the modules $M_i$ with $i \leq v$ are bipartite. Actually, for $A$ commutative, the existence of a non-zero acyclic minimal complex $P_\bullet$ of projective modules implies that $J^2 = \text{soc} A$. In general, for non-commutative short local rings, the existence of a non-zero acyclic minimal complex $P_\bullet$ of projective modules does neither imply that $J^2 = \text{soc} A$, nor that $J^2 = \text{soc} A_N$, see the examples in 9.3 and 9.4. In 15.3, we will show that if $J^2 = \text{soc} A$, then all the modules $M_i$ with $i \leq v$ are bipartite.

In case $a = e - 1$, we call $\delta(M) = et(M) - |M| = at(M) - |JM|$ the defect of the module $M$. The relevant properties of the defect can be found in section 7. For a complex of type I, all the images $M_i$ have defect zero; for a complex of type II the images $M_i$ with $i \leq v$ have defect zero, whereas $\delta(M_i) > 0$ for $i > v$.

Theorem 3 describes the structure of an acyclic minimal complex of projective modules, in case $A$ is not self-injective: there are just 2 possibilities, namely the complexes of type I and of type II. For the structure of an acyclic minimal complex of projective modules over a self-injective short local algebra, see Appendix A.7, Corollary.

1.4. Existence of semi-Gorenstein-projective and $\infty$-torsionfree modules. Both Theorems 2 and 3 imply: If there are Gorenstein-projective modules which are not projective, then $A_J$ and $J_A$ are solid modules and either $A$ is self-injective, or else $a = e - 1 \geq 2$. There is the following generalization.

Let us recall from [RZ1] that a module $M$ with $\text{Ext}^i(M, AA) = 0$ for all $i \geq 1$ is said to be semi-Gorenstein-projective, and that $M$ is $\infty$-torsionfree provided its transpose $\text{Tr} M$ is a semi-Gorenstein-projective right module. If $M$ is $\infty$-torsionfree, then $M$ is reflexive. If $M$ is semi-Gorenstein-projective, then $\Omega^2 M$ is semi-Gorenstein-projective and reflexive (and if $M$ is semi-Gorenstein-projective and not projective, also $\Omega^2 M$ is not projective). Thus, the existence of a non-projective module which is semi-Gorenstein-projective or $\infty$-torsionfree implies the existence of non-projective reflexive modules.

Theorem 4. Let $A$ be a short local algebra which is not self-injective. Assume that there exists a non-projective module $M$ which is semi-Gorenstein-projective or $\infty$-torsionfree. Then $A_J$ and $J_A$ are solid modules and $a = e - 1 \geq 2$.

Let $t = t(M)$. If $M$ is indecomposable, torsionless and semi-Gorenstein-projective, then $\dim \Omega^i M = (t, at)$ for all $i \geq 0$. If $M$ is indecomposable, and $\infty$-torsionfree, then $\dim \Omega^i M = (t, at)$ for all $i \geq 0$. If $M$ is both semi-Gorenstein-projective and reflexive, or if $M$ is $\infty$-torsionfree, then also $\dim M^* = (t, at)$.

Remark. If $M$ is semi-Gorenstein-projective, but not reflexive, with $\dim M = (t, at)$, then we may have $\dim M^* \neq (t, at)$, see 9.6.
For any natural number \( c \geq 0 \), we will exhibit in section 11 a short local algebra \( \Lambda_c \) of Hilbert type \((3 + c, 2 + c)\) with modules \( M, M', M'' \) of length \( a + 1 \) such that \( M \) is Gorenstein-projective, \( M' \) is semi-Gorenstein-projective and not torsionless, and \( M'' \) is \( \infty \)-torsionfree, with \( \Omega M'' \) having a simple direct summand. In particular, \( A \) has non-zero acyclic minimal complexes of projective modules both of type I and of type II. The algebra \( \Lambda_c \) is a \( k \)-algebra of dimension \( 6 + 2c \), where \( k \) is a field with an element \( q \neq 0 \) with infinite multiplicative order. The algebra \( \Lambda_0 \) has been considered already in \([RZ1]\) and \([RZ2]\), the general case is an easy modification.

1.5. TheAuslander-Reiten conjecture. Using Theorem 4 as well as the Appendix A, we get the following result.

**Theorem 5.** Let \( A \) be a short local algebra. If \( M \) is a non-projective semi-Gorenstein-projective module, then \( \text{Ext}^1(M, M) \neq 0 \).

Recall that the Auslander-Reiten conjecture \([AR]\) for an artin algebra \( A \) asserts: *if \( M \) is a non-projective semi-Gorenstein-projective module, then \( \text{Ext}^i(M, M) \neq 0 \) for some \( i \geq 1 \).* Thus, theorem 5 shows that the Auslander-Reiten conjecture holds true for short local algebras (and not just one of the groups \( \text{Ext}^i(M, M) \) with \( i \geq 1 \) is non-zero, but it is always already the first group \( \text{Ext}^1(M, M) \) which is non-zero). In case \( A \) is self-injective, Theorem 5 is due to Hoshion \([Ho1,Ho2]\). Note that in case \( A \) is a short local algebra which is not self-injective, and \( M \) is a non-projective semi-Gorenstein-projective module, then one even has \( \text{Ext}^i(M, M) \neq 0 \) for all \( i \geq 1 \), see 12.4.

1.6. Summary. Looking at Theorems 2, 3, 4, we should remark that the decisive assertions \( a(A) = e(A) - 1 \) and \( \dim M = (t, at) \) can be rewritten in the form

\[
|A J^2| = |A J/J^2| - 1 \quad \text{and} \quad |JM| = |A J^2| |M/JM|, 
\]

respectively, where \( J = J(A) \).

As in the commutative case, we see that there is an interesting trichotomy for short local algebras \( A \).

- **First**, there is the case \( a = 1 \). This includes the self-injective short local algebras with \( J^2 \neq 0 \).
- **Second**, there is the case \( a = e - 1 \).
- **Third**, there are the short local algebras with \( a \notin \{1, e - 1\} \) (this includes the case \( a = 0 \): the local algebras with \( J^2 = 0 \)). If \( A \) is a short local algebra with \( a \notin \{1, e - 1\} \), and \( (e, a) \neq (1, 0) \), then there are no non-zero acyclic minimal complexes of projective modules and also no \( \tilde{U} \)-paths of length 4.

For the first and the second case, there is the overlap \( a = 1, e = 2 \).

The short local algebras with \( a = e - 1 \) (the second case mentioned above) are of special interest. Examples of such algebras have been studied by Gasbarov-Peeva \([GP, 1990]\), Avramov-Gasharov-Peeva \([AGP, 1997]\), Veliche \([V, 2002]\), Yoshino \([Y, 2002]\), Jorgensen-Šega \([JS, 2006]\), Christensen-Veliche \([CV, 2007]\), Hughes-Jorgensen-Šega \([HJS, 2009]\). A certain non-commutative short local algebra \( A \) of Hilbert type \((3, 2)\) has been studied in
detail in [RZ1,RZ2] (the construction will be generalized in section 11 in order to obtain short local algebras of Hilbert type \((e, e - 1)\) with \(e \geq 3\) which have similar properties). For a further discussion of short local algebras of Hilbert type \((3, 2)\), see the forthcoming paper [RZ4].

As we have seen, the algebras \(A\) with \(a \notin \{1, e - 1\}\) do not have long \(\mathcal{U}\)-paths nor non-zero acyclic minimal complexes of projective modules. But we should stress that also algebras \(A\) with \(a \in \{1, e - 1\}\) may have neither \(\mathcal{U}\)-paths of length 2 nor non-zero acyclic minimal complexes of projective modules, see section 14.

**1.7. Outline of the paper.** Sections 2 and 8 are devoted to the simple module \(S\), its syzygies and the \(\mathcal{U}\)-component which contains \(S\). In particular, in 8.2 we show that \(\lim_n t_n(S) = \infty\), provided \(e \geq 2\). The proof of Theorems 1 and 2 are given in section 5 and 6, respectively. The proofs of theorems 3 and 4 can be found in section 9. The proof of theorem 5 is given in section 12.

Sections 3, 6, and 7 deal with the various possibilities for \(a\). There are the algebras with \(a \leq 1\). If \(A\) is self-injective, then either \(a = 0\) and \(e \leq 1\) or else \(a = 1\). These algebras are well-understood, see section 3 and the appendix A (the appendix A provides an overview over the relevant properties of self-injective short local algebras, as well as the related local algebras with radical square zero). The essence of sections 5 and 6 is: If one is interested in acyclic complexes of projective modules, or in long \(\mathcal{U}\)-paths, then the cases \(a \geq e\) and \(2 \leq a \leq e - 2\) can be discarded, and the case of interest is \(a = e - 1\). This case is considered in sections 7, 10, 11, 12 and in the examples 9.3 and 9.4 (but see also the forthcoming paper [RZ4]). In particular, we show in 10.2 that a commutative short local algebra of Hilbert type \((e, e - 1)\) has no complex of type II which involves a projective module of rank 1.

In section 14, we show that for \(e \geq 2\) and \(0 \leq a \leq e^2\), there are short local algebras of Hilbert type \((e, a)\) which have neither non-projective reflexive modules nor a non-zero acyclic minimal complex of projective modules. On the other hand, for \(1 \leq a \leq e - 1\), there are short local algebras of Hilbert type \((e, a)\) which have non-projective reflexive modules, see section 15.

The main tool in the paper will be the use of the transformation \(\omega_a^e\) on \(\mathbb{Z}^2\) as defined in section 4: it describes for suitable modules \(M\) in which way \(\dim M\) is changed when we apply \(\Omega_A\) (see the Main Lemma in sections 4 and 13, but also [RZ3]). The Main Lemma draws the attention to the possible equality \(t_2(M) = et_1(M) - at_0(M)\), see 4.4. Appendix B is devoted to the numbers \(b_n = b(e, a)_n\) defined recursively by the corresponding rule \(b_{n+1} = eb_n - ab_{n-1}\), starting with \(b_{-1} = 0, b_0 = 1\). It presents an explicit formula for these numbers \(b_n\) due to Avramov, Iyengar, Šega, provided \(a < \frac{1}{4}e^2\).

**1.8.** This paper and its successor [RZ3] want to outline some basic facts in the representation theory of short local algebras. For short local algebras \(A\), we have seen above the following trichotomy: there is the case \(a = 1\), second, there is the case \(a = e - 1\) and third, there are the algebras with \(a \notin \{1, e - 1\}\). The study of the \(\Omega\)-growth of modules and the study of Koszul modules in [RZ3] will show a further separation, namely between \(a \leq \frac{1}{4}e^2\) and \(a > \frac{1}{4}e^2\).

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2. The $\mathcal{U}$-component of the simple module $S$.

We start with some general observations concerning finite-dimensional local algebras $A$ which are not necessarily short. Let $S = k$ be the simple $A$-module. The following Lemma seems to be well-known (at least partially).

2.1. Lemma. Let $A$ be a finite-dimensional local algebra of Loewy-length $m$. The following assertions are equivalent:

(i) $\text{soc}_A A$ is simple.
(ii) $A$ is self-injective.
(iii) All modules are reflexive.
(iv) $S$ is reflexive.
(v) $\mathcal{U}S$ has Loewy length at most $m - 1$.

Proof: (i) $\implies$ (ii): If $\text{soc}_A A$ is simple, then the injective envelop of $A$ is indecomposable. But the indecomposable injective $A$-module has the same dimension as $A$, thus $A$ is injective. If (ii) is satisfied, then all modules are torsionless, therefore any module is reflexive. (iii) implies (iv) is trivial. (iv) $\implies$ (v): We assume that $S$ is reflexive, thus $\mathcal{U}S$ is torsionless. Of course, $S$ itself is torsionless, thus $\mathcal{U}S$ is indecomposable. Since $\mathcal{U}S$ is torsionless, indecomposable and not projective, it has Loewy length at most $m - 1$, thus condition (v) is satisfied.

(v) $\implies$ (i). Now assume that $\mathcal{U}S$ has Loewy length at most $m - 1$. Let $a = |J^m - 1|$. By assumption, $a \geq 1$. Since $S$ is torsionless, there is an $\mathcal{U}$-sequence $0 \to S \xrightarrow{u} P \xrightarrow{p} \mathcal{U}S \to 0$. Let $P$ be of rank $t$. Thus $t \geq 1$ and $|J^m - 1|P = at$. Since $\mathcal{U}S$ has Loewy length at most $m - 1$, $J^m - 1P$ is contained in the kernel of $p$, thus $at \leq 1$, and therefore $a = 1$ and $t = 1$.

Assume now that there is a simple submodule $U$ of $A$ which is not contained in $J^m - 1$. Let $v: U \to A$ be the inclusion map. Let $f: S \to U$ be an isomorphism. Since $u$ is a left $\text{add}(A)$-approximation, there is $f': P \to A$ with $f'u = vf$.

Let us assume that $f'$ is not surjective. Then the image of $f'$ is a module of Loewy length at most $m - 1$, thus $J^m - 1P$ is contained in the kernel of $f'$. We have $J^n - 1 \neq 0$. Since $J^n - 1 \subseteq \text{Ker}(p) = \text{Im}(u)$ and $\text{Im}(u)$ is simple, we see that $J^m - 1 = \text{Im}(u)$. It follows that $f'u = 0$ in contrast to $vf \neq 0$.

Thus we see that $f'$ is surjective. There is $f'': \mathcal{U}S \to A/U$ such that the following diagram commutes:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & S & \xrightarrow{u} & P & \xrightarrow{p} & \mathcal{U}S & \longrightarrow & 0 \\
\downarrow{f} & & \downarrow{f'} & & \downarrow{f''} & & \\
0 & \longrightarrow & U & \xrightarrow{v} & A & \longrightarrow & A/U & \longrightarrow & 0.
\end{array}
\]

Since $f'$ is surjective, also $f''$ is surjective. Since $J^m - 1$ is not contained in $U$, the module $A/U$ has Loewy length $m$. Therefore also $\mathcal{U}S$ has Loewy length $m$, a contradiction. This shows that $\text{soc}_A A \subseteq J^m - 1$. Since $a = 1$, it follows that $\text{soc}_A A$ is simple. \qed
Remark: Marczinzik [M1] has asked whether a finite-dimensional algebra is self-injective if all simple modules are reflexive.

The equivalence of (i) and (ii) in Lemma 2.1 implies the following: If $A$ is local and $e \leq 1$, so that $A$ is uniserial, $A$ has to be self-injective.

2.2. Lemma. Let $A$ be a local algebra. The following conditions are equivalent.
(i) $\operatorname{Ext}^1(S, AA) = 0$.
(ii) $A$ is self-injective.

Proof: Of course, (ii) implies (i). Conversely, assume that $\operatorname{Ext}^1(S, AA) = 0$. Then $\operatorname{Ext}^1(M, AA) = 0$ for all $A$-modules $M$, thus $AA$ is injective.

2.3. Corollary. Let $A$ be a local algebra which is not self-injective. Then the $\Omega$-component of $A$ which contains $S$ is of type $A_2$ with $[S]$ as its sink.

Proof. Since $S$ is torsionless, there is an arrow ending in $S$. Since $S$ is not reflexive, there is no path of length 2 ending in $S$. Since $\operatorname{Ext}^1(S, A) \neq 0$, no arrow starts in $S$.

We apply this to $\Omega$-sequences over short local algebras.

2.4. Corollary. Let $A$ be a short local algebra and $0 \to X \to P \to Z \to 0$ an $\Omega$-sequence.
(a) If $A$ is self-injective, then either $X$ is bipartite, or else $X$ is simple and then $Z = A/\operatorname{soc}_A A$.
(b) If $A$ is not self-injective, and $Z$ has Loewy length at most 2, then $Z$ is bipartite, and either $X$ is also bipartite or else $X$ is simple and $a = 0$, $e \geq 2$.

Proof. (a) The module $X$ is indecomposable and of Loewy length at most 2. Thus, if $X$ is not simple, then $X$ is bipartite. If $X = S$ is simple, then $Z = A/\operatorname{soc}_A A$.
(b) Both $X$ and $Z$ are indecomposable modules of Loewy length at most 2. Now $Z$ cannot be simple, since otherwise 2.3 asserts that $A$ is self-injective. Since $X$ is indecomposable, it is either bipartite or simple. If $X = S$ is simple, then 2.1 shows that the Loewy length of $A$ cannot be 3 (since we assume that $Z = \Omega S$ has Loewy length at most 2). Thus $a = 0$. Since $A$ is not self-injective, we have $e \geq 2$.

Let us add also the following observation.

2.5. Lemma. Let $A$ be a short local algebra. If $M$ is a reflexive module which is bipartite, then also $M^*$ is (reflexive and) bipartite.

Proof. We can assume that $M$ is indecomposable. If $M$ is projective, then $M = AA$ implies that $A$ has Loewy length of Loewy length 2, thus also $M^* = A_A$ is bipartite. Thus, we assume that $M$ is not projective. Of course, $M^*$ is torsionless. If $M^*$ would be projective, also $M$ would be projective. Thus $M^*$ has Loewy length at most 2. Also $M^*$ cannot be simple, since otherwise $A$ is self-injective and also $M$ is simple.

Proposition 6.1 will provide more information on the $A$-dual $M^*$ of a bipartite reflexive module $M$. 8
2.6. Example. If $M$ is torsionless and bipartite, then $M^*$ has Loewy length at most 2, but does not have to be bipartite.

Namely, if $M$ is bipartite, then $M$ is annihilated by $J^2$, thus any map $f: M \to A A$ maps into $J$. If $x \in J^2$, then the right multiplication on $r_x: A A$ by $x$ sends $J$ to 0, thus $r_x f = 0$. Thus shows that $M^*$ has Loewy length at most 2.

A typical example is given by the algebra $A = \Lambda_0$ considered in section 11 (and before in [RZ1]), namely the right $A$-module $m_1 A = (x - y) A$, as discussed in [RZ1]. Of course, $m_1 A$ is torsionless and bipartite, but $(m_1 A)^* = M(q)^{**} = \Omega M(1)$ (see 6.5 (8) and Theorem 1.6 in [RZ1]) has a simple direct summand.

3. Algebras with $a \leq 1$. Algebras with $e \leq 2$.

3.1. A short local algebra is self-injective if and only if either $a = 0$ and $e \leq 1$ or else $a = 1$ and $J^2 = \text{soc} A A$.

Proof. According to 2.1, $A$ is self-injective if and only if $\text{soc} A A$ is simple. First, assume that $A$ is self-injective. If $J^2 = 0$, and $J \neq 0$, then the socle of $A A$ is $J$, thus $a = 0$, $e = 1$; if $J^2 \neq 0$, then $J^2 \subseteq \text{soc} A A$, thus we must have $a = 1$ and $J^2 = \text{soc} A A$. Conversely, if either $a = 0$ and $e \leq 1$ or else $a = 1$ and $J^2 = \text{soc} A A$, then $\text{soc} A A$ is simple.

3.2. Short local algebras $A$ with $a \leq 1$. Of course, if $M$ is a module of Loewy length at most 2, then $JM$ is simple if and only if $M$ is the direct sum of a uniform module and a semisimple module. Thus, if $A$ is a short local algebra, then $a \leq 1$ and $e \geq 1$ if and only if $A J$ is the direct sum of a uniform module and a semisimple module.

Lemma. Let $A$ be a short local algebra with $a \leq 1$. The following assertions are equivalent:

(i) $A$ is self-injective and $J \neq 0$.
(ii) There exists a non-projective reflexive module.
(iii) $A J$ is solid.
(iv) $A J$ is indecomposable.
(v) $A J$ is uniform.
(vi) $A J$ is simple or bipartite.
(vii) Either $a = 0$ and $e = 1$, or else $a = 1$ and $J^2 = \text{soc} A A$.

The proof is straightforward: (i) $\implies$ (ii): If $J \neq 0$, then there are non-projective modules. For $A$ self-injective, all modules are reflexive. (ii) $\implies$ (iii): see Theorem 1. (iii) $\implies$ (iv): Solid modules are indecomposable. (iv) $\implies$ (v): An indecomposable module $M$ with $|JM| \leq 1$ is uniform. (v) $\implies$ (vi): Clear. (vi) $\implies$ (vii): If $A J$ is simple, then $a = 0$, $e = 1$. Otherwise $J^2$ is the socle of $A J$, and thus $a = 1$. (vii) $\implies$ (i): See 3.1. 

3.3. Short local algebras $A$ with $e \leq 2$.

Lemma. Let $A$ be a short local algebra with $e \leq 2$. The following assertions are equivalent:

(i) $A$ is self-injective and $J \neq 0$.
(ii) There exists a non-projective reflexive module.
(iii) $A J$ is uniform.
(iv) Either \( a = 0 \) and \( e = 1 \), or else \( a = 1 \) and \( J^2 = \soc_A J \).

Again, the proof is straightforward: (i) \( \implies \) (ii): If \( J \neq 0 \), then there are non-projective modules. For \( A \) self-injective, all modules are reflexive. (ii) \( \implies \) (iii): Since there exists a non-projective reflexive module, \( e \geq 1 \). If \( e = 1 \), then \( a = 0 \) or \( a = 1 \) and in both cases \( A \) \( J \) is of course uniform. Thus, according to Theorem 1, we can assume that \( a < e = 2 \) and that \( M = A J \) is solid. Since \( M \) is indecomposable, it follows that \( a \neq 0 \). But \( |JM| = a = 1 \) implies that \( M = A J \) is uniform. (iii) \( \implies \) (iv): Assume that \( A J \) is uniform. Either \( A J \) is simple, then \( a = 0 \) and \( e = 1 \), or else \( J^2 = \soc_A J \) and \( a = |J^2| = 1 \). (iv) \( \implies \) (i): See 3.1.

3.4. Example. The algebra \( A = k[x, y]/(x, y)^3 \) is a short local algebra with \( e = 2 \) such that \( A J \) is solid, thus indecomposable, but (of course) not uniform.

4. The Main Lemma.

Given arbitrary integers \( a \) and \( e \), let

\[
\omega^e_a = \begin{bmatrix} e & -1 \\ a & 0 \end{bmatrix}.
\]

4.1. Main Lemma. If \( M \) is a module of Loewy length at most 2, then there is a natural number \( w \) such that

\[ \dim \Omega M = \omega^e_a \dim M + (w, -w), \]

and such that \( \Omega M \) has a direct summand of the form \( S^w \). In particular, if \( \Omega M \) is bipartite, then

\[ \dim \Omega M = \omega^e_a \dim M. \]

Proof. Let \( M' = \Omega M \). There is an exact sequence \( 0 \to M' \to P \to M \to 0 \) with \( P \) projective and we can assume that the map \( M' \to P \) is an inclusion map. Let \( U = J^2 P \). Since \( M \) has Loewy length at most 2, \( U \) is mapped under \( P \to M \) to zero, thus \( U \subseteq M' \). Since \( U \) is semisimple, we have \( U \subseteq \soc M' \). Also, \( M' \) is a submodule of \( JP \), thus \( M'/U \) is a submodule of \( JP/J^2 P \) and therefore semisimple. This shows that \( J M' \subseteq U \). Let \( w = |U/JM'| \). Then

\[
\dim M' = (|M'/JM'|, |JM'|) = (|M'/U| + w, |U| - w) = (|M'/U|, |U|) + (w, -w).
\]

It remains to calculate \(|U|\) and \(|M'/U|\). Let \( \dim M = (t, s) \). Then \( P = A t \), thus \(|U| = |J^2 P| = at \). Also, \(|M'/U| = |JP/J^2 P| - |JM| = et - s \). This shows that \((|M'/U|, |U|) = \omega^e_a \dim M \). This completes the proof of the first formula.

Write \( M' = X \oplus X' \) with \( X \) bipartite and \( X' \) semisimple. Then \( \soc M' = \soc X \oplus \soc X' = JX \oplus X' \) (here we use that \( X \) is bipartite), and \( J M' = JX \oplus JX' = JX \). Altogether we get \( \soc M' = J X \oplus X' \). Since \( J M' \subseteq U \subseteq \soc M' \), the direct decomposition \( \soc M = J X \oplus X' \) yields \( U = JX \oplus (X' \cap U) \). As a submodule of \( X' \), the module \( X' \cap U \) is a direct sum of copies of \( S \). Since \( X' \cap U \) is isomorphic to \( U/JM' \), we have \(|X' \cap U| = w \),
thus $X' \cap U$ is isomorphic to $S^w$. Since $X'$ is semisimple, the submodule $X' \cap U$ is a direct summand of $X'$, thus a direct summand of $M'$. This shows that $M'$ has a direct summand of the form $S^w$.

It remains to show the second assertion: If $\Omega M$ is bipartite, then $\Omega M$ has no direct summand isomorphic to $S^w$, thus $w = 0$. □

**Remark.** The Main Lemma focuses the attention to a direct summand of $\Omega M$ which is of the form $S^w$. However, we should stress that $S^w$ may not be the largest semisimple direct summand of $\Omega M$, as 13.4 shows. Section 13 is devoted to a discussion of $\Omega M$ and its semisimple direct summands.

### 4.2. Aligned modules.

Let $A$ be a short local algebra of Hilbert type $(e, a)$. We say that a module $M$ of Loewy length at most 2 is *aligned* provided $\dim \Omega M = \omega_e \dim M$. Note that if $M$ is aligned, then $|J \Omega M| = a \cdot t(M)$. Here is a reformulation of part of the Main Lemma.

**Corollary.** Let $A$ be a short local algebra and $M$ a module of Loewy length at most 2. If $\Omega M$ is bipartite, then $M$ is aligned. □

**Remark.** The subsequent paper [RZ3] will provide several characterizations of the aligned modules.

### 4.3. Bipartite sequences and bipartite syzygy modules.

We say that an exact sequence

$$
\epsilon: \quad 0 \to X \to P \xrightarrow{p} Z \to 0
$$

is *bipartite*, provided $P$ is projective, both $X, Z$ have Loewy length at most 2 and $X$ is bipartite, or, equivalently, provided $Z$ has Loewy length at most 2, $p$ is a projective cover, and $S$ is not a direct summand of $X$. Note that if $M$ has Loewy length at most 2, then $\Omega M$ is bipartite if and only if the projective cover $p: P(M) \to M$ yields a bipartite sequence $0 \to \Omega M \to P(M) \xrightarrow{p} M \to 0$.

Starting with a module $M$ of Loewy length at most 2, we look at all its syzygy modules $\Omega^i M$ with $i \geq 1$. Of particular interest will be the case that the modules $\Omega^i M$ with $1 \leq i \leq n$ are bipartite (thus $S$ is not a direct summand of $\Omega^i M$ for all $1 \leq i \leq n$).

**Corollary.** Let $M$ be of Loewy length at most 2 and assume that there is $n \geq 1$ such that the modules $\Omega^i M$ with $1 \leq i \leq n$ are bipartite, then

$$
\dim \Omega^n M = (\omega_e)^n \dim M.
$$

□

### 4.4. Recursion formula.

Let $M$ be of Loewy length at most 2 and assume that both modules $M$ and $\Omega M$ are aligned. Then

$$
t_2(M) = et_1(M) - at_0(M).
$$

In case $a \neq 0$, we have $t_0(M) = \frac{1}{a}(et_1(M) - t_2(M))$. 

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Proof. We write \( t_i = t_i(M) = t(\Omega^i M) \) for \( 0 \leq i \leq 2 \). Let \( s_1 = |J\Omega M| \). Since \( M \) is aligned, \( s_1 = at_0 \). Since \( \Omega M \) is aligned, \( t_2 = et_1 - s_1 \). Thus \( t_2 = et_1 - s_1 = et_1 - at_0 \). \( \square \)

**Corollary.** Let \( M \) be of Loewy length at most 2 and assume that both modules \( \Omega M \) and \( \Omega^2 M \) are bipartite. Then

\[
t_2(M) = et_1(M) - at_0(M).
\]

In case \( a \neq 0 \), we have \( t_0(M) = \frac{1}{a}(et_1(M) - t_2(M)) \).

**Remark.** In Lescot [L], modules with Loewy length at most 2 such that the modules \( \Omega^i M \) with \( 1 \leq i \leq n \) are bipartite, are called “\( n \)-exceptional” modules; the modules which are \( n \)-exceptional for all \( n \geq 1 \) are called “exceptional”. See [RZ3] for a further discussion of these “exceptional” modules.


5.1. **Proof of Theorem 1.** Let \( A \) be a short local algebra and let \( M \) be an indecomposable reflexive module which is not projective. We show that \( _A J \) is solid and that either \( a = e = 1 \) or \( a < e \).

First of all, \( e > 0 \), since otherwise all modules are projective. If \( A \) is self-injective, then clearly \( _A J \) is solid and either \( a = 0 \), \( e = 1 \) or else \( a = 1 \) and \( e \geq 1 \) (see 3.1). Thus, we can assume that \( A \) is not self-injective. Then \( S \) is not reflexive, see 2.1. As a consequence, \( M \) is not simple. We must have \( a \geq 1 \). Namely, if \( J^2 = 0 \), then the only indecomposable non-projective torsionless module is \( S \), thus \( M = S \). But as we have mentioned, \( S \) is not reflexive.

Since \( M \) is indecomposable, torsionless and neither projective, nor simple, \( M \) has to be bipartite. Let \( \text{dim} M = (b, s) \). Since \( M \) is reflexive, \( \Omega M \) has Loewy length at most 2. Let \( 0 \longrightarrow M \overset{u}{\longrightarrow} P \overset{\rho}{\longrightarrow} \Omega M \rightarrow 0 \) be an \( \Omega \)-sequence, where \( P \) is projective of rank \( z \). We will assume that \( u \) is an inclusion map. Since \( \Omega M \) has Loewy length at most 2, we have \( J^2 P \subseteq \ker(p) = M \). Since \( J^2 P \) is semisimple, it follows that \( J^2 P \subseteq \soc M \). On the other hand, \( M \subseteq JP \) implies that \( \soc M = JM \subseteq J^2 P \), thus \( \soc M = J^2 P \). It follows that \( s = az \).

Let us show that \( J^2 = \soc A A \). If \( J^2 \neq \soc A A \), there is a simple submodule \( U \) of \( A A \) which is not contained in \( J^2 \). Let \( f: M \rightarrow U \) be a homomorphism with image \( f(M) = U \). and \( v: U \rightarrow A A \) the inclusion map. Since \( u: M \rightarrow P \) is a left add\((A)\)-approximation, there is \( f': P \rightarrow A A \) such that \( vf = f'u \). If \( f' \) is not surjective, then \( f'(P) \subseteq J \), thus \( f'(JP) \subseteq J^2 \) and therefore \( f'u(M) \subseteq J^2 \). But \( f'u = vf \) and \( vf(M) = v(U) = U \) is not contained in \( J^2 \). This shows that \( f' \) is surjective. There is the following commutative diagram

\[
\begin{array}{c}
0 \longrightarrow M \overset{u}{\longrightarrow} P \overset{\rho}{\longrightarrow} \Omega M \longrightarrow 0 \\
\downarrow f \quad \quad \downarrow f' \quad \quad \downarrow f'' \\
0 \longrightarrow U \overset{v}{\longrightarrow} A A \longrightarrow A/U \longrightarrow 0.
\end{array}
\]
Since \( f' \) is surjective, also \( f'' \) is surjective. Since \( U \) is not contained in \( J^2 \), the module \( A/U \) has Loewy length 3. Thus, also \( \mathcal{U}M \) has Loewy length 3. But we know already that \( \mathcal{U}M \) has Loewy length at most 2. This contradiction shows that \( J^2 = \text{soc}_A A \).

Next, we show that \( _A J \) is solid. Let \( \phi \) be an endomorphism of \( _A J \). Write \( P = \bigoplus_{i=1}^2 A_i \) with \( A_i = _A A \) and \( u \) as the transpose of \( [u_1, \ldots, u_z] \), where \( u_i : M \to A_i \). As we know, \( \text{soc} M = J^2 P = \bigoplus_{i=1}^2 J^2 A_i \), thus \( J^2 A_i = A_i \cap \text{soc} M \).

We denote the inclusion map \( JA_1 \subset A_1 \) by \( v_1 \) and write \( u_1 = v_1 u'_1 \), where \( u'_1 : M \to JA_1 \). Let \( f : M \to _A A \) be the composition

\[
M \xrightarrow{u'_1} JA_1 \xrightarrow{\phi} JA_1 \xrightarrow{v_1} A_1 = _A A.
\]

Since \( u \) is an add\(_{(A A)}\)-approximation, there are maps \( g_i : _A A \to _A A \) such that \( g = [g_1, \ldots, g_z] \) satisfies \( f = gu = \sum g_i u_i \). The map \( g_1 : _A A \to _A A \) is the right multiplication by some element \( \lambda \in A \).

Given \( x \in A_1 \cap \text{soc} M = J^2 A_1 \), we consider the element \( [x, 0, \ldots, 0] \in M \) and apply the map \( f = \sum g_i u_i \) to it. Since \( f = v_1 \phi u'_1 \), we have \( f([x, 0, \ldots, 0]) = \phi(x) \). On the other hand, we have \( u_i([x, 0, \ldots, 0]) = 0 \) for \( i \geq 2 \), thus \( \sum g_i u_i([x, 0, \ldots, 0]) = g_1(x) = x\lambda \). This shows that

\[
\phi(x) = f([x, 0, \ldots, 0]) = \sum g_i u_i([x, 0, \ldots, 0]) = x\lambda
\]

for all \( x \in J^2 A_1 \). Now \( J^2 A_1 \) is annihilated from the right by \( J \), thus \( x\lambda = \overline{\lambda} x \), where \( \overline{\lambda} = \lambda + J \) is an element of \( A/J = k \). This shows that the restriction of \( \phi \) to \( J^2 A_1 = J^2 \) is the scalar multiplication by \( \overline{\lambda} \). Since \( J^2 = \text{soc}_A A \), it follows that \( _A J \) is solid.

It remains to be shown that \( a < e \). Thus, assume that \( a \geq e \). Write \( P = A^{(z)} \) (here, given a module \( X \), we will write \( X^{(z)} \) for the direct sum of \( z \) copies of \( X \) in order to avoid confusion for example if \( X = J \)).

Let \( B = A/J^2 \) and \( P' \) the projective cover of \( M \) as a \( B \)-module. Of course, \( \dim P' = (b, eb) \). Since \( M \) is a factor module of \( P' \), it follows that \( az = s \leq eb \). Since \( e \leq a \), we have \( az \leq eb \leq ab \), thus \( z \leq b \).

The first case: we assume that \( z = b \). Then \( ez \leq az \leq eb = ez \) shows that \( a = e \) and therefore \( \dim M = (b, az) = (b, eb) = \dim P' \). This shows that \( P' = M \). Since \( M \) is indecomposable, we have \( b = 1 \). As we have mentioned, we consider \( u : M \to _A A \) as an inclusion map. Thus there is \( x \in J \) with \( M = Ax \). Let \( y \in J \setminus M \). Since \( Ax \) is a projective \( B \)-module and \( A y \) is a local \( B \)-module, there is a map \( f : Ax \to _A A \) with \( f(x) = y \). Since the embedding \( u : M \to _A A \) is an add\(_{(A A)}\)-approximation, there is an endomorphism \( f' \) of \( _A A \) with \( f = f'u \). Now \( f' \) is the right multiplication with an element \( a \in A \), and \( a = \lambda + a' \) for some scalar \( \lambda \) and \( a' \in J \). Thus \( y = f(x) = f'u(x) = x(\lambda + a') = \lambda x + xa' \). But \( xa' \in J^2 \subseteq Ax \), thus \( y \in Ax = M \), a contradiction.

The second case: let \( z < b \). Let us denote by \( u' : M \to J^{(z)} \), \( v : J \to A \) and \( w : J^2 \to A \) the canonical inclusion maps. Thus \( u = u^{(z)} u' \). Given \( a \in A \), we denote by \( r(a) : _A A \to _A A \) the right multiplication by \( a \). If \( a \in J \), then \( r(a) \) maps \( J \) into \( J^2 \) and the map \( r(a) : J \to J^2 \) depends only on the residue class \( \overline{a} \) of \( a \) modulo \( J^2 \). Thus we may write \( r(\overline{a}) = r(a) : J \to J^2 \).
and there is the following commutative diagram
\[ \begin{array}{ccc}
J & \xrightarrow{v} & A \\
\downarrow{r(\overline{a})} & & \downarrow{r(a)} \\
J^2 & \xrightarrow{w} & A \\
\end{array} \]
In this way, we obtain the following linear map
\[ \Phi: (J/J^2)^{(z)} \to \text{Hom}(M, J^2), \text{ defined by } \Phi(\overline{a_1}, \ldots, \overline{a_z}) = [r(\overline{a_1}), \ldots, r(\overline{a_z})]u'. \]

Let us show that \( \Phi \) is surjective. Let \( f: M \to J^2 \) be any homomorphism. By assumption, the inclusion map \( u = v^{(z)}u': M \to A^{(z)} \) is a left \( \text{add}(A) \)-approximation. Thus, there is \( f': A^{(z)} \to AA \) such that \( wf = f'u \). We write \( f' \) as \([r(a_1), \ldots, r(a_z)]\) with elements \( a_i \in A \). Since \( f \) vanishes on \( \text{soc} M = (J^2)^{(z)} \), we have \((J^2)a_i = 0\), thus \( a_i \in J \), for all \( 1 \leq i \leq z \).

Thus, we have the following diagram.
\[ \begin{array}{ccc}
M & \xrightarrow{u'} & (J^{(z)}) \\
\downarrow{f} & & \downarrow{f'} \\
J^2 & \xrightarrow{w} & AA \\
\end{array} \]
Here, the outer rectangle commutes by the choice of \( f' \). Since \( a_i \in J \), we have \( r(a_i)v = wr(\overline{a_i}) \), thus \([r(a_1), \ldots, r(a_z)]v^{(z)} = w[r(\overline{a_1}), \ldots, r(\overline{a_z})] \). Since \( w \) is a monomorphism, it follows that also the triangle on the left commutes: \( f = [r(\overline{a_1}), \ldots, r(\overline{a_z})]u' \). Thus, we see that
\[ f = [r(\overline{a_1}), \ldots, r(\overline{a_z})]u' = \Phi(\overline{a_1}, \ldots, \overline{a_z}). \]
In this way, we see that \( \Phi \) is surjective.

But a dimension argument shows that \( \Phi \) cannot be surjective. Any map \( M \to J^2 \) factors through the projection \( M \to \text{top} M \), thus \( \dim \text{Hom}(M, J^2) = \dim \text{Hom}(\text{top} M, J^2) = ba \). On the other hand, we have \( |(J/J^2)^{(z)}| = ze \). Since \( z < b \) and \( e < a \), we have \( ze < ba \). Thus \( |(J/J^2)^{(z)}| < \dim \text{Hom}(M, J^2) \), and therefore \( \Phi \) cannot be surjective. We obtain also in the second case a contradiction.

In both cases we have obtained contradictions. Thus, we see that \( a < e \).  

\[ \square \]

5.2. Remark. Note that an element \( z \in J \) belongs to \( \text{soc} A_A = \text{soc} J_A \) if and only if \( zJ = 0 \). As a consequence, \( J^2 = \text{soc} A_A \) if and only if \( A_1J \) is a faithful \( A/J^2 \)-module.

5.3. Example. A short local algebra with \( J^2 = \text{soc} A_A \subset \text{soc} A_A \). Since our general assumption is \( J^3 = 0 \), we always have \( J^2 \subseteq \text{soc} A_A \) as well as \( J^2 \subseteq \text{soc} A_A \). We may have \( J^2 = \text{soc} A_A \) and \( J^2 \neq \text{soc} A_A \) as the following example shows. Let \( A \) be the \( k \)-algebra with radical generators \( x, y \) and relations
\[ yx, y^2, x^3, x^2y. \]
Here, $J^2 = Ax^2 + Axy = \text{soc}_A A$ is of length 2, whereas $\text{soc}_A A = x^2A + yA + xyA$ is of length 3.

5.4. Examples. Short local algebras with $A_J$ indecomposable, but not solid. First example: here, $A_J$ has a non-zero nilpotent endomorphism.

Let $A$ be generated by $x, y, z$ with relations

$z^2, xy, yx, yz, zy, zx - xz, y^2 - xz, x^3$.

There is the endomorphism $f$ of $A_J$ given by $f(y) = f(z) = 0$ and $f(x) = z$.

Second example: Here we exhibit an $\mathbb{R}$-algebra such that $\text{End}(A_J) \sim \mathbb{C}$. We consider the $\mathbb{R}$-algebra with generators $x, y$, and the relations are

$xy - yx, x^2 + y^2$.

(Note that the 2-Kronecker module $\tilde{J}$ as mentioned in appendix A.2 is $(\mathbb{C}, \mathbb{C}; 1, i)$, where we write 1 for the identity map $\mathbb{C} \to \mathbb{C}$ and $i: \mathbb{C} \to \mathbb{C}$ for the multiplication by $i$; of course, $\text{End}(\mathbb{C}, \mathbb{C}; 1, i) = \mathbb{C}$.)

Note that both algebras are commutative.

5.5. Example. A short local algebra with $A_J$ solid, whereas $J_A$ is not solid. Let $A$ be generated by $x, y, z$ with relations

$x^2, y^2, z^2, xy, yz, zx - xy, zy - xz$.

Here, $A_J$ is solid, whereas $J_A$ is the direct sum of a module with dimension vector $(2, 2)$ and a simple module (generated by $y$). Note that $A_J$ is solid, but not faithful.

Note that Theorem 1 asserts that all the algebras exhibited in 5.3, 5.4 and 5.5 have no non-projective reflexive modules, thus all semi-Gorenstein-projective and all $\infty$-torsionfree modules are projective.


6.1. Proposition. Let $A$ be a short local algebra of Hilbert type $(e, a)$. Let $M$ be a bipartite module. If $M$ is reflexive and $\dim M = (t, s)$, then $a$ divides $s$ and $\dim M^* = (s/a, at)$. 15
Proof. Let $\dim M = (t, s)$. Since there exists a non-projective reflexive module $M$, we know that $A J$ is a solid $A$-module. Since $M$ is not simple, we also know that $a \geq 1$. Let $\mathcal{H}$ be the set of homomorphisms $f : M \to A A$ with semi-simple image (thus, these are the homomorphisms with image in $J^2$, and also the homomorphisms with kernel containing the socle of $M$). If $g : A A \to A A$ is the right multiplication by some element from $J$, then $g f = 0$. This shows that $\mathcal{H}$ is contained in the socle of $M^*$. Of course, $|\mathcal{H}| = at$. On the other hand, if $f : M \to A A$ is any element of $M^*$, then $g f (M) \subseteq g (J) \subseteq J^2$ shows that $g f$ belongs to $\mathcal{H}$. This shows that $M^*/\mathcal{H}$ is a semi-simple right $A$-module. Now $M^*$ is indecomposable and has no simple direct summand, thus $\mathcal{H} = \text{soc } M^*$.

Let $u_i : M \to A_i = A A$ be maps such that $u = [u_1, \ldots, u_z] : M \to \bigoplus_{i=1}^{z} A_i$ is a minimal left add$(A)$-approximation of $M$. We can assume that $u$ is an inclusion map. Since the cokernel of $u$ has Loewy length at most 2, we know that $J^2 P$ is contained in the socle of $M$ and actually equal to soc $M$. It follows that $s = |\text{soc } M| = az$. In particular, $s$ is divisible by $a$.

We claim that $u_1, \ldots, u_z$ is a basis of $M^*/\mathcal{H}$. First, we show the linear independence. Thus, let us assume that there are scalars $\lambda_i \in k$ such that $f = \sum_i \lambda_i u_i$ belongs to $\mathcal{H}$. We have to show that $\lambda_i = 0$ for all $i$. Thus, assume that some $\lambda_i$ is non-zero, say let $\lambda_1 \neq 0$. Let $0 \neq x \in J^2 A_1$. We apply $f$ to $[x, 0, \ldots, 0]$ and get $f([x, 0, \ldots, 0]) = \lambda_1 x \neq 0$. But this means that $f$ does not vanish on soc $M$, thus $f \notin \mathcal{H}$, a contradiction.

Second, we have to show that $u_1, \ldots, u_z$ generate $M^*$ modulo $\mathcal{H}$. Let $f : M \to A A$ be any homomorphism. Since $u$ is a left add$(A)$-approximation, there are maps $f_i : A A \to A A$ such that $f = \sum_i f_i u_i$. Write $f_i = \lambda_i \cdot 1_M + g_i$ where $\lambda_i \in k$ and $g_i$ maps into $J$. Then $f = \sum_i f_i u_i = \sum_i \lambda_i u_i + g$, with $g = \sum_i g_i u_i$. The image of any $u_i$ is contained in $J$, thus the image of $g_i u_i$ is contained in $J^2$. This shows that $g \notin \mathcal{H}$.

Since $u_1, \ldots, u_z$ is a basis of $M^*/\mathcal{H}$, we see that $|M^*/\mathcal{H}| = z = s/a$. \hfill $\Box$

Remark. Let us stress that for a bipartite reflexive module $M$ with $\dim M = (t, at)$, we get $\dim M^* = \dim M$.

6.2. Proposition. Let $A$ be a short local algebra of Hilbert type $(e, a)$ and assume that $A$ is not self-injective. Let $M$ be an indecomposable module which is not projective. If both $M$ and $\mathcal{U} M$ are reflexive, then

$$|J M| = \frac{a(a+1)}{e} t(M) \quad \text{and} \quad |J \mathcal{U} M| = \frac{ae}{a+1} t(\mathcal{U} M).$$

Proof. Since $A$ is not self-injective, the modules $M$ and $\mathcal{U} M$ are not simple. Also, we know that $a \geq 2$ according to Corollary of Theorem 1. Let $\dim \mathcal{U} M = (x, ay)$, therefore $\dim M = (ex - ay, ax)$, according to 4.1. By 6.1, we have $\dim (\mathcal{U} M)^* = (y, ax)$ and $\dim M^* = (x, aex - a^2 y)$. According to [RZ1], 4.2 (b), the $A$-dual of the $\mathcal{U}$-sequence $0 \to M \to P \to \mathcal{U} M \to 0$ is an $\mathcal{U}$-sequence $0 \to (\mathcal{U} M)^* \to P^* \to M^* \to 0$, and 4.1 asserts that

$$(y, ax) = \dim (\mathcal{U} M)^* = \omega_a \dim M^* = \omega_a (x, aex - a^2 y) = (ex - aex + a^2 y, ax).$$

thus $ex - aex + a^2 y = y$ and therefore $e(1 - a)x = (1 - a^2)y$. Since $a \neq 1$, we see that $y = \frac{e}{a+1} x$ and therefore $|J \mathcal{U} M| = ay = \frac{ae}{a+1} x = \frac{ae}{a+1} t(\mathcal{U} M)$. 16
Also, we have \( t(M) = ex - ay = ex - \frac{ae}{a+1}x = \frac{e}{a+1}x \), thus \( x = \frac{a+1}{e}t(M) \) and therefore \( |JM| = ax = \frac{a(a+1)}{e}t(M) \).

6.3. Proof of Theorem 2. Let \( A \) be a short local algebra of Hilbert type \((e, a)\) and assume that \( A \) is not self-injective. As we know, \( a \geq 2 \). Assume that there exists a path of length 4 in the \( \mathcal{Q} \)-quiver of \( A \), say ending at \( M \). Then the modules \( M, \mathcal{Q}M, \mathcal{Q}^2M \) are reflexive (see Theorem 1.5 (2) in [RZ1]). Let \( \dim \mathcal{Q}M = (t, s) \). According to 6.2, the pair \( M, \mathcal{Q}M \) shows that \( s = \frac{ae}{a+1}t \), whereas the pair \( \mathcal{Q}M, \mathcal{Q}^2M \) shows that \( s = \frac{a(a+1)}{e}t \). Since \( t \neq 0 \), it follows that \( \frac{ae}{a+1} = \frac{a(a+1)}{e} \), thus \( e^2a = a(a+1)^2 \) and therefore \( e = a + 1 \). □

7. The special case \( a = e - 1 \geq 1 \).

Since the cases \( a = 1 \) do not provide any challenge, the interesting cases are those with \( a \geq 2 \). But we include the case \( a = 1 \) in order to point out that the special cases \( a = e - 1 \) may be seen as having features which are similar to the (tame) self-injective rings of Hilbert type \((2, 1)\), thus also to \( L(2) \).

If \( a = e - 1 \) and \( M \) is a module of Loewy length at most 2 with \( \dim M = (t, s) \), let \( \delta(M) = at - s \). We call \( \delta(M) \) the defect of \( M \).

7.1. Lemma. Let \( a = e - 1 \geq 1 \). Let \( 0 \to X \to P \to Z \to 0 \) a bipartite sequence. Then \( \dim X = \dim Z + \delta(Z)(1, 1) \) and \( \delta(X) = a\delta(Z) \).

Proof. We have
\[
\dim X = (et - s, at) = ((a + 1)t - s, at) = (t, s) + (at - s, at - s) = \dim Z + \delta(Z)(1, 1),
\]
and \( \delta(X) = a((a + 1)t - s) - at = a^2t - as = a(at - s) = a\delta(Z) \). □

7.2. Lemma. Let \( a = e - 1 \geq 1 \). Let \( 0 \to X \to P \to Z \to 0 \) be bipartite. Then the following conditions are equivalent:
(i) \( \delta(X) = 0 \).
(ii) \( \delta(Z) = 0 \).
(iii) \( \dim X = \dim Z \).
(iv) \( t(X) = t(Z) \).
(v) \( |JX| = |JZ| \).

Proof. Since \( \delta(X) = a\delta(Z) \), the conditions (i) and (ii) are equivalent. Since \( \dim X = \dim Z + \delta(Z)(1, 1) \), the conditions (ii) and (iii) are equivalent. Of course, (iii) implies both (iv) and (v). Now \( \dim X = \dim Z + \delta(Z)(1, 1) \) means that \( t(X) = t(Z) + \delta(Z) \) and \( |JX| = |JZ| + \delta(Z) \). Thus, if (iv) or (v) is satisfied, then \( \delta(Z) = 0 \), thus (ii) holds. □

7.3. Lemma. Let \( a = e - 1 \geq 1 \). If \( \delta(M) = 0 \), then either \( t(\Omega M) = t(M) \) and \( \delta(\Omega M) = 0 \), or else \( t(\Omega M) > t(M), \delta(\Omega M) > 0 \) and \( \Omega M \) is not bipartite.

If \( \delta(M) > 0 \), then \( t(\Omega M) > t(M) \) and \( \delta(\Omega M) > 0 \). Thus, if \( \delta(M) \geq 0 \) and \( \delta(\Omega M) > 0 \), then
\[
\cdots > t_{i+1}(M) > t_i(M) > \cdots > t_1(M) > t(M).
\]
Proof. The Main Lemma asserts that \( \dim \Omega M = \omega^e \dim M + (w, -w) \) for some \( w \geq 0 \).

First, let \( \delta(M) = 0 \), then \( \dim M = (t, at) \) for some \( t > 0 \). Now, \( \omega^e(t, at) = (t, at) \).

We have \( \dim \Omega M = (t, at) + (w, -w) \) for some \( w \geq 0 \). If \( w = 0 \), then trivially \( t(\Omega M) = t \) and \( \delta(\Omega M) = 0 \). If \( w > 0 \), then \( t(\Omega M) = t + w > t = t(M) \) and \( \delta(\Omega M) = a(t + w) - (at - w) = (a + 1)w > 0 \). Also, \( \Omega M \) is not bipartite, according to 4.1.

Second, assume that \( at - s = \delta(M) > 0 \), thus \( at > s \). Now \( \dim \Omega M = (et - s + w, at - w) \) for some \( w \geq 0 \). Then \( t(\Omega M) = et - s + w = at + t - s + w > t + w \geq t = t(M) \). Also, \( a(et - s + w) = a(t + at - s + w) > a(t + w) \geq at \geq at - w \), thus \( \delta(\Omega M) > 0 \).

The last assertion follows by induction. \( \square \)

Remark. For further considerations concerning short algebras with \( a = e - 1 \), we refer to sections 10, 11, 12, as well as to [RZ4].

8. The syzygy modules of \( S \).

8.1. Lemma. If \( a \geq e \), and \( 0 \to X \to P \to Z \to 0 \) is a bipartite sequence, then \( |\soc X| > |\soc Z| \).

Proof. Let \( \dim X = (t, s) \) and \( \dim Z = (t', s') \). The Main Lemma asserts that \( (t, s) = \omega^e(t', s') = (et' - s', at') \). Thus \( |\soc X| = s = at' \geq et' > s' \), since \( t = et' - s' > 0 \). \( \square \)

If \( (a_n)_n \) is a sequence of real numbers, we write (as usual) \( \lim_n a_n = \infty \) provided for every integer \( b \) there is \( N = N(b) \) such that \( a_n > b \) for all \( n \geq N \).

8.2. Proposition. Let \( A \) be a short local algebra with \( e \geq 2 \). Then \( \lim_n t_n(S) = \infty \), thus also \( \lim_n |\Omega^n S| = \infty \). If, in addition, \( a < e \), then the sequence of the Betti numbers \( t_n(S) \) of \( S \) is strictly increasing: \( t_n(S) < t_{n+1}(S) \) for all \( n \in \mathbb{N} \).

Prrof. For any module \( M \), we have \( t(M) \leq |M| \leq \max(e + 1) t(M) \), thus \( \lim_n t_n(M) = \infty \) if and only if \( \lim_n |\Omega^n(M)| = \infty \).

Let \( t_n = t_n(S) = t(\Omega^n S) \). For \( a < e \), we show that the sequence \( (t_n)_n \) is strictly increasing.

First, let \( a = 0 \). Then \( \Omega^n S = S^{\infty} \) for all \( n \geq 0 \). Since \( e \geq 2 \), we have \( e^{n+1} > e^n \), thus \( t_n < t_{n+1} \).

Second, let \( 1 \leq a < e - 1 \). We have \( t_0 = 1, t_1 = e \). We show by induction that \( t_{n+1} > t_n \) for all \( n \geq 0 \). For \( n = 0 \), this holds true since \( e \geq 2 \). Thus, let \( n \geq 1 \). We assume that \( t_{n+1} > t_n \). The Main Lemma asserts that \( t_{n+2} \geq et_{n+1} - at_n = (e - 1)t_{n+1} + at_n \geq at_{n+1} + at_n = a(t_{n+1} - t_n) > 0 \), where we use that \( a \geq 1 \). This shows that \( t_{n+2} > t_{n+1} \).

Finally, let \( a \geq e \). We show that \( \lim_n |\Omega^n S| = \infty \). If all the modules \( \Omega^n S \) are bipartite, then 8.1 asserts that \( |\soc \Omega^{n+1} S| > |\soc \Omega^n S| \) for all \( n \geq 0 \), thus \( |\Omega^n S| \geq |soc \Omega^n S| > n \) for all \( n \).

It remains to consider the case that there is some \( \Omega^n S \) which is not bipartite. Let \( m \) be minimal. We claim that \( \Omega^m S \) is not simple.

If \( m = 1 \), then \( \Lambda J = \Omega S \) is of course not simple. Let \( m \geq 2 \). The minimality of \( m \) implies that \( Z = \Omega^{m-1} S \) is bipartite. Let \( p: P \to Z \) be a projective cover, thus \( \Omega^m S \) is
the kernel of \( p \). Since \( Z \) is of Loewy length 2, we see that \( J^2P \) is contained in the kernel \( \Omega^mS \) of \( p \). We have \(|J^2P| \geq |J^2| = a \geq 2\), thus \(|\Omega^mS| \geq 2\). This shows that \( \Omega^mS \) is not simple.

Thus, there is \( m \geq 1 \) such that \( \Omega^mS \) is neither bipartite nor simple. We have \( \Omega^mS \simeq S \oplus X \) for some \( X \neq 0 \). By induction, we have \( \Omega^{bm}S \simeq S \oplus \bigoplus_{i=0}^{b-1} \Omega^mX \), for all \( b \geq 0 \), thus \( \Omega^{bm}S \) is the direct sum of \( b+1 \) non-zero modules. As a consequence, \( \Omega^{bm+i}S \) is the direct sum of \( b+1 \) non-zero modules, for all \( i \geq 0 \), and therefore \(|\Omega^{bm+i}S| > b \) for all \( i \geq 0 \). Thus, let \( N(b) = bm \).

### 8.3. Example

A short local algebra \( A \) with \( t_1(S) = t_2(S) \). In general, the Betti numbers are not strictly increasing, as the following example shows. Let \( A \) be the \( k \)-algebra generated by \( x, y \) with relations

\[
xy, \ x^2 - y^2, \ x^3
\]

It is a short local algebra of Hilbert type \((2,2)\). We have \( \Omega S = A^J \) with dimension vector \((2,2)\). As \( \Omega(AJ) \) we can take the submodule of \( AA^2 \) generated by \([y,x]\) and \([0,y]\), and this is a free \( L(2) \)-module of rank 2, thus \( \dim \Omega^2S = (2,4) \). We see that \( t_1(S) = 2 = t_2(S) \).

### 9. Proof of Theorems 3 and 4

For the proof of Theorem 3, we will use the following result by Christensen and Veliche.

#### 9.1. Christensen-Veliche Lemma

Let \( e > 0 \) and \( a > 1 \) be integers and let \((c_i)_{i \geq 0}\) be a sequence of positive integers with

\[
c_i = ec_{i+1} - ac_{i+2} \quad \text{for all} \quad i \geq 0.
\]

Then \( a = e - 1 \) and \( c_i = c_0 \) for all \( i \).

See the appendix of [CV].

#### 9.2. Proof of Theorem 3

Let \( A \) be a short local algebra which is not self-injective. Since \( A \) is not self-injective, we have \( e \geq 2 \). Let \( P_\bullet \) be a non-zero minimal complex of projective modules which is acyclic. Let \( t_i \) be the rank of \( P_i \) and \( M_i \) the image of \( d_i \). Since \( P_i \) is a projective cover of \( M_i \), we have \( t(M_i) = t_i \).

Note that we have \( a \geq 1 \). Namely, if \( a = 0 \), then the modules \( M_i \) are semisimple and \( \Omega S = S^e \) shows that the sequence \( \cdots, t_{i+1}, t_i, \cdots \) is strictly decreasing. Impossible.

Next, we show that \( M_i \) is bipartite for \( i \ll 0 \). Let \( t = |M_0| \). According to 8.2, there is \( N = N(b) \) such that \(|\Omega^nS| > b \) for all \( n \geq N \). Let \( n \geq N \) and assume that \( S \) is a direct summand of \( M_{-n} \). Then \( \Omega^nS \) is a direct summand of \( \Omega^nM_{-n} = M_0 \), and therefore \(|\Omega^nS| \leq |M_0| = b \), a contradiction. This shows that all the modules \( M_{-n} \) with \( n \geq N \) are bipartite.

Using, if necessary, an index shift, we can assume that all the modules \( M_i \) with \( i \leq 0 \) are bipartite. Let \( c_i = t_{-i} = t(M_{-i}) \) for \( i \geq 0 \). Since all the modules \( M_{-i} \) are bipartite, there is the recursion formula 4.4 which asserts that

\[
c_i = ec_{i+1} - ac_{i+2}
\]
for all \( i \geq 0 \). Thus we can use the Christensen-Veliche Lemma 9.1 in order to conclude that 
\( a = e - 1 \) and that the sequence \( c_0, c_1, \cdots \) is constant, thus that the sequence \( t_0, t_{-1}, t_{-2}, \ldots \) is constant.

There are two possibilities: First, all the modules \( M_i \) may be bipartite. In this case, 
\( t_i = t_{i+1} \) for all \( i \in \mathbb{Z} \).

Second, not all modules \( M_i \) are bipartite, thus there is a minimal index \( u \) such that 
\( M_{u+1} \) is not bipartite. As we have seen, this implies that \( t_u = t_i \) for all \( i \leq u \).

Since \( S \) is a direct summand of \( M_{u+1} \), we use again 8.2 in order to see that there is 
some \( i \geq u \) such that \( t_{i+1} > t_i \). Let \( v \) be the minimal index \( i \) with this property. Thus we 
have 
\[ t_{v+1} > t_v = t_{v-1} = \cdots . \]

We apply Lemma 7.2 to the bipartite sequence 
\[ 0 \rightarrow M_u \rightarrow P_{u-1} \rightarrow M_{u-1} \rightarrow 0. \]
Since \( t(M_u) = t_u = t_{u-1} = t(M_{u-1}) \), it follows that \( \delta(M_u) = 0 \). The first part of Lemma 7.3 
yields by induction that \( \delta(M_i) = 0 \) for \( u \leq i \leq v \) and then that \( \delta(M_{u+1}) > 0 \). The last 
part of Lemma 7.3 asserts that 
\[ \cdots > t(M_{i+1}) > t(M_i) > \cdots > t(M_{v+1}) > t(M_v) \]
(with \( i \geq v \)). This completes the proof. \( \square \)

We will say that a complex \( P_\bullet \) is of type I, provided it is a non-zero minimal complex 
of projective modules which is acyclic, and all the modules \( P_i \) have the same rank.

We will say that a complex \( P_\bullet \) is of type II, provided it is a non-zero minimal complex 
of projective modules \( P_i \) which is acyclic, and there is some integer \( u \) such that 
\[ \cdots > t_{u+2} > t_{u+1} > t_u = t_{u-1} = t_{u-2} = \cdots , \]
where \( t_i \) is the rank of \( P_i \).

**9.3. Example.** An algebra \( A \) of Hilbert type \((2, 1)\) with \( J^2 \subset \text{soc } A A \) and \( J^2 \subset \text{soc } A A \) 
with a complex of type I.

In contrast to the commutative case, we cannot assert in Theorem 3 that \( J^2 = \text{soc } A A \) 
or that \( J^2 = \text{soc } A A \), as the following examples shows: Let \( A \) be the \( k \)-algebra with 
generators \( x, y \) and relations \( x^2, xy, y^2 \).

\[
\begin{align*}
A J & \longrightarrow \quad x & y \\
\downarrow y & \quad y x
\end{align*}
\]

(note that \( y \) belongs to \( \text{soc } A A \) and \( x \) belongs to \( \text{soc } A A \), but neither \( x \) nor \( y \) belong to \( J^2 \)).
The complex 
\[ \cdots \longrightarrow x \rightarrow AA \longrightarrow x \rightarrow AA \longrightarrow \cdots \]
is non-zero, minimal and acyclic (here, \( x \) denotes the right multiplication by \( x \), thus all the 
images are equal to \( M = Ax = A/Ax \)). On the other hand, \( J^2 = kyx \) is 1-dimensional, 
whereas both \( \text{soc } A A = kyx + ky \) and \( \text{soc } A A = kyx + kx \) are 2-dimensional.
9.4. Example. An algebra $A$ of Hilbert type $(3, 2)$ with $J^2 \neq \text{soc}_A A$, with a complex of type II.

The algebra $A$ will be similar to the algebra $A_0$ considered in section 11 (and before in [RZ1], but with the relation $xz = 0$ instead of $xz = zx$. To be precise: $A$ is generated by $x, y, z$, subject to the relations:

$$x^2, y^2, z^2, xy + qyx, xz, yz, zy - zx,$$

with $q \in k$ having infinite multiplicative order. Following [RZ1], we may visualize the algebra as follows:

![Diagram of algebra A]

The algebra $A$ has the basis $1, x, y, z, yx, zx$. We have $|\text{soc}_A A| = 3$ with basis $yx, zy, z$, whereas, of course, $|J^2| = 2$. We get a complex of type II by taking the projective covers of the modules $A(x - \alpha y)$ where $\alpha = q^{-i}$ with $i \geq 2$, and a minimal projective resolution of $A(x - q^{-1}y)$. Note that $\Omega A(x - q^{-1}y) = A(x - y) \oplus S$.

9.5. Proof of Theorem 4.

We assume again that $A$ is a short local algebra which is not self-injective and we assume that there exists a module $M$ which is indecomposable, non-projective and either semi-Gorenstein-projective or $\infty$-torsionfree. Thus, there is a reflexive module which is not projective and therefore Corollary to Theorem 1 asserts that $a \geq 2$. Also, there exists an $\Omega$-path of length 6, thus Theorem 2 asserts that $a = e - 1$ and $J^2 = \text{soc}_A A = \text{soc} A_A$.

We have seen in 2.3 that $S$ is neither semi-Gorenstein-projective, nor $\infty$-torsionfree. If $M$ is $\infty$-torsionfree, then $M$ has Loewy length at most 2. Since $M$ cannot be isomorphic to $S$, we see that $M$ (and all the modules $\Omega^n M$ with $n \geq 0$) are bipartite. In case $M$ is semi-Gorenstein-projective, we have to assume in addition that the Loewy length of $M$ is at most 2 in order to conclude that $M$ (but also all the modules $\Omega^n M$ with $n \geq 0$) are bipartite.

Now assume that $M$ is $\infty$-torsionfree. According to 4.4, there is the recursion formula which is needed in order to use 9.1. It follows that $\dim \Omega^i M = (t, at)$ for all $i \geq 0$.

Next, we assume that $M$ is semi-Gorenstein-projective, and of Loewy length 2. Let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution and let $M_i$ be the cokernel of the map $P_i+1 \to P_i$, thus we have $M = M_0$. The sequences $0 \to M_{i+1} \to P_i \to M_i \to 0$ are $\Omega$-sequences. Since $M_0$ is indecomposable, all the modules $M_i$ are indecomposable. For $i \geq 1$, the modules $M_i$ have Loewy length at most 2, and by assumption, this also holds for $M_0$. Note that all the modules $M_i$ are bipartite, since $S$ is not semi-Gorenstein-projective.

Since the modules $M_i$ with $i \geq 2$ are reflexive, the module $M_2^*$ is $\infty$-torsionless, $(M_{2+j})^* \simeq \Omega^j M_2^*$ and the projective cover of $(M_{2+j})^*$ is $(P_{1+j})^*$, for all $j \geq 0$. We have already discussed the case of an $\infty$-torsionfree module and know that $t(P_2^*) = t(P_3^*)$, say
equal to $t$. Thus $t(P_2) = t = t(P_3)$, and therefore $t(M_2) = t(M_3)$. This shows that condition (iv) is satisfied for the sequence $0 \to M_3 \to P_2 \to M_2 \to 0$. We consider the bipartite sequences $0 \to M_{i+1} \to P_i \to M_i \to 0$ with $0 \leq i \leq 2$ and use several times the equivalence of (i), (ii), (iii) and (iv) in the Lemma 7.2, in order to see that 

$$\dim M_1 = \dim M_0 = (t, at).$$

It remains to look at $M^*$. First, assume that $M$ is $\infty$-torsionfree. There is an $\mathcal{U}$-sequence $0 \to M \to P \to \mathcal{U}M \to 0$. Since both $M$ and $\mathcal{U}M$ are reflexive, the $A$-dual sequence $0 \leftarrow M^* \leftarrow P^* \leftarrow (\mathcal{U}M)^* \leftarrow 0$ is also an $\mathcal{U}$-sequence. As we know, $t(\mathcal{U}M) = t$, thus $P$ has rank $t$, therefore $P^*$ has rank $t$. This implies that $t(M^*) = t$, and therefore 

$$\dim M^* = (t, at),$$

since $M^*$ is semi-Gorenstein-projective and bipartite.

Second, assume that $M$ is semi-Gorenstein-projective and reflexive. We consider an $\mathcal{U}$-sequence $0 \to \Omega M \to P \to M \to 0$. Since both $M$ and $\Omega M$ are reflexive, also the $A$-dual $0 \leftarrow (\Omega M)^* \leftarrow P^* \leftarrow M^* \leftarrow 0$ is an $\mathcal{U}$-sequence. Now, the rank of $P$ is $t$, thus the rank of $P^*$ is $t$ and therefore $|\top(\Omega M)^*| = t$. Now, $(\Omega M)^* = \mathcal{U}M^*$. Since $M^*$ is $\infty$-torsionfree, 

$$\dim M^* = \dim \mathcal{U}M^* = (t, at).$$

9.6. Example. A short local algebra with an indecomposable module $M$ which is semi-Gorenstein-projective and torsionless, with $\dim M^* \neq \dim M$. Let $A = \Lambda_0$ as discussed in section 11 (and before in [RZ1]) and let $M$ be the right module $m_1 A = (x-y)A$ (as above in 2.6). The module $M$ is indecomposable, semi-Gorenstein-projective, and torsionless (but not reflexive). We have $(m_1 A)^* = M(q)^* = \Omega M(1)$, see 6.7 in [RZ1]. Therefore 

$$\dim m_1 A = (1, 2),$$

whereas $\dim (m_1 A)^* = (2, 1)$.

10. Some complexes of type I.

First, let us consider local modules. Note that a module $M$ with Loewy length at most 2 is local iff $\dim M = (1, s)$ for some natural number $s$.

10.1. Lemma. Let $A$ be a short local algebra with $a = e - 1$ and assume that $A$ is not self-injective. If $0 \to X \to P \to Z \to 0$ is a bipartite sequence, with $X$ a local module, then $\dim X = \dim Z = (1, a)$. In particular, also $Z$ is local.

Proof. First, let $e = 2$, thus $a = 1$. Since $A$ is not self-injective, 5.2 asserts that $J^2 \subseteq \soc_A A$, thus $A J = I \oplus S$, where $I$ is indecomposable and of length 2. Let $B$ be the factor algebra of $A$ modulo the annihilator of $I$, thus of $A J$. Then $a(B) = 0$, $e(B) = 1$, thus $I$ and $S$ are the only indecomposable $B$-modules. Since $X$ is cogenerated by $A J$, it is a $B$-module. Since $X$ is bipartite, we have $X = I$, thus $\dim X = (1, 1)$. Since the cokernel of the embedding $X \to P$ has Loewy length at most 2, we see that the projective module $P$ has rank 1, thus $\dim Z = (1, 1)$.

Second, let $e \geq 3$. Since $X$ is local and not simple, $\dim X = (1, s)$ for some $s$ with $1 \leq s \leq e$. According to the Main Lemma, $\dim Z = (\frac{s}{a}, -1 + \frac{s}{a}(a + 1))$. It follows that $\frac{s}{a}$ has to be a natural number. Since $a \leq s \leq a + 1$ and $a \geq 2$, it follows that $s = a$ and therefore $\dim X = (1, a) = \dim Z$.

Remark. Let $A$ be a short local algebra with $a = e - 1$ and assume that $A$ is not self-injective. Let $0 \to X \to P \to Z \to 0$ be a bipartite sequence. If $Z$ is a local module, then $X$ does not have to be local. For an example, take an algebra of the form $A = \Lambda_0$ as
discussed in section 11 (and before in [RZ1]. Let $X$ be the submodule of $P = _AA$ generated by $x$ and $y$ and $Z = P/X$. Then both $X$ and $Z$ are indecomposable of Loewy length 2. We have $\dim X = (2, 2)$, and $\dim Z = (1, 1)$, thus $Z$ is local whereas $X$ is not local. Note that $\delta(X) = 2$, and $\delta(Z) = 1$.

**Corollary.** Let $A$ be a short local algebra with $a = e - 1$ and assume that $A$ is not self-injective. If $X$ is a local reflexive module, then $\dim X = \dim \Omega X = (1, a)$.

Proof. Since $X$ is torsionless, there is an exact sequence $\epsilon: 0 \to X \to P \to \Omega X \to 0$. Since $X$ is even reflexive, we know that $\Omega X$ has Loewy length at most 2, thus $\epsilon$ is a bipartite sequence.

We consider now the case of a commutative short local algebra with $a = e - 1$. First, let $A$ be an arbitrary commutative artinian ring.

**10.2. Lemma.** Let $A$ be a commutative artinian ring. If $M$, $\Omega M$ and $\Omega^2 M$ are local modules, then $\Omega^3 M \cong \Omega M$.

Proof. Let $p: A \to M$, $p': A \to \Omega M$, $p'': A \to \Omega^2 M$ be projective covers. Let $u: \Omega M \to A$ be the kernel of $p$ and $u': \Omega^2 M \to A$ be the kernel of $p'$. Then we have $(u'p') (u'p'') = 0$. Now $u'p', u'p''$ are right multiplications by elements of $A$. Since $A$ is commutative, we have $(u'p'')(u'p') = 0$, thus $p''u = 0$ (since $p'$ is epi and $u'$ is mono). The sequence $0 \to \Omega M \xrightarrow{u} A \xrightarrow{p''} \Omega^2 M \to 0$ is a short exact sequence, since $u$ is mono, $p''$ epi, $p''u = 0$, and $|\Omega M| + |\Omega^2 M| = |_AA|$. Thus $\Omega^3 M = \Omega M$.

**Corollary.** Let $A$ be a commutative short local algebra. Then any complex of type I involving a projective module of rank 1 is periodic of period 2, and there is no complex of type II involving a projective module of rank 1.

If $A$ is a non-commutative short local algebra, then there may exist non-periodic complexes of type I involving a projective module of rank 1, as well as with complexes of type II involving a projective module of rank 1. A typical example is the algebra $A = \Lambda_0$ discussed in section 11 (and before in [RZ1]).

**10.3. Proposition.** Let $A$ be a commutative short local algebra with $a = e - 1$ and assume that $A$ is not self-injective. If $X$ is a local module and an $\Omega$-path of length 4 ends in $X$, then $X$ is Gorenstein-projective with $\Omega$-period 2 and $\dim \Omega X = \dim X$.

Proof. The $\Omega$-path shows that the modules $X$, $\Omega X$, $\Omega^2 X$ are reflexive. Corollary 10.1 shows successively that the modules $\Omega X$, then $\Omega^2 X$, finally $\Omega^3 X$ are local. We apply Lemma 10.2 to $M = \Omega^3 X$ (with $\Omega M = \Omega^2 X$, $\Omega^2 M = \Omega X$, $\Omega^3 M = X$) and see that $X \cong \Omega^2 X$. This shows that $X$ is Gorenstein-projective with $\Omega$-period 2. Also we see that $\dim \Omega X = \dim X$.

**10.4.** Next, let us draw the attention to acyclic minimal complexes $P_\bullet$ such that $H_i(P_\bullet) \neq 0$ for all $i \in \mathbb{Z}$. Answering questions in [CV], Hughes-Jorgensen-Şega [HJS] provided examples of such complexes over a commutative ring $A$, namely over a short local algebra of Hilbert type $(5, 4)$. In the non-commutative setting, there are such examples already over short local algebras of Hilbert type $(2, 1)$ and $(3, 2)$.
Examples. Short local algebras with an acyclic minimal complexes $P_\bullet$ such that $H_i(P_\bullet^*) \neq 0$ for all $i \in \mathbb{Z}$.

As first example, take the algebra $A$ of Hilbert type $(2, 1)$ exhibited in 9.3 and the complex $P_\bullet$ mentioned there, with $d_i : AA \to AA$ the multiplication by $y$ for all $i \in \mathbb{Z}$. All images are equal to $Ay$, thus 2-dimensional, and therefore $P_\bullet$ is acyclic. In the $A$-dual complex $P_\bullet^*$, the images are equal to $yA$, thus 1-dimensional. Therefore $H_i(P_\bullet^*) \neq 0$ for all $i \in \mathbb{Z}$.

An example $A$ with Hilbert type $(3, 2)$ is the algebra $A = \Lambda_0$ as discussed in section 11 (but also in [RZ1]; actually, one may take any algebra of the form $\Lambda(q)$ as considered in [RZ1], with arbitrary $q$). Let $M = Ay$. Then $\Omega M \simeq M$. If $P_\bullet$ is the complex with $P_i = AA$ and with all maps $d_i : P_i \to P_{i-1}$ being the right multiplication by $y$, then $P_\bullet$ is acyclic and minimal, with all images isomorphic to $Ay$ (thus bipartite), whereas all the images of $P_\bullet^*$ are isomorphic to the 2-dimensional right module $yA$ and therefore $\dim H_i(P_\bullet^*) = 2$ for all $i \in \mathbb{Z}$.

10.5. Any $\infty$-torsionfree module $M$ has a projective coresolution which is the concatenation of $\mathcal{O}$-sequences, we may call it its $\mathcal{O}$-coresolution. We may concatenate the $\mathcal{O}$-coresolution of $M$ with a minimal projective resolution of $M$ and obtain in this way an acyclic minimal complex $P_\bullet(M)$ of projective modules. Given an $\infty$-torsionfree module $M$, one may ask whether $P_\bullet(M)$ is a complex of type I or of type II.

Let us stress that both cases are possible, as the algebra $A = \Lambda_0$ considered in section 11 (and before in [RZ1]) shows. The $\Lambda_0$-module $M(1)$ is $\infty$-torsionfree, and $\Omega M(1)$ has a simple direct summand, thus the minimal projective resolution of $M(1)$ consists of projective modules whose rank is not bounded (see Proposition 8.2), thus $P_\bullet(M(1))$ is a complex of type II.

On the other hand, if $M$ is even Gorenstein-projective, then $P_\bullet(M)$ is a complete projective resolution, thus it is a complex of type I.

But there are also $\infty$-torsionfree modules which are not Gorenstein-projective, such that $P_\bullet(M)$ is a complex of type I. For example, the $\Lambda_0^{op}$-module $m_q^2 \Lambda_0$ is $\infty$-torsionfree. Here, for $\alpha \in k$, we define $m_\alpha = x - \alpha y \in \Lambda_0$. The syzygies of $m_q^2 \Lambda_0$ are the modules $m_q \Lambda_0$ with $q \leq 1$, thus of rank 1. We see that $P_\bullet(m_q^2 \Lambda_0)$ is a complex of type I.

In addition, let us remark that there are complexes $P_\bullet = (P_i, d_i)$ of type I such that the image $M$ of some $d_i$ is semi-Gorenstein-projective, but not Gorenstein-projective. An example is the $\Lambda_0^{op}$-module $M = m_1 \Lambda_0$ in [RZ1].

11. Some short local algebras of Hilbert type $(a + 1, a)$.

Let $q$ be an element of $k$ with infinite multiplicative order. Let $c \geq 0$ be an integer. We define a local algebra $\Lambda = \Lambda_c$ by generator and relations. Note that for $c = 0$ we deal with the algebra $\Lambda_0 = \Lambda(q)$ constructed in [RZ1] (and considered also in [RZ2]).

The algebra $\Lambda = \Lambda_c$ is generated by $x, y, z, u_1, \ldots, u_c$, subject to the relations:

$$x^2, y^2, z^2, yz, xy + qyx, xz - zx, zy - zx,$$

as well as

$$xu_i - u_ix, yu_i, u_iy, zu_i, u_iz, u_iu_j,$$
for all $1 \leq i, j \leq c$. We obtain in this way a short local algebra of Hilbert type $(3 + c, 2 + c)$ say with radical $J$, such that $yx, zx, u_1x, \ldots, u_cx$ is a basis of $J^2 = \text{soc} \Lambda J = \text{soc} J_\Lambda$.

We may visualize (the coefficient quiver of) $\Lambda J$ as follows:

```
  y   z   u_1   \ldots   u_c
   \downarrow\downarrow \downarrow\downarrow
   x   x   x   \ldots   x
   \downarrow\downarrow \downarrow\downarrow
  yx zx u_1x \ldots u_cx
   \downarrow\downarrow \downarrow\downarrow
   y z u_1 u_c
```

using the usual convention that a solid arrow $v \rightarrow v'$ labeled say by $x$ means that $xv = v'$, a dashed arrow $v \rightarrow\rightarrow v'$ labeled by $x$ means that $xv$ is a non-zero multiple of $v'$ (in our case, $xy = -qyx$). Here, the middle layer with the vertices $yx, zx, u_1x, \ldots, u_cx$ is the basis of $J^2$ mentioned already.

We are interested in the modules $M(\alpha)$ with $\alpha \in k$ with basis $v, v', v'', v_1, \ldots, v_c$, such that $xv = \alpha v'$, $yv = v'$, $zv = v''$, $u_iv = v_i$, for all $1 \leq i \leq c$ and such that $v', v'', v_1, \ldots, v_c$ are annihilated by all generators.

```
  v   \ldots   \ldots
   \downarrow\downarrow \downarrow\downarrow
   \downarrow\downarrow \downarrow\downarrow
   x   u_1 u_c
   \downarrow\downarrow \downarrow\downarrow
  x   \downarrow\downarrow \downarrow\downarrow
   \downarrow\downarrow \downarrow\downarrow
  v' v'' v_1 \ldots v_c
```

The modules $M(\alpha)$ with $\alpha \in k$ are pairwise non-isomorphic indecomposable $\Lambda$-modules of length $3 + c$. As in [RZ1] one shows:

1. The module $M(0)$ is Gorenstein-projective and $\Omega$-periodic with period 1. In particular, there are non-zero acyclic minimal complexes of projective modules of type I.

2. The module $M(q)$ is semi-Gorenstein-projective and not torsionless.

3. The module $M(1)$ is $\infty$-torsionfree and $\Omega M(1)$ has a simple direct summand. As a consequence, $P_*(M(1))$ is a non-zero acyclic minimal complex of projective modules of type II.

Of course, also the further properties of $\Lambda_0$ shown in [RZ1] carry over to the algebras $\Lambda_c$ with arbitrary $c \geq 0$. Here, we only want to stress that for any $a \geq 2$, there does exist a short local algebra $A$, namely $A = \Lambda_{a-2}$, of Hilbert type $(a + 1, a)$ which has modules $M, M', M''$ of length $a + 1$ such that $M$ is Gorenstein-projective, $M'$ is semi-Gorenstein-projective and not torsionless, and $M''$ is $\infty$-torsionfree, with $\Omega M''$ having a simple direct summand.
12. The Auslander-Reiten conjecture (proof of theorem 5).

We need some preliminary considerations (they are well-known, see for example Iyama [I], section 2.1, and also [M2]).

12.1. Lemma. Let $\text{Ext}^1(Z, A) = 0$. Then, for any module $N$, we have

(a) $\text{Ext}^1(Z, N) \simeq \text{Hom}(\Omega Z, N)$,

(b) $\text{Hom}(Z, N) \simeq \text{Hom}(\Omega Z, \Omega N)$.

Proof. Let $0 \to X \xrightarrow{u} PZ \to Z \to 0$ be exact, where $PZ$ is a projective cover of $Z$, thus $X = \Omega Z$.

(a) We get the exact sequence

$$\text{Hom}(PZ, N) \xrightarrow{\text{Hom}(u, N)} \text{Hom}(X, N) \xrightarrow{\delta} \text{Ext}^1(Z, N) \to 0$$

Of course, the image of $\text{Hom}(u, N)$ always lies in $\text{Hom}(X, N)_{\text{add} A}$ (the set of homomorphisms $X \to N$ which factor through $\text{add} A$). Since $\text{Ext}^1(Z, A) = 0$, the map $u$ is a left $\text{add}(A)$-approximation, thus any homomorphism $X \to N$ which factors through $\text{add}(A)$ factors through $u: X \to PZ$. This shows that the image of $\text{Hom}(u, N)$ is equal to $\text{Hom}(X, N)_{\text{add} A}$. By definition, $\underline{\text{Hom}}(X, N) = \text{Hom}(X, N)/\text{Hom}(X, N)_{\text{add} A}$, thus $\delta$ yields an isomorphism $\underline{\text{Hom}}(X, N) \simeq \text{Ext}^1(Z, N)$.

(b) Let $0 \to \Omega N \to PN \to N \to 0$ be exact. Any map $f: Z \to N$ lifts to a map $f': PZ \to PN$ and thus yields by restriction a map $f'': X \to \Omega N$. If $f$ factors though $\text{add} A$, then $f''$ factors also through $\text{add} A$. In this way, we obtain an additive map $\eta: \underline{\text{Hom}}(Z, N) \to \underline{\text{Hom}}(X, \Omega N)$. Since $u$ is a left $\text{add}(A)$-approximation, the map $\eta$ is bijective. \qed

12.2. Lemma. If $\text{Ext}^i(Z, A) = 0$ for $i = 1, 2$, then, for any module $N$

$$\text{Ext}^1(Z, N) \simeq \text{Ext}^1(\Omega Z, \Omega N).$$

Proof. Since $\text{Ext}^1(Z, A) = 0$, we have $\text{Ext}^1(Z, N) \simeq \text{Hom}(\Omega Z, N)$. Since $\text{Ext}^1(\Omega Z, A) = 0$, we have $\underline{\text{Hom}}(\Omega Z, N) \simeq \underline{\text{Hom}}(\Omega^2 Z, \Omega N)$ and $\underline{\text{Hom}}(\Omega^2 Z, \Omega N) \simeq \text{Ext}^1(\Omega Z, \Omega N)$. \qed

12.3. Corollary. If $M$ is semi-Gorenstein-projective, and $N$ is an arbitrary module, we have $\text{Ext}^i(M, N) \simeq \text{Ext}^i(\Omega M, \Omega N)$, for all $i \geq 1$.

Proof. We apply 12.2 to $\Omega^{i-1} M$ and see:

$$\text{Ext}^i(M, N) \simeq \text{Ext}^1(\Omega^{i-1} M, N) \simeq \text{Ext}^1(\Omega^i M, \Omega N) \simeq \text{Ext}^i(\Omega M, \Omega N).$$ \qed

12.4. Proposition. Let $A$ be a short local algebra which is not self-injective, and let $M$ be a non-projective semi-Gorenstein-projective module. Then $\text{Ext}^i(M, M) \neq 0$ for all $i \geq 1$. 26
Proof. We can assume that $M$ is indecomposable, then also all the modules $\Omega^i M$ are indecomposable with $i \geq 0$. Let $(e, a)$ be the Hilbert-type of $A$. Let $t = t(\Omega M)$. According to Theorem 4, we have $a = e - 1 \geq 2$ and $\dim \Omega^i M = (t, at)$ for all $i \geq 1$. We have for $i \geq 1$

$$\text{Ext}^i(M, M) = \text{Ext}^i(\Omega M, \Omega M) = \text{Ext}^i(\Omega^i M, \Omega M)$$

where we use 12.3. Now $\dim \Omega^i M = (t, at) = \dim \Omega M$, thus both modules $\Omega^i M$ and $\Omega M$ are regular modules, see A.2. Since $e \geq 3$, it follows that $\text{Ext}_L^1(\Omega^i M, \Omega M) \neq 0$. But then also $\text{Ext}_A^1(\Omega^i M, \Omega M) \neq 0$.

12.5. Proof of Theorem 5. Let $A$ be a short local algebra and let $M$ be a non-projective semi-Gorenstein-projective module. We want to show that $\text{Ext}^1(M, M) \neq 0$. We can assume that $M$ is indecomposable. According to Theorem 4, $A$ is either self-injective, or we have $a = e - 1 \geq 2$. If $A$ is self-injective, then Proposition A.5 asserts that $\text{Ext}^1(M, M) \neq 0$. If $a = e - 1 \geq 2$, then we use Proposition 12.4.

13. The Main Lemma, revisited.

13.1. Main Lemma in the case $J^2 = \text{soc}_A A$. Let $A$ be a short local algebra with $J^2 = \text{soc}_A A$. Let $M$ be a module of Loewy length at most 2. Let $\Omega M = X \oplus S^w$ with $X$ bipartite and $w \in \mathbb{N}$. Then

$$\dim \Omega M = \omega_a^e \dim M + (w, -w).$$

Proof. Let $M' = \Omega M$ and take an exact sequence $0 \to M' \to P \to M \to 0$ with $P$ projective and with an inclusion map $M' \to P$. Let $U = J^2 P$. As in the proof of 4.1, we see that $JM' \subseteq U \subseteq \text{soc} M'$ and that

$$\dim M' = \omega_a^e \dim M + (w, -w).$$

where $w = |U/JM'|$.

Now $J^2 = \text{soc}_A A = \text{soc}_A J$ means that $\text{soc}_A J^2$ is bipartite, thus also $JP$ is bipartite. Therefore $M' \subseteq JP$ implies that $\text{soc} M' \subseteq \text{soc} JP = J^2 P = U$, and therefore $U = \text{soc} M'$.

Write $M' = X \oplus W$ with $X$ bipartite and $W$ semisimple. Then $U = \text{soc} M' = JX \oplus W$, and $JM' = JX \oplus JW = JX$. Altogether, we get $U = JM' \oplus W$. It follows that $w = |U/JM'| = |W|$. Thus, $W$ is isomorphic to $S^w$ and therefore $M' = X \oplus W = X \oplus S^w$ with $X$ bipartite.

13.2. Recall that a module $M$ of Loewy length at most 2 is said to be aligned (see 4.2), provided $\dim \Omega M = \omega_a^e \dim M$.

Corollary. Let $A$ be a short local algebra with $J^2 = \text{soc}_A A$. Then a module $M$ of Loewy length at most 2 is aligned if and only if $\Omega M$ is bipartite.

Proof. Let $M$ be a module of Loewy length at most 2. We have seen in 4.2 that if $\Omega M$ is bipartite, then $M$ is aligned. For the converse, we need the assumption that $J^2 = \text{soc}_A A$. By 13.1, we know that $\Omega M = X \oplus S^w$ with $X$ bipartite and $\dim \Omega M =$.
\( \omega_d \dim M + (w, -w) \). If \( M \) is aligned, then \( \dim \Omega M = \omega_d \dim M \), thus \( w = 0 \), and therefore \( \Omega M \) is bipartite.

Using 13.1, we are able to improve Theorem 3 in case \( J^2 = \soc_A A \).

**13.3. Corollary.** Let \( A \) be a short local algebra of Hilbert type \((e,e-1)\) which is not self-injective and assume that \( J^2 = \soc_A A \).

Let \( P_i = (P_i, d_i)_i \) be a non-zero acyclic minimal complex of projective modules of type II, let \( M_i \) be the image of \( d_i \) and \( t_i = t(P_i) = t(M_i) \). As we know, there is \( v \in \mathbb{Z} \) with \( t_{v+1} > t_v = t_{v-1} \). Let \( t = t_v \). Then all the modules \( M_i \) with \( i \leq v \) are bipartite, whereas \( M_{v+1} \) is not bipartite.

Proof. By Theorem 3, we know that \( M_{v+1} \) is not bipartite and that \( \dim M_i = (t, at) \) for all \( i \leq v \). Suppose that \( M_i \) is not bipartite, say \( M_i = U \oplus S^w t \) with \( U \) bipartite and \( w \geq 1 \). Let \( M = M_{i-1} \). According to 13.1, we have \( \dim M_i = \dim M = \omega_d \dim M + (w, -w) \). Thus \( t(M_i) = t + w > t \) and therefore \( i > v \).

**13.4. Remark.** Let us return to the Main Lemma itself. Let \( M \) be a module of Loewy length at most 2. If we use covering theory, the number \( w \) provided by the Main Lemma can be understood well. Thus, let \( A \) be a \( \mathbb{Z} \)-cover of \( A \) (we assume that the set of vertices of the quiver of \( A \) is \( \mathbb{Z} \), and that the arrows go from \( i \) to \( i+1 \), for all \( i \)). Let \( \pi \) be the push-down functor. Let \( \tilde{M} \) be a module with \( \pi(\tilde{M}) = M \), such that top \( \tilde{M} \) is a direct sum of copies of \( S(0) \) (we recall the definition of \( \tilde{M} \) in A.2). Then \( \Omega \tilde{M} = U \oplus S(2)^w \oplus S(1)^{w'} \), with \( U \) being bipartite (and having support equal to \( \{1,2\} \)) provided \( U \neq 0 \). It follows that \( \Omega M = \pi(\Omega \tilde{M}) = \pi(U) \oplus S^{w+w'} \). Here we see the number \( w \) which is mentioned in the Main Lemma. If we consider \( \Omega \tilde{M} \) as a representation of the \( e \)-Kronecker quiver with vertices 1, 2, then \( S(2)^w \) is a maximal direct summand of \( \Omega \tilde{M} \) which is semisimple and projective, whereas \( S(1)^{w'} \) is a maximal direct summand of \( \Omega \tilde{M} \) which is semisimple and injective.

**14. Algebras without non-projective reflexive modules and without non-zero acyclic minimal complexes of projective modules.**

**14.1. Proposition.** Let \( e \geq 2 \). For any \( 0 \leq a \leq e^2 \), there exists a short local algebra of Hilbert type \((e,a)\) such that any reflexive module is projective and such that the only acyclic minimal complex of projective modules is the zero complex.

Proof. Let \( E \) be a vector space of dimension \( e \) say with basis \( x_1, \ldots, x_e \) and let \( T \) be the truncated tensor algebra \( T = k \oplus E \oplus (E \otimes E) \). Of course, \( T \) is a short local algebra with \( J(T) = E \oplus (E \otimes E) \) and \( J(T)^2 = E \otimes E \), thus \( e(T) = e \), \( a(T) = e^2 \).

Let \( 0 \leq a \leq e^2 \). We will choose a suitable subspace \( U \subseteq E \otimes E \) with \( \dim U = e^2 - a \) and define \( A = T/U \). Then \( J(A) = J(T)/U \). Always, \( J(A) = J(T)/U \) will be decomposable, thus Theorem 1 asserts that \( A \) has no non-projective reflexive modules.

If \( a = 0 \), then we have to take \( U = E \otimes E \) and obtain \( A = L(e) \). Since \( e \geq 2 \), \( J(A) = E \) is a semisimple \( A \)-module of length \( e \), thus decomposable.

Let \( E' \) be the subspace of \( E \) with basis \( x = x_1 \), and \( E'' \) the subspace generated by \( x_2, \ldots, x_e \). Thus \( E = E' \oplus E'' \).
If \( a \geq e \), then \( E \otimes E' \) has dimension \( e(e - 1) \geq e^2 - a \), thus there is a subspace \( U \subseteq E \otimes E'' \) of dimension \( e^2 - a \). Then, for \( A = T/U \), we have \( J(A) = J' \oplus J'' \), where \( J' = E' \oplus (E \otimes E') \) and \( J'' = E'' \oplus (E \otimes E'')/U \) are non-zero submodules of \( A \), \( J(A) \) is decomposable. Note that \( \dim J(A)^2 = \dim(E \otimes E') + \dim(E \otimes E'')/U = e + (e(e - 1) - (e^2 - a)) = a \).

Finally, let \( 1 \leq a < e \). Let \( U' \) be the subspace of \( E \otimes E \) with basis \( x_{a+1} \otimes x, \ldots, x_e \otimes x \), and let \( U'' = E \otimes E'' \). Let \( U = U' \oplus U'' \). By abuse of notation, we will denote the residue class of \( z \in T \) modulo \( U \) just by \( z \) again. We note that \( A \) is the direct sum of the local module \( N \) generated by \( x = x_1 \) (with basis \( x, x \otimes x, \ldots, x_a \otimes x \), thus \( \dim N = (1, a) \)) and a semisimple module with basis \( x_2, \ldots, x_e \), thus \( J(A) \simeq N \oplus S^{e-1} \). In particular, \( A \) is again decomposable.

We claim that the only acyclic minimal complex of projective \( A \)-modules is the zero complex. According to Theorem 3, we only have to look at the case \( a = e - 1 \). Note that \( J(A) \) has the basis \( x_1, \ldots, x_e; x_1 \otimes x, \ldots, x_a \otimes x \).

The only indecomposable modules cogenerated by \( A \) are \( N \) and \( S \) (namely, the annihilator \( C \) of \( A \) is the ideal generated by \( J' \) and the element \( x_e \), thus \( A'' = A/C \) is of the form \( L(a) \), and \( A'' N \) is the indecomposable projective \( L(a) \)-module).

We have \( \Omega S = A \). And we have \( \Omega N = S^e \) (namely, the map \( f : A \to N \) with \( f(1) = x \) maps \( x_i \) to \( x_i \otimes x \), thus its kernel has basis \( x_1 \otimes x, \ldots, x_a \otimes x \) and \( x_e \), thus \( \Omega N \) is of the form \( S^e \).)

Assume now that \( P \) is an acyclic minimal complex of projective modules and that \( M \) is one of the images. Then \( M \) is torsionless of Loewy length at most 2, thus of the form \( M = N^u \oplus S^v \) for some natural numbers \( u, v \). We have \( t(M) = u + v \). Since

\[
\Omega M = \Omega(N^u \oplus S^v) = S^{eu} \oplus N^v \oplus S^{e-1}v,
\]

we have \( t(\Omega M) = eu + v + (e - 1)v = e(u + v) \). It follows that \( t(P_1) = et(P_i) \) for all \( i \in \mathbb{Z} \). Since \( e \geq 2 \), this is only possible if \( t(P_i) = 0 \) for all \( i \in \mathbb{Z} \), thus \( P \) is the zero complex. \( \square \)

**Remark.** The assumption \( e \geq 2 \) is necessary, since all short local algebras with \( e = 1 \) are self-injective and not semisimple (thus, the simple module is non-projective and reflexive and occurs as an image in an acyclic minimal complex of projective modules).

### 15. Algebras with a non-projective reflexive module.

**15.1. Proposition.** Let \( 1 \leq a \leq e - 1 \). There exists an (even commutative) short local algebra \( A \) of Hilbert type \((e, a)\) with a reflexive module of Loewy length 2 with dimension vector \((1, a)\).

Proof. Let \( c = e - a - 1 \). Let \( A \) be the commutative algebra with generators

\[
x, y_1, \ldots, y_a, z_1, \ldots, z_c\]

and relations

\[
x^2, xz_j, y_iy_j, y_iy_j, z_j^2 - xy_a, z_jz_j'\]
for all \( i, i' \in \{1, \ldots, a\} \) and all \( j, j' \in \{1, \ldots, c\} \) with \( j' \neq j \). The elements \( xy_1, \ldots, xy_a \) form a basis of the vector space \( J^2 = \operatorname{soc}_A A = \operatorname{soc} A_A \). For \( a = c = 2 \), the module \( _A J \) looks as follows

\[
\begin{array}{c}
  x \\
  y_1 \\
  y_2 \\
  z_1 \\
  z_2 \\
  xy_1 \\
  xy_2
\end{array}
\]

Let \( M = Ax \), this is a module with Loewy length 2 and \( \dim M = (1, a) \). Let us show that the embedding \( \iota : Ax \to A_A \) is a left add\((A_A)\)-approximation.

First, consider an element \( m = \alpha x + \sum \beta_i y_i + \sum \gamma_j z_j \) with coefficients \( \alpha, \beta_i, \gamma_j \in k \) and assume that there is a surjective map \( Ax \to Am \). We have \( xm = \sum \beta_i xy_i \). Since the element \( x \) annihilates \( Ax \), we must have \( xm = 0 \), thus \( \beta_i = 0 \) for all \( i \). We have \( z_j m = \gamma_j xy_a \). Since the element \( z_j \) annihilates \( x \), we must have \( \gamma_j = 0 \). It follows that \( m = \alpha x \). This shows that for any homomorphism \( f : Ax \to A_J \), there is a scalar \( \alpha \in k \) such that \( f - \alpha \iota \) maps into \( J^2 \).

Second, we show that all the maps \( g : Ax \to A_J^2 \) factor through \( \iota \). Let \( g(m) = \sum \delta_i xy_i \) with \( \delta_i \in k \). Let \( g' \) be the right multiplication on \( A_A \) with \( \sum \delta_i y_i \). Since

\[
g'(m) = g'(x) = x \sum \delta_i y_i = \sum \delta_i xy_i = g(m),
\]

it follows that \( g' \iota = g \). Altogether, we see that \( \iota \) is a left add\((A_A)\)-approximation.

It remains to show that the factor module \( UM = A A / Ax \) is cogenerated by \( A_J \). Now \( A A / Ax \) maps onto \( Ax \) as well as onto all the modules \( A z_j \) with \( 1 \leq j \leq c \) and the intersection of the kernels of these maps is zero. This shows that \( A A / Ax \) can be embedded into \( Ax \oplus \bigoplus_{j} A z_j \). \( \square \)

Note that the element \( x \) constructed in the proof is a Conca generator of \( J \) (following [AIS], we say that an element \( x \in A \) is a Conca generator of \( J \) provided \( x^2 = 0 \neq x \) and \( J^2 = Jx \), see [RZ3]).

16. Final remarks.

16.1. The modules we have been interested in are mainly torsionless modules, namely syzygy modules; therefore we often have restricted the attention to the \( A \)-modules of Loewy length at most 2, thus to \( L(e) \)-modules, or, better, to the factor category \( \operatorname{mod} L(e) / \operatorname{add} A_A \) (here, we factor out the ideal of \( \operatorname{mod} L(e) \) given by all maps which factor through a projective \( A \)-module). Of course, the syzygy functor \( \Omega_A \) has also to be taken into account; it is an endo-functor of the category \( \operatorname{mod} L(e) / \operatorname{add} A_A \).

Note that the syzygy modules in \( \operatorname{mod} A \) are the modules cogenerated by \( W = A_J \). This means: We start with an \( L(e) \)-module \( W \) (namely the radical \( W = A_J \) of \( A \)) and look at the category \( \operatorname{sub} W \) of all \( L(e) \)-modules cogenerated by \( W \), as well as at the endo-functor \( \Omega_A \) of \( \operatorname{sub} W / \operatorname{add} A_A \).

In dealing with \( L(e) \)-modules \( M \), the main invariant is the dimension vector \( \dim M \); it is a pair of natural numbers, thus an element of \( \mathbb{Z}^2 \). Here, \( \mathbb{Z}^2 \) is the Grothendieck group of the \( L(e) \)-modules with respect to the exact sequences of the form \( 0 \to JM \to M \to \ldots \).
$M/JM \to 0$, where $M$ is any $L(e)$-module (equivalently, given an $L(e)$-module, we may consider the corresponding $K(e)$-module $\widetilde{M}$, see Appendix A.2, and take as $\dim M$ the usual dimension vector of $M$). As we have mentioned, the main tool in this paper has been the transformation $\omega_{a}^{e}$ on $\mathbb{Z}^{2}$, since it describes for the modules $M$ in sub $W$ the dimension vector $\dim \Omega_{A}M$ in terms of $\dim M$, at least roughly. The transformation $\omega_{a}^{e}$ plays a role quite similar to the usual use of $\omega_{a}^{e}$ or better of $(\omega_{a}^{e})^{2}$ in the representation theory of the $e$-Kronecker quiver (where $(\omega_{a}^{e})^{2}$ describes the change of the dimension vectors of indecomposable non-projective modules when we apply the Auslander-Reiten translate $\tau$). A decisive difference if of course that fact that $\omega_{1}^{e}$ is invertible, whereas, for $a \geq 2$, $\omega_{a}^{e}$ is not invertible over $\mathbb{Z}$.

16.2. Part of the paper has been devoted to the study of acyclic minimal complexes of projective modules, thus to the study of minimal projective coresolutions (of a module without non-zero projective direct summands): note that a minimal projective coresolution determines uniquely an acyclic minimal complex of projective modules (see 10.5) and any acyclic minimal complex of projective modules is obtained in this way. As we have seen, a minimal projective coresolution of a module seldom does exist. Also, if it exists, then it may not be unique (see for example the module $M(0,0,1)$ mentioned in [RZ2], 1.7). However, if it exists, then its structure may be very restricted: If $A$ is a short local algebra, and $P_{0} \to P_{-1} \to P_{-2} \to \cdots$ is a non-zero minimal projective coresolution of some module, let $t_{i} = t_{i}(P_{i})$. Then either $t_{i} = t_{i-1}$ for $i \ll 0$ (and $a = e - 1$) or else $t_{i+1} + t_{i-1} = et_{i}$ for all $i \ll 0$ (and $A$ is self-injective with $a = 1$), see Theorem 3 and the appendix A.7.

Appendix A. Radical square zero algebras and self-injective algebras.

We want to describe the categories mod $A$ where $A$ is a local algebra with radical square zero or a self-injective short local algebra. We start in A.2 with the radical square zero $k$-algebra $A = L(e)$ (with radical $J$ of dimension $e$ and $A/J = k$). In order to do so, we look in A.1 at a related algebra, the path algebra of the $e$-Kronecker quiver.

A.1. The structure of mod $K(e)$.

We denote by $K(e)$ the $e$-Kronecker quiver with $e$ arrows (or its path algebra):

\[
\begin{array}{c}
\circ \\
0 \\
\end{array} \xrightarrow{\langle e \rangle} \begin{array}{c}
\circ \\
1 \\
\end{array}
\]

(here and also later, we will depict a set of $e$ arrows with same source and same sink by a single arrow endowed with the symbol $\langle e \rangle$). A representation (or module) $V$ of $K(e)$ will be written in the form $V = (V_{0}, V_{1}; \phi: k^{e} \otimes V_{0} \to V_{1})$. There are two simple representations, namely $S(0) = (k, 0; 0)$ and $S(1) = (0, k; 0)$.

The Grothendieck group of mod $K(e)$ (with respect to exact sequences) is $\mathbb{Z}^{2}$. Given a representation $V$ of $K(e)$, the corresponding element in the Grothendieck group is the dimension vector $\dim V = (\dim V_{0}, \dim V_{1})$ of $V$. On $\mathbb{Z}^{2}$, we consider the quadratic form $q(x, y) = x^{2} + y^{2} - ey$. We have $q(\dim V) = \dim \text{End}(V) - \dim \text{Ext}^{1}(V, V)$ for every module $V$ (see [R]); more generally, given modules $V, V'$ with $\dim V = \dim V'$, we have $q(\dim V) = \dim \text{Hom}(V, V') - \dim \text{Ext}^{1}(V, V')$. We can use $q$ in order to distinguish

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between the regular indecomposable and the non-regular indecomposable modules: An indecomposable module V is regular, provided \( \text{Ext}^1(V, V) \neq 0 \), and this happens if and only if \( q(\text{dim} V) \leq 0 \). The remaining indecomposable modules are the indecomposable modules with \( q(\text{dim} V) = 1 \) and then \( \text{dim} \text{End}(V) = 1 \). (An element \( (x, y) \in \mathbb{Z} \) is said to be a real root of \( q \) provided \( q(x, y) = 1 \) and an imaginary root in case \( q(x, y) \leq 0 \).) If \( V \) is a regular indecomposable module, then there exists an indecomposable module \( V' \) with \( \text{dim} V' = \text{dim} V \) such that \( V \) and \( V' \) are not isomorphic. If \( V \) is indecomposable with \( q(\text{dim} V) = 1 \), then any indecomposable module \( V' \) with \( \text{dim} V' = \text{dim} V \) is isomorphic to \( V \). A non-regular indecomposable module \( V \) with \( \text{dim} V = (x, y) \) is said to be preprojective provided \( x < y \), otherwise it is said to be preinjective (and then \( x > y \)).

For \( e = 1 \), there are just 3 indecomposable representations, namely \( S(1), P(0), S(0) \), with \( \text{dim} S(1) = (0, 1), \text{dim} P(1) = (1, 1) \) and \( \text{dim} S(0) = (1, 0) \).

We assume now that \( e \geq 2 \). The indecomposable preprojective modules can be labeled \( P_0, P_1, P_2, \ldots \), with \( P_0 = S(1) \), \( P_1 \) the indecomposable projective representation corresponding to the vertex 0 (thus \( \text{dim} P_1 = (1, e) \)) and \( \text{dim} P_{i+1} = e \text{dim} P_i - \text{dim} P_{i-1} \) for \( i \geq 1 \). Similarly, the indecomposable preinjective modules can be labeled \( Q_0 = S(0), Q_1, Q_2, \ldots \); with \( Q_0 = S(0) \), \( Q_1 \) the indecomposable injective representation corresponding to the vertex 1 (thus \( \text{dim} Q_1 = (e, 1) \)) and \( \text{dim} Q_{i+1} = e \text{dim} Q_i - \text{dim} Q_{i-1} \) for \( i \geq 1 \). If we define \( b_n \) for \( n \geq -1 \) recursively by \( b_{-1} = 0, b_0 = 1 \) and \( b_{n+1} = eb_n - b_{n-1} \) for \( n \geq 0 \), then \( \text{dim} P_n = (b_{n-1}, b_n) \) and \( \text{dim} Q_n = (b_n, b_{n-1}) \) (for example, for \( e = 3 \), the sequence \( b_1, b_2, b_3, \ldots \) is just the sequence of the even-index Fibonacci numbers \( 0, 1, 3, 8, 21, 55, 144, \ldots \)). An explicit formula for the numbers \( b_n \) due to Avramov, Iyengar and Šega will be exhibited in Appendix B.

The global structure of \( \text{mod} K(e) \) can be seen by looking at the Auslander-Reiten quiver of \( K(e) \). It has the following shape:

![Auslander-Reiten Quiver](image)

There are two Auslander-Reiten components of non-regular modules, the preprojective component (seen on the left) and the preinjective component (seen on the right). Non-zero maps between preprojective modules (and between preinjective modules, respectively) go from left to right. Also, there are no non-zero maps from a regular module to a preprojective module, and no non-zero maps from a preinjective module to a preprojective or a regular module.

**History.** Here are at least some hints. The representations of \( K(2) \) are called Kronecker modules, since they have been classified by Kronecker in 1890, completing earlier partial results by Jordan and Weierstrass, as mentioned for example in [ARS]. This classification plays an important role in many parts of mathematics. A standard reference for
the matrix approach (in the language of matrix pencils) is Gantmacher’s book on matrix theory \cite{Gm}. There is the equivalent theory of coherent sheaves over the projective line, where the usual reference is the splitting theorem of Grothendieck (but one should be aware that this result can be traced back to Hilbert (1905), Plemelj (1908), and G. D. Birkhoff (1913), see \cite{OSS}).

Of course, the fact that there are just 3 indecomposable representations of $K(1)$ is a basic statement of elementary linear algebra.

The representation theory of $K(e)$ with $e \geq 3$ has attracted a lot of interest in the last 40 years, but is still very mysterious.

**A.2. The push-down functor** $\pi : \text{mod } K(e) \to \text{mod } L(e)$.

We recall that $L(e)$ is the local $k$-algebra with radical $J$ such that $J^2 = 0$, $\dim J = e$ and $L(e)/J = k$. We assume here that $|J| = e \geq 2$ and identify $J = k^e$.

We denote by $\pi : \text{mod } K(e) \to \text{mod } L(e)$ the push-down functor: it sends $V = (V_0, V_1; \phi : k^e \otimes V_0 \to V_1)$ to the representation

$$\pi V = \pi(V_0, V_1; \phi : k^e \otimes V_0 \to V_1) = (V_0 \oplus V_1; \begin{bmatrix} 0 & 0 \\ \phi & 0 \end{bmatrix}).$$

Under the functor $\pi$, the two simple representations of $K(e)$ are sent to the unique simple $L(e)$-module $S$. The indecomposable $K(e)$-modules of length at least 2 correspond under $\pi$ bijectively to the indecomposable $L(e)$-modules of length at least 2, thus to the indecomposable bipartite $L(e)$-modules. We have $\dim \pi V = \dim V$ for any $K(e)$-module $V$ without a simple projective direct summand.

Conversely, given an $L(e)$-module $M$, we denote by $\tilde{M}$ the $K(e)$-module

$$\tilde{M} = (\text{top } M, \text{rad } M; \overline{\mu} : J \otimes \text{top } M \to \text{rad } M),$$

where $\overline{\mu}$ is induced by the multiplication map $\mu : J \otimes M \to M$ (note that $J \otimes \text{rad } M$ is contained in the kernel of $\mu$ and that the image of $\mu$ is rad $M$). We have $\dim \tilde{M} = \dim M$ for any $L(e)$-module $M$.

We have $\pi \tilde{M} \simeq M$ for any $L(e)$-module $M$, and conversely, we have $\pi \tilde{V} \simeq V$ for any $K(e)$-module $V$ without a simple projective direct summand. Altogether we see: $\pi$ and $\sim$ provide inverse bijections between isomorphism classes as follows:

$$\begin{array}{cccccc}
\{ \text{indecomposable } \\
K(e)\text{-modules } V \\
\text{different from } S(1) \} \\
\end{array} \xrightarrow{\pi} \xleftarrow{\sim} \begin{array}{c}
\{ \text{indecomposable } \\
L(e)\text{-modules} \} \\
\end{array}$$

An indecomposable $L(e)$-module $M$ will be said to be regular provided $\tilde{M}$ is a regular $K(e)$-module. The Auslander-Reiten quiver for $L(e)$ is obtained from the Auslander-Reiten
quiver of $K(e)$ by identifying the vertices $S(1)$ and $S(0)$ in order to obtain the vertex $S$.

\[
\begin{tikzpicture}
\node (S) at (0,0) {$S$};
\node (P1) at (1,1) {$\pi P_1$};
\node (P2) at (-1,1) {$\pi P_2$};
\node (Q1) at (1,-1) {$\pi Q_1$};
\node (Q2) at (-1,-1) {$\pi Q_2$};
\node (P3) at (0,2) {$\pi P_3$};
\draw (S) -- (P1) node [midway, above] (TextNode) {\(e\)};
\draw (S) -- (P2) node [midway, above] (TextNode) {\(e\)};
\draw (S) -- (Q1) node [midway, above] (TextNode) {\(e\)};
\draw (S) -- (Q2) node [midway, above] (TextNode) {\(e\)};
\draw (P1) -- (Q1) node [midway, above] (TextNode) {\(e\)};
\draw (P2) -- (Q2) node [midway, above] (TextNode) {\(e\)};
\draw (P3) -- (S) node [midway, above] (TextNode) {\(e\)};
\end{tikzpicture}
\]

**Proposition.** If $M, M'$ are $L(e)$-modules, then $\pi$ yields an injective map

\[
\Hom_{K(e)}(\widetilde{M}, \widetilde{M}') \xrightarrow{\pi} \Hom_{L(e)}(M, M')
\]

and

\[
\Hom_{L(e)}(M, M') = \pi \Hom_{K(e)}(\widetilde{M}, \widetilde{M'}) \oplus \Hom_{L(e)}(\text{top} M, \rad M').
\]

Proof. It is easy to show this directly. But one also may invoke the general covering theory as developed by Gabriel and his students. We use the $\mathbb{Z}$-cover $Q$ of $L(e)$ with vertex set $\mathbb{Z}$, with $e$ arrows $z \rightarrow z+1$ for all $z \in \mathbb{Z}$ and with all paths of length 2 as relations. We identify the full subquiver of $Q$ with vertices 0, 1 with $K(e)$.

If $V$ is a representation of $Q$ and $j \in \mathbb{Z}$, let $V[j]$ be the shifted representation with $V[j]_i = V_{i+j}$. The push-down functor $\pi$ can be extended to a functor $\pi: \mod Q \rightarrow \mod L(e)$ and covering theory asserts that $\pi$ yields a bijection between $\bigoplus_{j \in \mathbb{Z}} \Hom_Q(V, V[j])$ and $\Hom_{L(e)}(\pi V, \pi V')$.

It remains to consider the indecomposable representations $V, V'$ of $Q$ which are either bipartite with support $\{0, 1\}$, or equal to $S(0)$. For example, if both $V, V'$ are bipartite with support in $\{0, 1\}$, then $\Hom_Q(V, V'[1]) = \Hom_k(V_0, V'_1) = \Hom_{L(e)}(\text{top} V, \rad V')$. \(\square\)

**Remarks.** Let us mention some consequences which play a role in our discussion of short local algebras.

(1) **Extensions.** Let $M$ be an indecomposable regular $L(e)$-module. Then

\[
\Ext^1_{L(e)}(M, M) \neq 0.
\]

If $e \geq 3$ and $M, M'$ are indecomposable regular $L(e)$ modules with $\dim M = \dim M'$, then

\[
\Ext^1_{L(e)}(M, M') \neq 0.
\]

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Proof. We have
\[ \dim \text{End}(\tilde{M}) - \dim \text{Ext}^1_{K(e)}(\tilde{M}, \tilde{M}) = q(\dim M) \leq 0. \]
Thus \( \text{End}(\tilde{M}) \neq 0 \) implies that \( \dim \text{Ext}^1_{K(e)}(\tilde{M}, \tilde{M}) \). Of course, a non-split self-extension of \( \tilde{M} \) yields under \( \pi \) a non-split self-extension of the \( L(e) \)-module \( M \).

The second assertion is shown in the same way, now using that for \( e \geq 3 \) we have \( q(\dim M) < 0. \) \( \Box \)

(2) Solid modules. Let \( M \) be an \( L(e) \)-module. The following conditions are equivalent:
(i) \( M \) is solid.
(ii) \( M \neq 0 \) and \( \text{End} M = k \cdot 1_M + \{ \phi \in \text{End} M | \text{Im} \phi \subseteq \text{rad} M \subseteq \text{Ker} \phi \} \).
(iii) \( \dim \text{End} M = 1 + |\text{top}\, M| \cdot |\text{rad} M| \).
(iv) \( \text{End}(\tilde{M}) = k \).
If these conditions are satisfied, then \( M \) is indecomposable.

Proof. (i) \( \implies \) (ii). Assume that \( M \) is solid. An endomorphism of \( M \) which does not vanish on \( \text{soc} M \) has to be invertible. In particular, \( M \) has to be indecomposable: namely if \( M = M' \oplus M'' \) is a direct decomposition, then the projection onto \( M' \) maps \( \text{soc} M' \) onto itself and vanishes on \( \text{soc} M'' \). Thus, either \( M = S \) or else \( M \) is bipartite. If \( \phi \) is an endomorphism of \( M \) and its restriction to \( \text{soc} M \) is the scalar multiplication by \( \lambda \in k \), then \( \phi - \lambda 1_M \) maps \( M \) into \( \text{rad} M \). This shows that \( \text{End}(M) = k \cdot 1_M \oplus \text{Hom}(\text{top} M, \text{rad} M) \), thus (ii) is satisfied.

(ii) \( \implies \) (iii) is trivial. The implication (iii) \( \implies \) (iv) is a direct consequence of the proposition.

(iv) \( \implies \) (i). Since \( \tilde{M} \) is indecomposable, also \( M \) is indecomposable. If \( M = S \), then clearly \( M \) is solid. Thus, we can assume that \( M \) is bipartite. The proposition shows that any endomorphism \( \phi \) of the form \( \phi = \lambda 1_M + \tilde{\phi} \), where \( \text{soc} M = \text{rad} M \subseteq \text{Ker}(\phi') \). This shows that the restriction of \( \phi \) to \( \text{soc} M \) is the scalar multiplication by \( \phi \). \( \Box \)

(3) Modules without self-extensions. Let \( e \geq 2 \). Let \( M \) be an indecomposable \( L(e) \)-module. The following conditions are equivalent.
(i) \( M \) is isomorphic to \( \pi P_i \) or \( \pi Q_i \) for some \( i \geq 1 \),
(ii) \( M \) is not simple and \( q(\dim M) = 1 \).
(iii) \( \text{Ext}^1_{L(e)}(M, M) = 0 \).

Proof. An indecomposable \( K(e) \)-module \( V \) satisfies \( q(\dim V) = 1 \) if and only if \( V \) is preprojective or preinjective. This yields the equivalence of (i) and (ii).

(iii) \( \implies \) (i): If \( M \) is regular, then (1) asserts that \( \text{Ext}^1_{L(e)}(M, M) \neq 0 \). If \( M = S \), then, of course, \( \dim \text{Ext}^1_{L(e)}(M, M) = e > 0 \). This shows that an indecomposable module \( M \) with \( \text{Ext}^1_{L(e)}(M, M) = 0 \) is isomorphic to \( \pi P_i \) or \( \pi Q_i \) for some \( i \geq 1 \),

(i) \( \implies \) (iii). Let \( M \) be a bipartite module with \( \dim M = (x, y) \). We define \( g(M) = \dim \text{End}(M) - 1 - xy \). Since \( xy = |\text{top}\, M| \cdot |\text{rad} M| \), we see that \( g(M) \geq 0 \) and that \( g(M) = 0 \) if and only if \( M \) is solid.
The projective cover of $M$ is isomorphic to $L(e)^x$, and $\Omega M$ is semi-simple, namely isomorphic to $S^z$ with $z = ex - y$. We apply $\text{Hom}(-, M)$ to the exact sequence $0 \to S^z \to L(e)^x \to M \to 0$ and obtain the exact sequence

$$0 \to \text{Hom}(M, M) \to \text{Hom}(L(e)^x, M) \to \text{Hom}(S^z, M) \to \text{Ext}_1^{L(e)}(M, M) \to 0.$$  

We have $\dim \text{Hom}(M, M) = xy + 1 + g(M)$, $\dim \text{Hom}(L(e)^x, M) = x(x + y)$ and finally $\dim \text{Hom}(S^z, M) = zy = (ex - y)y$. Thus

$$\dim \text{Ext}_1^{L(e)}(M, M) = xy + 1 + g(M) - x(x + y) + (ex - y)y$$

$$= 1 + g(M) - x^2 + exy - y^2 = 1 - q(x, y) + g(M).$$

If $M$ is isomorphic to $\pi P_i$ or $\pi Q_i$ for some $i \geq 1$, then $q(x, y) = 1$ and $M$ is solid, thus $g(M) = 0$, and therefore $\text{Ext}_1^{L(e)}(M, M) = 0$. \qed

**Historical remark.** The algebra $K(e)$ is obtained from $L(e)$ by a process which has been called “separation of a node” by Martinez [MV1] (a node is a simple module $S$ which never occurs as a composition factor of $\text{rad} M/(\text{rad} M \cap \text{soc} M)$, for any module $M$; if the algebra is given by a quiver with relations, then a vertex $v$ is a node iff the composition of any arrow ending in $v$ with any arrow starting in $v$ is a relation). It seems that the first systematic separation of nodes was used in Gabriel’s paper [Gb]: he showed that using separation of the nodes, the representations of a radical-square-zero algebra over an algebraically closed field can be obtained from the representations of a corresponding hereditary algebra (note that for a radical-square-zero algebra, all simple modules are nodes). The separation of nodes yields algebras which are stably equivalent, as later described in Auslander-Reiten-Smalø [ARS, Chapter X].

**A.3. The self-injective short local algebras $A$ with $e \geq 2$.**

Let $A$ be a self-injective short algebra with $e \geq 2$. We obtain the Auslander-Reiten quiver for $A$ from the Auslander-Reiten quiver of $A/J^2$ by inserting the vertex $A$. 

![Diagram](image_url)
The modules $\pi P_i$ with $i \geq 1$ are the indecomposable $A$-modules which are different from $A A$ and preprojective in the sense of Auslander-Smalø [AS]. The modules $\pi Q_i$ with $i \geq 1$ are the indecomposable $A$-modules which are different from $A A$ and preinjective in the sense of Auslander-Smalø.

Finally, let us present the Auslander-Reiten quiver of the triangulated category $\text{mod} \ A$.

\begin{center}
\includegraphics[width=0.8\textwidth]{quiver.png}
\end{center}

\textbf{A.4. The cases $e = 1$.}

If $A$ is a self-injective short algebra with $e = 1$, then either $a = 0$ or $a = 1$. In both cases, $A$ is uniserial, thus its module category is well understood.

It may be of interest to draw the four relevant pictures in case $e = a = 1$, so that one may compare them with the pictures for $e \geq 2$ exhibited above. Note that the last three categories live (again) on a cylinder. For a unified presentation, we also show $\text{mod} \ K(1)$ as embedded into a cylinder — a rather unusual display of a single triangle. Always, the dashed vertical lines are lines which have to be identified.

\begin{center}
\includegraphics[width=0.8\textwidth]{quiver_cases.png}
\end{center}

\textbf{A.5. Extensions of modules over self-injective algebras.}

In the following proposition, the first assertion is due to Hoshino [Ho1].

\textbf{Proposition.} Let $A$ be a self-injective short local algebra. If $M$ is a non-projective module, then $\text{Ext}^1(M, M) \neq 0$. 

If $e \neq 2$ and $M, M'$ are non-projective indecomposable modules with $\dim M = \dim M'$. Then $\text{Ext}^1(M, M') \neq 0$.

Proof. Since $M, M'$ are non-projective indecomposable modules, they have Loewy length at most 2. Since there are non-projective modules, we must have $e \geq 1$ and thus $\text{Ext}^1(S, S) \neq 0$, where $S$ is the simple $A$-module.

If $e = 1$, then either $M$ is simple, thus $M' \simeq M$ and $\text{Ext}^1(M, M') \neq 0$, or else $a = 1$ and $M$ is of length 2. Then again $M' \simeq M$ and there is an exact sequence $0 \to M \to A \otimes S \to M \to 0$, which shows that $\text{Ext}^1(M, M) \neq 0$, thus $\text{Ext}^1(M, M') \neq 0$.

Thus, we can assume that $e \geq 2$. If $M$ and $M'$ are regular, then $\text{Ext}^1_{K(e)}(M, M') \neq 0$, see A.2. Thus, there is an exact sequence $0 \to M' \to M'' \to M \to 0$ in mod $L(e)$ which does not split. This sequence shows that $\text{Ext}^1_A(M, M') \neq 0$.

If $M$ is not regular, then $\dim M = \dim M'$ implies that $M \simeq M'$ and $M$ belongs to the orbit of $S$ under $\Omega$ and $\Omega^{-1}$. The corollary 12.3 asserts that $\text{Ext}^1(M, M) \simeq \text{Ext}^1(S, S) \neq 0$. □

**Example.** There exists a self-injective short local algebra $A$ with a non-projective indecomposable module $M$ having the following properties:

(a) $\text{Ext}^i(M, M) = 0$ for all $i \geq 2$.

(b) If $M'$ is a non-projective, indecomposable module such that $\dim M' = \dim M$, and $M' \neq M$ and $M' \neq \Omega M$, then $\text{Ext}^1(M, M') = 0$.

Of course, in (b) the converse also holds: Namely, as we saw above, $\text{Ext}^1(M, M) \neq 0$. Also, for any non-projective module $M$, we have $\text{Ext}^1(M, \Omega M) \neq 0$.

Here is the example. Let $A$ be the the quantum exterior algebra in two variables (see for example [Sm], but also [RZ1]), this is the $k$-algebra generated by $x, y$ with the relations $x^2, y^2, xy + qyx$, where $q \in k$ has infinite multiplicative order. Note that the elements $1, x, y$, and $yx$ form a basis for $A$. Let $M = A(x - y)$, this is an indecomposable module of length 2.

Proof of (a) and (b). We have $\dim M = (1, 1)$. The indecomposable modules $M'$ with $\dim M' = (1, 1)$ are the indecomposable modules of length 2, they are of the form $M_\alpha = A(x - \alpha y)$ with $\alpha \in k$ and $M_\infty = Ay$. These modules are pairwise non-isomorphic, thus $\dim_k \text{Hom}(M_\alpha, M_\beta) = 1$ for $\alpha \neq \beta$ in $k \cup \{\infty\}$. The equality $(x - qy)(x - y) = 0$ shows that $\Omega M_\alpha = M_{q\alpha}$ for all $\alpha \in k$. In particular, we have $M = M_1$ and $\Omega^i M = M_{q^i}$ for all $i \geq 0$.

For the proof of (b), let $M'$ be non-projective, indecomposable module such that $\dim M' = \dim M$, thus $M' = M_\alpha$ for some $\alpha \in k \cup \{\infty\}$. We assume that $M'$ is not isomorphic to $M$ or $\Omega M$, thus $\alpha \notin \{1, q\}$. We want to show that $\text{Ext}^1(M, M') = 0$. Now $\text{Ext}^1(M, M') = \text{Hom}(\Omega M, M') = \text{Hom}(M_q, M_\alpha)$ (where we use 12.1(a)). Since $\alpha \neq q$, we know that $\dim_k \text{Hom}(M_q, M_\alpha) = 1$. In order to show that $\text{Hom}(M_q, M_\alpha) = 0$, we have to show that there is a non-zero map $M_q \to M_\alpha$ which factors through a projective module. There is the inclusion map $u: M_q \subset A A$. Also, there is a projection map $p: A A \to M_\alpha$ with kernel $\Omega M_\alpha = M_{q\alpha}$. If the composition $pu$ is zero, then $u$ maps into the kernel $M_{q\alpha}$ of $p$, thus $u$ yields an isomorphism $M_q \to M_{q\alpha}$ and therefore $\alpha = 1$, a contradiction. Thus, we see that the composition $pu$ is non-zero, and therefore $\text{Hom}(M_q, M_\alpha) = 0$. This completes the proof of (b).
Proof of (a). Let \( i \geq 2 \). Using 12.1(a), we see that \( \text{Ext}^i(M, M) = \text{Ext}^1(\Omega^{i-1}M, M) = \text{Ext}^1(M_{q_i-1}, M_1) \). Since \( M_1 \not\cong M_{q_i-1} \) and \( M_1 \not\cong \Omega M_{q_i-1} = M_q \), the assertion (b) implies that \( \text{Ext}^i(M_{q_i-1}, M_1) = 0 \).  

\[ \square \]

### A.6. The BGP-functors.

We want to show that for a self-injective short local algebra \( A \) of Hilbert-type \((e, 1)\), the syzygy functor \( \Omega = \Omega_A \) corresponds to a BGP-reflection functor for the \( K(e) \)-modules, as considered in [DR].

A BGP-functor \( \sigma_\beta \) for the representations of \( K(e) \) starts with two \( k \)-\( k \)-bimodules \( _0B_1, _1B_0 \) of dimension \( e \) and a non-degenerate bilinear form \( \beta: _0B_1 \otimes _1B_0 \to k \). By definition,

\[
\sigma_\beta(V_0, V_1; \phi: _1B_0 \otimes V_0 \to V_1) = (\text{Ker} \phi, \phi': _0B_1 \otimes \text{Ker} \phi \to V_0),
\]

where \( \phi' \) is the composition

\[
_0B_1 \otimes \text{Ker} \phi \xrightarrow{1 \otimes u} _0B_1 \otimes_1 B_0 \otimes V_0 \xrightarrow{\beta \otimes 1} k \otimes V_0 = V_0,
\]

with \( u: \text{Ker} \phi \to _1B_0 \otimes V_0 \) the canonical inclusion map. We have \( \sigma_\beta(S(1)) = 0 \). Let \( \text{mod}_0 K(e) \) (and \( \text{mod}_1 K(e) \)) be the full subcategory of all \( K(e) \)-modules without simple projective (and injective, respectively) direct summands. The restriction of \( \sigma_\beta \) to \( \text{mod}_0 K(e) \) is an equivalence \( \text{mod}_0 K(e) \to \text{mod}_1 K(e) \). If we denote the matrix \( \omega_\beta \) just by \( \sigma \), then \( \text{dim} \sigma_\beta M = \sigma \text{dim} M \), for any indecomposable \( K(e) \)-module \( M \) which is not simple projective.

If \( M \) is indecomposable and not isomorphic to \( S(1) \), then \( \text{dim} \sigma_\beta M = \sigma \text{dim} M \). It follows that for \( e \geq 2 \), we have

\[
\sigma_\beta P_i = \begin{cases} P_{i-1} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0, \end{cases}
\]

\[
\sigma_\beta Q_i = Q_{i+1} \text{ for all } i \geq 0.
\]

Now we fix a self-injective algebra \( A \) of Hilbert-type \((e, 1)\) and an embedding of \( k^e \) as a complement of \( J^2 \) in \( J \), thus we identify \( J/J^2 \) with \( B = k^e \). Let \( _1B_2 = _2B_1 = B \) and take as bilinear form \( \beta: B \otimes B \to k \) the multiplication map \( J/J^2 \otimes J/J^2 \to J^2 = k \). Since \( A \) is self-injective, \( \beta \) is non-degenerate and we write \( \sigma_A = \sigma_\beta \).

For any \( A \)-module \( M \), let \( \Omega_A M \) be its first syzygy module. We claim that for \( M \) in \( \text{mod}_0 K(e) \), the module \( \pi(\sigma_A M) \) is isomorphic to \( \Omega_A \pi(M) \) (of course, we have to exclude \( S(1) \), since \( \pi(\sigma_A S(1)) = 0 \), whereas \( \Omega_A \pi S(1) = \Omega_A S = _AS \)). Namely, let us start with the \( A \)-module \( M = \pi(T, \phi: B \otimes T \to JM) \), where \( T = \text{top} M = M/JM \) (thus, we identify \( M \) with \( T \oplus JM \), this is the right column in the following diagram). Its projective cover is \( P(M) = A \otimes T = (k \oplus B \oplus J^2) \otimes T \) (this is the middle column) with canonical map \( p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi & 0 \end{bmatrix}: P(M) \to M \). This yields \( \Omega_A M \) (namely the left column) as the kernel of \( p \).

Altogether, we deal with five exact sequences of vector spaces (displayed in the upper five
rows), organized in two commutative diagrams. In this way, we obtain the exact sequence of representations of \(K(e)\) exhibited as the lowest row:

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow & k \otimes T & \stackrel{1}{\rightarrow} & T & \rightarrow & 0 \\
0 & \rightarrow & B \otimes 0 & \rightarrow & B \otimes T & \stackrel{1}{\rightarrow} & B \otimes T & \rightarrow & 0 \\
\downarrow & & \downarrow & & 1 & & \phi & & \\
0 & \rightarrow & \text{Ker}(\phi) & \stackrel{u}{\rightarrow} & B \otimes T & \stackrel{\phi}{\rightarrow} & JM & \rightarrow & 0 \\
0 & \rightarrow & B \otimes \text{Ker}(\phi) & \stackrel{1\otimes u}{\rightarrow} & B \otimes B \otimes T & \stackrel{1\otimes \phi}{\rightarrow} & B \otimes JM & \rightarrow & 0 \\
\downarrow & & \downarrow & & \beta \otimes 1 & & \downarrow & & \\
0 & \rightarrow & J^2 \otimes T & \stackrel{1}{\rightarrow} & J^2 \otimes T & \rightarrow & 0 & \rightarrow & 0 \\
0 & \rightarrow & \Omega_A M & \rightarrow & P(M) & \stackrel{p}{\rightarrow} & M & \rightarrow & 0
\end{array}
\]

There is the following commutative diagram of functors:

\[
\begin{array}{ccc}
\text{mod}_0 K(e) & \xrightarrow{\sigma_A} & \text{mod } K(e) \\
\downarrow & & \mu \pi \\
\text{mod } A / \text{add}(A) & \xrightarrow{\Omega_A} & \text{mod } A / \text{add}(A)
\end{array}
\]

where \(\mu\): mod \(L(e) \rightarrow \text{mod } A\) is the canonical embedding.

**Historical remark.** Reflection functors for quivers were introduced by Bernstein-Gelfand-Ponomarev [BGP] and play an important role in the representation theory of quivers. They have been generalized to species in [DR]. As we have seen above, this generalization is also of interest for quivers, for example for the \(e\)-Kronecker quiver \(K(e)\), since one avoids in this way the use of a fixed basis of the arrow space. But we should stress that the account given here deviates from the usual convention (say used in [BGP] and [DR]) which is based on changing the orientation of arrows. Indeed, the BGP-reflection functors considered in [BGP] and [DR] send a representation of the \(e\)-Kronecker quiver \(\circ \xrightarrow{\langle e \rangle} \circ\) to a representation of the quiver \(\circ \xleftarrow{\langle e \rangle} \circ\) (with opposite orientation). In contrast, we relabel the vertices in order to obtain \(\sigma_\beta\) as an endo-functor of mod \(K(e)\). As a consequence, the change of the dimension vector under \(\sigma_\beta\) is described by the product \(\sigma\) of the usual BGP-reflection matrix \(\begin{bmatrix} 1 & 0 \\ e & -1 \end{bmatrix}\) and the matrix \(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) (corresponding to the exchange of the coordinate axes):

\[
\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ e & -1 \end{bmatrix} = \begin{bmatrix} e & -1 \\ 1 & 0 \end{bmatrix} = \omega_1^e.
\]
A.7. The $\mathcal{U}$-quiver. Acyclic minimal complexes of projective modules.

Let $A$ be a self-injective short local algebra of Hilbert-type $(e, 1)$, with $e \geq 2$. As a consequence of A.6, we want to describe the $\mathcal{U}$-quiver of $A$.

First, there are the $\mathcal{U}$-components which contain only indecomposable $A$-modules of the form $M = \pi X$, where $X$ is a regular $K(e)$-module:

\[
\cdots \quad \omega A M \quad \omega M \quad \omega A^{-1} M \quad \omega A^{-2} M \quad \cdots \\
\sigma^2 \dim M \quad \sigma \dim M \quad \dim M \quad \sigma^{-1} \dim M \quad \sigma^{-2} \dim M
\]

(below any module, we show the corresponding dimension vector). In general, such an $\mathcal{U}$-component is of type $\mathbb{Z}$ (for the definition of the type of an $\mathcal{U}$-component, see 1.5 in [RZ1]). Only for $e = 2$, $M$ may be $\omega A$-periodic, and then, of course, we deal with an $\mathcal{U}$-component of type $\tilde{A}_n$ for some $n \geq 0$.

In addition, there is just one further $\mathcal{U}$-component, namely the component containing the simple module $S$. It is always of type $\mathbb{Z}$ and consists of $S$ and the modules $\pi P_i$ and $\pi Q_i$ with $i \geq 1$. We have $\pi Q_i = \omega A S$ and $\pi P_i = \omega A S$; in particular, we have $\pi Q_1 = A J$, and $\pi P_1 = A A / J^2$.

\[
\cdots \quad \pi Q_2 \quad \pi Q_1 \quad S \quad \pi P_1 \quad \pi P_2 \quad \cdots \\
\cdots \quad \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} \quad \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \cdots
\]

(again, we show below any module the corresponding dimension vector). Since for $i \geq 0$, we have $\omega A S = \pi Q_i$ and $\dim \pi Q_i = \dim \pi P_i = (b_i, b_{i-1})$, we see that

\[t_i(S) = b_i\]

for all $i \geq 0$. This means that the numbers $b_i$ for $i \geq 0$ are just the Betti numbers of $S$.

In the display of the $\mathcal{U}$-component of $S$ we have inserted a dashed vertical line between the dimension vectors of $S$ and of $\pi P_1$. This separation line should stress that $\omega (\pi P_1) = S$, whereas $\sigma (\dim \pi P_1) = \sigma \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dim S$. There is just one $\mathcal{U}$-sequence which is not bipartite, namely the sequence starting in $S$ (as mentioned already in 2.4(a)):

\[0 \to S \to A \to \pi P_1 \to 0.\]

It is this sequence which is marked by the separation line.

**Proposition.** Let $M_1 \leftrightarrow M_0 \leftrightarrow M_{-1}$ be an $\mathcal{U}$-path. If $M_0$ is not isomorphic to $S$ nor to $\pi P_1 = A A / J^2$, then

\[t(M_1) + t(M_{-1}) = et(M_0),\]

Proof. The only $\mathcal{U}$-sequence which is not bipartite is the sequence $0 \to S \to A \to A / S \to 0$. Thus, if $M_0$ is not isomorphic to $S$ nor to $A / J^2$, then both sequences $0 \to M_1 \to P(M_0) \to M_0 \to 0$ and $0 \to M_0 \to P(M_{-1}) \to M_{-1} \to 0$ are bipartite. Let
\textbf{dim} \(M_{-1} = (t, s)\). Then \textbf{dim} \(M_0 = (et - s, t)\) and \textbf{dim} \(M_1 = (e(et - s) - t, et - s)\). Since \(t(M_1) = e(et - s) - t, t(M_0) = et - s, t(M_{-1}) = t\), we see that \(t(M_1) + t(M_{-1}) = eM_0\). \(\square\)

There are the two remaining \(\mathfrak{U}\)-paths \(M_1 \leftarrow M_0 \leftarrow M_{-1}\) with \(M_0 = S\) and \(M_0 = A^eA/J^2\). Both are part of the \(\mathfrak{U}\)-component which contains \(S\). This \(\mathfrak{U}\)-component has been displayed above. Let us show again the relevant part:

\[
\pi Q_1 \xrightarrow{\cdots} S \xrightarrow{\pi P_1} \pi P_2 \xrightarrow{\cdots} \pi A/J \xrightarrow{\cdots} \pi (A^eA/J^2)
\]

and recall that \(t(AJ) = b_1 = e, t(S) = b_0 = 1, t(A^eA/J^2) = b_0 = 1, t(\pi(A^2A/J^2)) = b_1 = e\).

\textbf{Corollary}. Let \(e \geq 2\). Let \(P_{\bullet}\) be an acyclic minimal complex of projective modules and let \(t_i = t(P_i)\). If \(S\) is not an image in \(P_{\bullet}\), then

\[
(*) \quad t_{i-1} + t_{i+1} = et_i
\]

for all \(i \in \mathbb{Z}\). If \(S\) is the image of \(P_0 \rightarrow P_{-1}\), then (*) holds for all \(i \notin \{0, -1\}\) and \(t_{-1} = t_0 = 1, t_{-2} = t_1 = e\). \(\square\)

\textbf{Historical Remark}. For a self-injective algebra \(A\), the \(\mathfrak{U}\)-quiver just depicts the graph of the operation \(\Omega\) on the set of isomorphism classes of indecomposable non-projective modules, thus it visualizes a basic concept which has been used since the early days of homological algebra.

\textbf{A.8. Koszul modules.}

The forthcoming paper [RZ3] will draw the attention to Koszul modules as defined by Herzog and Iyengar [HI], see also [AIS]. If \(A\) is a short local algebra, then an \(A\)-module \(M\) of Loewy length at most 2 is a Koszul module if and only if all the modules \(\Omega^nM\) with \(n \geq 0\) are aligned, see [RZ3].

Since for a self-injective algebra \(A\), any \(A\)-module is Gorenstein-projective, the minimal projective resolutions of all indecomposable non-projective modules are displayed by the \(\mathfrak{U}\)-quiver. It follows:

\textbf{Corollary [Sj, MV2, AIS]}. Let \(A\) be a self-injective short local algebra with \(e \geq 2\). If \(M\) is indecomposable, then \(M\) is Koszul if and only if \(M\) is not preprojective in the sense of Auslander-Smalo (thus not of the form \(\pi P_1, \pi P_2, \ldots\)).

\textbf{Addendum}. Let \(A\) be a self-injective short local algebra. If \(e \geq 2\), then the simple module \(S\) is a Koszul module, and for any module \(M\), there exists \(m \geq 0\) such that \(\Omega^mM\) is Koszul. If \(e = 1\), and \(a = 1\), then the only Koszul modules are the projective modules.

Proof. We can assume that \(M\) is an indecomposable module. First, let \(e \geq 2\) and assume that \(M\) is not Koszul, then \(M = \pi P_m\) for some \(m \geq 1\) and therefore \(\Omega^m(\pi P_m) = S\) is Koszul. If \(e = 1\), and \(a = 1\), then \(A\) is uniserial, thus \(M\) is isomorphic to \(k, AJ\) or \(AA\), and, of course, the modules \(k\) and \(AJ\) are not Koszul. \(\square\)

\textbf{Historical Remark}. The Koszul modules over a self-injective short local algebra have been determined by Sjödin [Sj], Martínez-Villa [MV2] and Avramov-Iyengar-Şega
We hope that our outline of the general setting explains what is considered as a surprising behavior in [AIS] (see the last line in the introduction and the lead text for Theorem 4.6).

Note that already in 1979, Sjödin [Sj] has looked for indecomposable non-projective modules $M$ at the power series $P_M^A = \sum_{n \geq 0} t_n(M)T^n$ (called the Poincaré series of $M$). He showed that for a self-injective short local algebra $A$, the series $P_M^A$ is rational (this follows from the fact that $\Omega^m M$ is Koszul for some $m \geq 0$).

Appendix B. A formula of Avramov-Iyengar-Șega.

B.1. Let $e, a$ be real number. We define recursively the sequence $b_n = b(e, a)_n$ with $n \geq -1$ as follows: $b_{-1} = 0$, $b_0 = 1$ and

\begin{equation}
(*) \quad b_{n+1} = eb_n - ab_{n-1},
\end{equation}

for $n \geq 0$. For the theme of this paper, it is the case that $e, a$ are natural numbers and $a \leq e^2$ which is of interest. Namely, if $A$ is a short local algebra with Hilbert type $(e, a)$, then $e, a$ are natural numbers with $a \leq e^2$, and the recursion rule $(*)$ has popped up in 4.4, when dealing with a module $M$ such that both $M$ and $\Omega^1 M$ are aligned.

As a consequence, we see the relevance of the numbers $b_n = b(e, a)_n$: We have $t_n(S) = b_n$ for all $0 \leq n \leq N$ if and only if the modules $\Omega^n S$ with $0 \leq n < N$ are aligned. As we have mentioned in A.8 (with reference to [RZ3]), the module $S$ is a Koszul module in the sense of [HI] iff all the modules $\Omega^n M$ with $n \geq 0$ are aligned. Thus $S$ is a Koszul module iff $t_n(S) = b_n$ for all $n \geq 0$ (and then $\dim \Omega^n S = (b_n, b_{n-1})$).

The paper [AIS] aimed to provide a concise formula for the numbers $b(e, 1)$ with $e \geq 3$, but the formula presented there was slightly distorted and usually did not even give integers. We are indebted to Avramov, Iyengar and Șega for communicating to us a proper revision and to allow us to include it here.

B.2. Theorem (Avramov, Iyengar, Șega). If $a < \frac{1}{4}e^2$, then for all $n \geq 0$

\[ b(e, a)_n = \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} (e^2 - 4a)^j e^{n-2j}. \]

Proof ([AIS2]): Since we assume that $a < \frac{1}{4}e^2$, the roots of the polynomial $1 - eT + aT^2$ are real numbers, and do not coincide. The roots are

\[ \lambda = \frac{e - q}{2}, \quad \text{and} \quad \rho = \frac{e + q}{2}, \quad \text{where} \quad q = \sqrt{e^2 - 4a} > 0. \]

Starting with the factorization

\[ 1 - eT + aT^2 = (1 - \rho T)(1 - \lambda T), \]
we may look at the power series expansion of the rational function \((1 - eT + aT^2)^{-1}\):

\[
\frac{1}{1 - eT + aT^2} = \frac{1}{(\rho - \lambda)} \left( \frac{\rho}{1 - \rho T} - \frac{\lambda}{1 - \lambda T} \right) = \frac{1}{q} \sum_{n \geq 0} (\rho^{n+1} - \lambda^{n+1}) T^n
\]

Of course, we have

\[
\frac{1}{1 - eT + aT^2} = \sum_{n \geq 0} b(e, a)_n T^n,
\]

therefore

\[
b(e, a)_n = \frac{1}{q} (\rho^{n+1} - \lambda^{n+1}).
\]

The binomial expansions of \(\rho^{n+1}\) and \(\lambda^{n+1}\) yield

\[
\rho^{n+1} - \lambda^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} \frac{1}{2^{n+1}} (e^{n+1-i} q^i - (-1)^i e^{n+1-i} q^i)
\]

\[
= \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} q^{2j+1} e^{n-2j}
\]

Altogether, one gets that

\[
b(e, a)_n = \frac{1}{q} (\rho^{n+1} - \lambda^{n+1}) = \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} q^{2j+1} e^{n-2j},
\]

\[
= \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} (e^2 - 4a)^j e^{n-2j}.
\]

\[\square\]

Note that the formula exhibited above is already of interest in the case \(e = 3\) and \(a = 1\). In this case the numbers \(b_n = b(3, 1)_n\) are just the even-index Fibonacci numbers (see A.1).

References.


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