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Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure

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Abstract. These notes are devoted to a single invariant, the Gabriel-Roiter measure of finite length modules: this invariant was introduced by Gabriel (under the name 'Roiter measure') in 1972 in order to give a combinatorial interpretation of the induction scheme used by Roiter in his 1968 proof of the first Brauer-Thrall conjecture. It is strange that this invariant (and Roiter's proof itself) was forgotten in the meantime. One explanation may be that both Roiter and Gabriel pretend that their considerations are restricted to algebras of bounded representation type which are shown to be of finite representation type, thus restricted to algebras of **finite** representation type. But, as we are going to show, this invariant is of special interest when dealing with algebras of **infinite** representation type! And there may be a second explanation: in the early seventies, it was possible to calculate this invariant only for few examples, whereas nowadays there is a wealth of methods available. Looking at such examples, we are convinced that the Gabriel-Roiter measure has to be considered as a very important invariant and that it can be used to lay the foundation of the representation theory of artin algebras.

The notes are based on lectures given at Hirosaki (2003), Queretaro (2004), Hangzhou and Beijing (2005) looking at some basic questions in the representation theory of artin algebras. The main emphasis in the Hirosaki lectures was to develop a direct approach to the representation theory of artin algebras, using the Gabriel-Roiter measure, independent of Auslander-Reiten theory. Indeed, the Gabriel-Roiter measure should be considered as being elementary: only individual modules are considered (as abelian groups with a set of prescribed endomorphisms), whereas the notions of Auslander-Reiten theory relate from the beginning the given module to the whole module category. In the later lecture series (at Queretaro, Hangzhou and Beijing), some useful connections between the Gabriel-Roiter measure and Auslander-Reiten theory have been included. In the present notes we do not hesitate to mingle the different approaches whenever it seems suitable.

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The presentation centers on some old topics in representation theory, and we hope to convince the reader of the usefulness of the Gabriel-Roiter measure when dealing with results of the following kind:

- The first Brauer-Thrall conjecture established by Roiter and Auslander, which asserts that infinite representation type implies unbounded representation type.
- Any module for an artin algebra of finite representation type is a direct sum of finitely generated modules.
- Auslander's theorem asserting that artin algebras of infinite representation type have indecomposable modules which are not finitely generated.

Of course, it is not surprising that the Gabriel-Roiter measure is useful when dealing with the Brauer-Thrall conjectures, since it was introduced in this context. But it seems that the relationship to the other two topics was not realized before.

This report has three parts. The first part develops general properties of the Gabriel-Roiter measure, the second part deals with what we call the "take-off part" of the module category of an artin algebra, the third one with the "landing part". With respect to the Gabriel-Roiter measure, the category $\operatorname{mod}\Lambda$ for any artin algebra Λ is divided into three different subcategories: the take-off part, the central part and the landing part. Of highest concern should be the central part — however, at present, there is not yet a single result available in print concerning the central part.

We do not touch at all the dual construction, the Gabriel-Roiter "comeasure". Concerning the Gabriel-Roiter comeasure, and the interrelation between the measure and the comeasure, we propose to look at [R5]: the rhombic picture seems to provide a fascinating description of the module category, but also here we have to wait for future research in order to get a full understanding of what is going on.

The core of the notes is self-contained, but for some of the proofs we refer to [R5]. There are several remarks and examples (usually marked by a star: Example*, Remark*) which involve notions which are unexplained. They may be helpful for some (or most) of the readers, but can be skipped in a first reading.

Notation: $\mathbb{N}_1 = \{1, 2, \dots\}$ denotes the positive natural numbers. Given two integers $a \leq b$, let $[a, b]$ be the set of integers z with $a \leq z \leq b$.

0. Preliminaries

Let R be an arbitrary ring, we consider left R-modules and call them just modules. We will assume some basic results from module theory and homological algebra. All our consideration are related to finite length modules: we only deal with modules which are unions of modules of finite length. Recall that finite length modules are modules which are both artinian and noetherian; they have a finite chain of submodules which cannot be refined (such a chain is called a composition series). The Jordan-Hölder theorem asserts that the length of such a chain depends only on the module, it is called the *length* of the module. We denote the length of a module M by $|M|$. A module M is said to be *indecomposable*, provided it is non-zero and cannot be written as a direct sum $M = M_1 \oplus M_2$ of two non-zero modules (a direct sum decomposition $M = M_1 \oplus M_2$ is given by two submodules M_1, M_2 of M such that $M_1 + M_2 = M$ and $M_1 \cap M_2 = 0$.) It is obvious that any finite length module is a finite direct sum of indecomposable modules (and they are again of finite length). The Fitting Lemma tells us that the endomorphism ring of an indecomposable module of finite length is a local ring (the set of non-invertible endomorphisms forms an ideal), and this implies the Krull-Remak-Schmidt theorem: A decomposition of a finite length module into indecomposable modules is unique up to isomorphism.

0.1. Questions. Given a natural number d, are there indecomposable modules of length d? And how many such modules are there? In particular: are there infinitely many isomorphism classes of indecomposable modules of length d? Let $a(d)$ be the number of isomorphism classes of indecomposable modules of length d (recall: we mean left R-modules, the R being fixed; of course we could write $a_R(d)$ in order to specify the ring). Thus, we consider the questions:

- When is $a(d) \neq 0$?
- When is $a(d) = \infty$?

EXAMPLE* 1. Let R be a discrete valuation ring, for example $R = k[[T]]$, the power series ring in one variable, or $R = \mathbb{Z}_{(p)}$ the ring of p-adic integers. The maximal ideal of R is a principal ideal, say generated by the element π. The module $R/R\pi^d$ is indecomposable and of length d, and is, up to isomorphism the only indecomposable module of length d. Thus we see that $a(d) = 1$ for any $d \in \mathbb{N}_1$.

EXAMPLE 2. Let k be a field and let $R = T_n(k)$ be the ring of upper triangular $(n \times n)$-matrices with coefficients in k. Then $a(d) = n - d + 1$, for $1 \leq d \leq n$ and 0 for $n < d$, thus altogether we obtain $\binom{n+1}{2}$ isomorphism classes. Using "quivers" (as we will often do), we can interprete $R = T_n(k)$ as the path algebra of the linearly ordered quiver of type A_n, with vertices $1 \to 2 \to \cdots \to n$, and the indecomposables of length d correspond bijectively to the intervals $[i, i + d - 1]$ with $1 \leq i \leq n - d$.

REMARK*. More generally, Gabriel's theorem asserts that the following quivers are of **finite representation type** (this means: $\sum_d a(d) < \infty$) and that the number of indecomposables is as follows (and in particular independent of the orientation):

type	A_n	D_n	E_6	E_7	E_8
$\sum_d a(d)$	$\binom{n+1}{2}$	$n(n-1)$	36	63	120

The types Δ which occur here are just the simply laced Dynkin diagrams (which occur in Lie theory, but also in many other parts of mathematics); and the numbers below are known as the number of positive roots of Δ: in fact, there is a natural bijection between the indecomposable representations and the positive roots.

EXAMPLE 3. The Kronecker quiver: it has two vertices x, y and two arrows, say going from x to y. Then one knows that $a(n) = 2$ for n odd, and $a(n) \geq 3$ for n even. For n even, the number $a(n)$ depends on the base field k. In case we deal with representations over an infinite field k, then $a(n) = \infty$ for n even. (Here we have an example of what has been called **strongly unbounded representation type**: there are infinitely many natural numbers $d_1 < d_2 < \ldots$ such that $a(d_i) = \infty$ for all i.)

The main problem of representation theory is to find invariants for modules and to describe the isomorphism classes of all the indecomposable modules for which such an invariant takes a fixed value. A typical such invariant is the length of a module: the simple modules are those of length 1 (and there is just a finite number of such modules), the information concerning the indecomposable modules of length 2 is stored in the quiver (in case we deal with a finite dimensional algebra over some algebraically closed field) or the "species" of Λ. Given any invariant γ, as a first question one may look for values *of finite type:* these are those values v such that there are only finitely many isomorphisms classes of indecomposable modules M with $\gamma(M) = v$. The invariant to be discussed here is the Gabriel-Roiter measure.

In most parts of these lectures we will assume that R is an *artin algebra.* This means that the center $z(R)$ of R is artinian and that R is a finitely generated $z(R)$-module. Typical examples are the finite-dimensional k-algebras, where k is a field, but also all finite rings. Some elementary properties of the Gabriel-Roiter measure hold for finite length modules over an arbitrary ring R, or, more generally, for arbitrary finite length categories (a finite length category is an abelian category such than any object is both artinian and noetherian, thus has a composition series; for any ring R, the category of all finite length R-modules is of course a length category).

0.2. The Brauer-Thrall conjectures. As we have mentioned the Gabriel-Roiter measure was introduced by Gabriel in [G] in order to clarify the intricate induction scheme used by Roiter [Ro] in his proof of the first Brauer-Thrall conjecture. Roiter's proof of this conjecture marks the beginning of the new representation theory of finite dimensional algebras. Let us recall the precise statement of both Brauer-Thrall conjectures.

BTh 1. *Bounded representation type (that means $a(d) = 0$ for large d) implies finite representation type.* (Under the assumption that $a(1) < \infty$.)

If $a(d) < \infty$ for all d, then bounded representation type implies finite representation type, trivially. Thus we can reformulate BTh 1 as follows:

BTh 1′ *If $a(d) = \infty$ for some d, then $a(d) \neq 0$ for infinitely many d.*

BTh 2. *Unbounded representation type implies strongly unbounded representation type.* (Under the assumption that we deal with finite dimensional algebras over an infinite field, or more generally, with infinite (and connected) artin algebras.)

The assumption is quite essential, since there are plenty of finite length categories which are of unbounded type, but not of strongly unbounded type. Example 1 above is a typical such example.

BTh 1 is true for R a semiprimary ring: for finite dimensional algebras this was shown by Roiter, for left artinian rings by Auslander. Note: Bounded representation type means that every simple module has a relative projective cover and a relative injective envelope in the category of finite length modules; thus a semiprimary ring of bounded representation type is necessarily left artinian.

Remark*. One should be aware of the V-rings as constructed by Cozzens: here all simple modules are injective, thus also relative projective in the category of finite length modules. These rings satisfy $a(d) = 0$ for $d \geq 2$, however they are not semiprimary. Note that there are such examples with $a(1)$ being finite, as well as examples with $a(1)$ being infinite.

0.3. Smalø [S] has shown the following:

THEOREM. *Let R be an artin algebra with $a(d) = \infty$ for some d, then $a(d') = \infty$ for some $d' > d$.*

We see that in order to show that an artin algebra is of strongly unbounded representation type, it is sufficient to show that $a(d) = \infty$ for some d.

Thus we can reformulate BTh 2 as follows:

BTh 2$'$. *Let R be an infinite artin algebra. If $a(d) \neq 0$ for infinitely many d, then $a(d) = \infty$ for some d.*

We therefore see: The two assertions BTh 1$'$ and BTh 2$'$ are inverse to each other. They claim that there is a strong interrelation between large indecomposables on the one hand, and families of indecomposables of the same length on the other hand.

The investigations presented below (and those in [R5]) yield a lot of insight into BTh 1, but there is not yet any corresponding result concerning BTh 2. But it seems that the Gabriel-Roiter measure should also be helpful in dealing with BTh 2.

I. General Results.

1. The Gabriel-Roiter measure

In the first three sections of the paper, all the modules considered are finite length modules.

1.1. The definition. We define the Gabriel-Roiter measure $\mu(M)$ for modules M of finite length by induction on the length. It will be a rational number. For the zero module 0, let $\mu(0) = 0$. Given a module M of length $n > 0$, we may assume by induction that $\mu(M')$ is already defined for any proper submodule M' of M. Let

$$\mu(M) = \max \mu(M') + \begin{cases} 2^{-n} & \\ & \text{in case } M \text{ is} \\ 0 & \end{cases} \begin{array}{c} \text{indecomposable,} \\ \\ \text{decomposable,} \end{array} .$$

Here, the maximum is taken over all proper submodules M' of M; in order to see that this maximum exists, we have to observe inductively: If M is of length $n \geq 1$, there is a set of natural numbers $I(M) \subseteq [1, n]$ such that $\mu(M) = \sum_{i \in I(M)} 2^{-i}$. (The set $I(M)$ will be analyzed in more detail below.)

1.2. Here are some elementary properties:

PROPERTY 1. *For any non-zero module M, there is an indecomposable submodule M' of M with $\mu(M') = \mu(M)$.*

Thus, when taking the maximum $\mu(M')$ in the definition of $\mu(M)$, it is sufficient to consider only proper submodules M' of M which are indecomposable.

PROPERTY 2. *Let Y be a module and $X \subseteq Y$ a submodule. Then $\mu(X) \leq \mu(Y)$. If Y is indecomposable and X is a proper submodule of Y, then $\mu(X) < \mu(Y)$.*

PROPERTY 3. *For any module M of length $n \geq 1$, we have*

$$\frac{1}{2} \;\leq\; \mu(M) \;\leq \sum_{i=1}^{n} 2^{-i} = \frac{2^n-1}{2^n} \;<1.$$

The lower bound is clear, since M contains a simple submodule S and $\mu(S) = 2^{-1}$. The upper bound follows from the fact that there is the subset $I(M) \subseteq [1,n]$ with $\mu(M) = \sum_{i\in I(M)} 2^{-i}$.

1.3. Some examples:

- If M is a **local** module of length n, then $\mu(M) = \mu(\operatorname{rad} M) + 2^{-n}$.
- If M is a **simple** module, then $\mu(M) = \frac{1}{2}$.
- If M is of **length two,** then $\mu(M)$ is equal to:
  $\frac{1}{2}$ if M is decomposable
  $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ if M is indecomposable,
- If M is of **length three,** there are already four possibilities: $\mu(M)$ may be one of the following numbers:
  $\frac{1}{2}$ if M is semisimple,
  $\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$ if M is indecomposable with socle of length 2,
  $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ if M has an indecomposable direct summand of length 2,
  $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$ if M has simple socle.
- In general, *if M is of length $n \geq 1$, there are 2^{n-1} possibilities for $\mu(M)$,* namely $I(M)$ may be an arbitrary subset of $[1,n]$ containing 1. Proof, by induction, that all these possibilities actually do occur (for suitable finite dimensional algebras): The assertion is clear for $n=1$. Let $n \geq 2$. Let $J \subseteq [1,n]$ be an arbitrary subset containing 1. By induction there is a finite-dimensional algebra R and an R-module N of length $n-1$, such that $I(N) = J \cap [1, n-1]$. If $n \notin J$, let S be a simple module. Then $I(N \oplus S) = J$, see property 5 below. If $n \in J$, let $R' = \begin{bmatrix} R & N \\ 0 & k \end{bmatrix}$, this is again a finite-dimensional k-algebra called the one-point-extension. There is an indecomposable projective R'-module P with radical equal to N, and $I(P) = I(N) \cup \{n\} = J$.

1.4. Gabriel-Roiter submodules. If M is an indecomposable module, and M' is an indecomposable submodule of M with $\mu(M')$ maximal, then we call M' a *Gabriel-Roiter submodule of M* (and the embedding $M' \subset M$ a *Gabriel-Roiter inclusion*). We may reformulate this as follows: If $X \subset Y$ is an inclusion of indecomposable modules, then X is a Gabriel-Roiter submodule of Y if and only if $\mu(X) = \mu(Y) - 2^{-|Y|}$.

PROPERTY 4. *If M is indecomposable and $|M| = n$, then $\mu(M) = \frac{a}{2^n}$ with a an odd natural number, such that $2^{n-1} \leq a < 2^n$. In particular, the Gabriel-Roiter measure of an indecomposable module M determines the length of M.*

Proof, by induction on the length of M. If M is simple, then $\mu(M) = \frac{1}{2}$. Otherwise, take a Gabriel-Roiter submodule M' of M. Let $n' = |M'|$, thus $n' < n$. By induction $\mu(M') = \frac{a'}{2^{n'}}$ with a' odd. Thus $\mu(M) = \mu(M') + 2^{-n} = \frac{a}{2^n}$ with

$a = a'2^{n-n'} + 1$. Since $n - n' \geq 1$, it follows that a is odd. The bounds on a follow from the bounds on $\mu(M)$ mentioned in Property 3.

COROLLARY. *Let X, Y be modules with $\mu(X) = \mu(Y)$. If X is indecomposable, then $|X| \leq |Y|$.*

Proof: According to Property 1, there is an indecomposable submodule Y' of Y with $\mu(Y') = \mu(Y)$. Now X, Y' are indecomposable modules with same Gabriel-Roiter measure, thus they have the same length. Therefore $|X| = |Y'| \leq |Y|$.

1.5. Gabriel-Roiter filtrations. A chain

$$X_1 \subset X_2 \subset \cdots \subset X_t \tag{$*$}$$

will be called a *Gabriel-Roiter filtration* (of X_t), provided X_1 is simple and all the inclusions $X_{i-1} \subset X_i$ are Gabriel-Roiter inclusions, for $2 \leq i \leq t$. Note that all the modules X_i involved in a Gabriel-Roiter filtration are indecomposable.

1.6. There is a another possibility for introducing and calculating the Gabriel-Roiter measure of a module M of length n, concentrating on the set $I(M) \subseteq [1, n]$ such that $\mu(M) = \sum_{i \in I(M)} 2^{-i}$.

This approach deals with the set $\mathcal{U}$ of all possible chains

$$U_\bullet = (U_1 \subset U_2 \subset \cdots \subset U_t)$$

of indecomposable submodules U_i of M. Given such a chain $U_\bullet$, we consider the set $|U_\bullet| = \{|U_i| \mid 1 \leq i \leq t\}$ of the length of these submodules (or finally the corresponding rational number $\sum_{s \in |U_\bullet|} 2^{-s}$). Thus we deal with finite sets of natural numbers, let us denote by $\mathcal{P}_f(\mathbb{N}_1)$ the set of finite subsets of $\mathbb{N}_1$. We want to consider this set as a totally ordered set. We introduce the following order relation: Let $I \neq J$ be elements of $\mathcal{P}_f(\mathbb{N}_1)$. Set $I < J$ provided the smallest element in the symmetric difference $(I \setminus J) \cup (J \setminus I)$ belongs to J (since we assume that $I \neq J$, the set $(I \setminus J) \cup (J \setminus I)$ cannot be empty, thus there is a smallest element). It is easy to see that this relation is transitive, and of course it is anti-symmetric. It follows that $\mathcal{P}_f(\mathbb{N}_1)$ with this ordering is totally ordered. For any module M, let $I(M)$ be the maximum of the sets $|U_\bullet|$ in the totally ordered set $(\mathcal{P}_f(\mathbb{N}_1), \leq)$, where $U_\bullet$ belongs to $\mathcal{U}$. Note that if M is of length n, then all the sets $|U_\bullet|$ are subsets of the set $[1, n] = \{1, 2, ..., n\}$, thus there are only finitely many possible $|U_\bullet|$.

The relationship between $I(M)$ and $\mu(M)$ is established via the equality:

$$\mu(M) = \sum_{i \in I(M)} 2^{-i}.$$

Note that if I, J belong to $\mathcal{P}_f(\mathbb{N}_1)$, then

$$I < J \iff \sum_{i \in I} 2^{-i} < \sum_{j \in J} 2^{-j}.$$

This shows that the order introduced on $\mathcal{P}_f(\mathbb{N}_1)$ and the usual ordering of rational numbers are compatible.

If

$$U_\bullet = (U_1 \subset U_2 \subset \cdots \subset U_t)$$

belongs to $\mathcal{U}$, then $I(U_t) = |U_\bullet|$ is and only if $U_\bullet$ is a Gabriel-Roiter filtration.

In future, when referring to the Gabriel-Roiter measure of module M, we will deal either with $\mu(M)$ or with $I(M)$, whatever is more suitable.

When Gabriel introduced the (Gabriel-)Roiter measure, he used this second approach of dealing with $I(M)$ as an element in the totally ordered set $(\mathcal{P}_f(\mathbb{N}_1), \leq)$. We should stress that the set $I(M)$ may be considered to be more intrinsic than the rational number $\mu(M)$. After all, the reference to the base number 2 used in the definition of $\mu(M)$ is really arbitrary, and 2 could be replaced by any other prime number. On the other hand, to deal with an invariant which takes values in the well-known set of rational numbers seems to be quite satisfactory for psychological reasons – in contrast to an invariant which takes values in the rather strange totally ordered set $(\mathcal{P}_f(\mathbb{N}_1), \leq)$ (by the way, it is more the totally ordering which seems to be horrifying than the set itself). Of course, it is well-known (and easy to see) that any countable totally ordered set can be realized as a subset of the totally ordered set $\mathbb{Q}$, however the embedding used here (mapping I to $\sum_{i\in I} 2^{-i}$) seems to be really manageable.

1.7. Main property (Gabriel). *Let $X, Y_1, \dots, Y_n$ be indecomposable modules, and let $u\colon X \to \bigoplus_i Y_i$ be a monomorphism. Then*
(a) $\mu(X) \leq \max_i \mu(Y_i)$.
(b) *If $\mu(X) = \max_i \mu(Y_i)$, then u is a split monomorphism.*

For the proof we refer to [R5], or see [G].

Property 5. *If X, X' are modules, then*

$$\mu(X \oplus X') = \max(\mu(X), \mu(X')).$$

For the proof we first show: If $X_1, \dots, X_n$ are indecomposable modules, then $\mu(\bigoplus_i X_i) = \max_i \mu(X_i)$. The inequality $\geq$ follows directly from the definition. The inequality $\leq$ follows from Gabriel's main property (a) and the definition. Now, if X, X' are modules, write them as direct sums of indecomposables and apply the first assertion.

Corollary. *If X is an indecomposable submodule of M with $\mu(X) = \mu(M)$, then X is a direct summand.* In particular, if $M \neq 0$, then M has an indecomposable direct summand X with $\mu(X) = \mu(M)$.

Proof: Write $M = \bigoplus M_i$ with indecomposable modules M_i. Property 5 asserts that $\mu(M) = \max \mu(M_i)$. Gabriel's main property (b) shows that the embedding $X \to M = \bigoplus M_i$ splits. The second assertion follows from property 1.

1.8. Before we proceed, let us insert here several characterizations of modules with simple socle. Of special interest seems to be condition (5) and the equivalence of this condition with the other ones is again an immediate consequence of Gabriel's main properties.

Lemma. *Let M be a module of length n. Then the following conditions are equivalent:*
(1) *The socle of M is simple.*
(2) *Any non-zero submodule of M is indecomposable.*
(3) *There exists a composition series of M with all terms indecomposable.*
(4) $I(M) = [1, n]$.

(5) $\mu(M') < \mu(M)$, *for any proper factor module* M' *of* M.

Modules with these properties are often called *uniform* modules.

Proof: The implications (1) $\implies$ (2) $\implies$ (3) are obvious. If there exists a composition series $U_\bullet$ of M with all terms indecomposable, then clearly $|U_\bullet| = [1, n]$ is the maximal possibility, thus $U_\bullet$ is a Gabriel-Roiter filtration and $I(M) = [1, n]$. This shows that (3) implies (4). If we assume (4), and M' is a proper factor module, say of length $n' < n$, then $I(M') \subseteq [1, n'] \subset [1, n] = I(M)$, thus also $I(M') < I(M)$. It remains to show that (5) implies (1). Assume M has two different simple submodules S and S'. Then we can form the factor modules M/S and M/S' and the canonical maps give rise to an embedding $M \to M/S \oplus M/S'$. Main property (a) yields $I(M) \leq \max(I(M/S), I(M/S'))$, but by assumption both $I(M/S) < I(M)$ and $I(M/S') < I(M)$, thus also $\max(I(M/S), I(M/S')) < I(M)$, a contradiction.

We may reformulate the essential implication (5) $\implies$ (1) as follows: If M is a module, then either the socle of M is simple, or else there is an indecomposable factor module M' of M with $\mu(M) \leq \mu(M')$. Indeed we have:

Property 6. *If M is indecomposable, then either M has simple socle, or else there is a factor module M' of M with simple socle such that $\mu(M) < \mu(M')$.*

Proof. If the socle of M is not simple, then write $\operatorname{soc} M = \bigoplus S_i$ with S_i simple. For any i, choose a submodule U_i of M with $S_i \cap U_i = 0$ and such that U_i is maximal with this property. Then M/U_i has simple socle and the canonical maps $M \to M/U_i$ combine to an embedding $M \to \bigoplus_i M/U_i$. Gabriel's main property (a) asserts that $\mu(M) \leq \max \mu(M/U_i)$. We even must have $\mu(M) < \max \mu(M/U_i)$, since otherwise M would be isomorphic to M/U_i for some i by property (b). Thus there is $M' = M/U_i$ with $\mu(M) < \mu(M/U_i)$.

1.9. We consider now maps $f\colon X \to Y$, where X, Y are indecomposable modules with $\mu(X) > \mu(Y)$.

Property 7. *For any indecomposable module M, let M' be the intersection of the kernels of all maps $M \to Z$ with $\mu(Z) < \mu(M)$. Then*
- *$\mu(M/M') < \mu(M)$, and*
- *if U is a submodule of M, such that $\mu(M/U) < \mu(M)$, then $M' \subseteq U$.*

This means that M/M' is the largest factor module of M with Gabriel-Roiter measure smaller than the $\mu(M)$. Of course, $M' \neq 0$, since $\mu(M/M') < \mu(M)$.

Proof: Let M' be the intersection of the kernels of all maps $f\colon M \to Z$ with $\mu(Z) < \mu(M)$. There are finitely many maps $f_i\colon M \to Z_i$, say $1 \leq i \leq t$, such that $\mu(Z_i) < \mu(M)$, and such that the intersection of the kernels of these maps f_i is equal to M' (since M is of finite length, thus artinian). So $(f_1, \ldots, f_t)\colon M/M' \to \bigoplus Z_i$ is injective. But this implies, by Gabriel's main property, that $\mu(M/M') \leq \max(\mu(Z_i)) < \mu(M)$. If $M' = 0$, then we get $\mu(M) < \mu(M)$, a contradiction. Thus $M' \neq 0$. Of course, for any $f\colon M \to X$ with $\mu(X) < \mu(M)$, the kernel of f contains M'. In particular, if M'' is a submodule of M such that $\mu(M/M'') < \mu(M)$, then $M' \subseteq M''$.

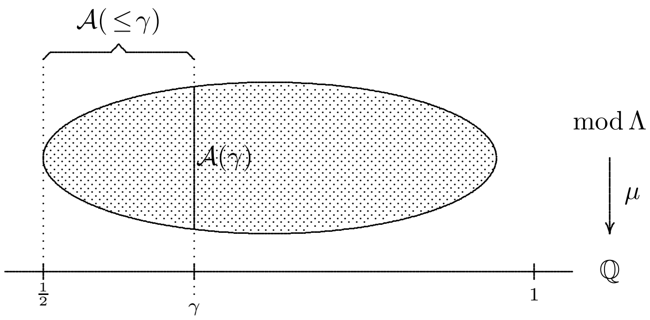
Note that property 7 conversely implies Gabriel's main property (a): Namely, assume that indecomposable modules $X, Y_1, \ldots, Y_n$ and a monomorphism $u\colon X \to$

$\bigoplus Y_i$ are given. Denote by $u_i \colon X \to Y_i$ the composition of u with the corresponding projections. If $\mu(X) > \mu(Y_i)$, then property 7 asserts that $X' \subseteq \operatorname{Ker}(u_i)$. But since u is a monomorphism, $0 \neq X'$ cannot be contained in $\operatorname{Ker}(u_i)$ for all i, thus $\mu(X) \leq \mu(Y_i)$ for some i.

1.10. Definition. For every rational number γ, we denote by $\mathcal{A}(\gamma)$ the class of all indecomposable modules (of finite length) with Gabriel-Roiter measure γ. Similarly, let us denote by $\mathcal{A}(\leq \gamma)$ the class of all indecomposable modules (of finite length) with Gabriel-Roiter measure $\leq \gamma$ (here we may start even with a real number γ). We say that γ is a *Gabriel-Roiter measure* (for Λ, in case there could be a doubt) provided $\mathcal{A}(\gamma)$ is not empty.

There always is a smallest Gabriel-Roiter measure $I_1 = \frac{1}{2}$ (provided Λ is non-zero) and $\mathcal{A}(I_1)$ is the class of all simple modules. This holds for any ring R. In our case of an artin algebra Λ, there also is a largest Gabriel-Roiter measure (again provided Λ is non-zero), namely $I^1 = [1, q]$, where q is the maximal length of an indecomposable injective module. (For a general ring R, there may exist arbitrary large finite length modules with simple socle, as in the case of a discrete valuation ring; then there is no largest Gabriel-Roiter measure.)

We will use the Gabriel-Roiter measure μ in order to visualize the category $\operatorname{mod}\Lambda$.



What really matters is the fact that all the subcategories $\operatorname{add}\mathcal{A}(\leq\gamma)$ are closed under submodules (or, equivalently, that the categories $\mathcal{A}(\leq\gamma)$ are closed under cogeneration). By the way, there are a lot of investigations dealing with subcategories which are closed under submodules — but often one requires in addition that the subcategory is also closed under extensions. In our setting, this latter assumption is satisfied only in trival cases: Namely, as soon as $\gamma \geq \frac{1}{2}$, the subcategory $\operatorname{add}\mathcal{A}(\leq\gamma)$ contains all the simple objects. Thus, if we require that such a subcategory is closed under extension, we will just obtain all the finite length modules!

The set of Gabriel-Roiter measures which occur for a given Λ gives a lot of information about Λ (or better, its Morita equivalence class). But note that different types of rings may yield the same Gabriel-Roiter measures. For example, there are four different quivers of type A_4, they are distinguished by the number of sinks and sources. The linear orientation leads to just one sink and just one source, the corresponding path algebra is serial, and the Gabriel-Roiter measures are

$$\{1\} < \{1,2\} < \{1,2,3\} < \{1,2,3,4\}.$$

The same Gabriel-Roiter measures occur for the second orientation with precisely one sink (and two sources). For the remaining two orientations (with two sinks, and one or two sources), the list of all Gabriel-Roiter measures is

$$\{1\} < \{1,3\} < \{1,2\} < \{1,2,4\} < \{1,2,3\}.$$

2. Gabriel-Roiter inclusions

An indecomposable module M has a Gabriel-Roiter submodule if and only if M is not simple (this is obvious: on the one hand, a simple module has no proper indecomposable submodule; but any module which is neither zero nor simple has indecomposable proper submodules). If M is not simple, M may have several Gabriel-Roiter submodules, but all have the same length.

2.1. The mono-irreducibility of Gabriel-Roiter inclusions. *If $X \subset Y$ is a Gabriel-Roiter inclusion, and if U is a proper submodule of Y which contains X, then the inclusion map $X \to U$ splits.*

We call a monomorphism u *mono-irreducible,* provided first, it does not split, and second, for every factorization $u = u''u'$ with u'' a monomorphism, either u' is a split monomorphism or u'' is an isomorphism. The statement asserts that a Gabriel-Roiter inclusion is mono-irreducible.

Proof: We assume $X \subseteq U \subset Y$ and that X is a Gabriel-Roiter submodule of Y. We decompose $U = \bigoplus U_i$ such that all U_i are indecomposable. Now $\mu(U_i) < \mu(Y)$, since $U_i \subseteq U \subset Y$ is a proper submodule of Y and Y is indecomposable. Since X is a Gabriel-Roiter submodule of Y, it follows that $\mu(U_i) \le \mu(X)$, thus $\max_i \mu(U_i) \le \mu(X)$. However X is a submodule of $\bigoplus_i U_i$, thus by part (a) of Gabriel's main property we see that also $\mu(X) \le \max_i \mu(U_i)$. Thus $\mu(X) = \max_i \mu(U_i)$, and therefore the inclusion $X \to U$ splits, according to part (b) of Gabriel's main property.

Remark*. In order to explain the terminology, one should compare this definition with that of an irreducible map in the sense of Auslander and Reiten: there, one considers arbitrary factorizations, not just factorizations using monomorphisms.

Irreducible monomorphisms are mono-irreducible; however, there are obvious mono-irreducible maps which are not irreducible: for example consider the path algebra of the quiver:

$$\begin{tikzcd} a & \\ & c \arrow[ul] \arrow[dl] \\ b & \end{tikzcd}$$

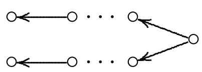
The inclusion map of the simple module $S(a)$ into its injective envelope is mono-irreducible, however it factorizes through the projective cover of $S(c)$, thus it is not irreducible. Also, there is the following phenomenon: Given indecomposable modules X, Y, there may be irreducible monomorphisms $f\colon X \to Y$ and also a monomorphism $g\colon X \to Y$ which is not even mono-irreducible. For example, take the hereditary algebra $\widetilde{A}_{21}$, let S be simple projective and P the indecomposable projective of length 4. Then $\mathrm{Hom}(S, P)$ is 2-dimensional and the non-zero maps are monomorphisms. Thus the monomorphisms (up tp scalar multiplication) $S \to P$ form a projective line; one of these equivalence classes is not mono-irreducible (it factorizes through an indecomposable length 2 submodule), the remaining ones are irreducible, thus mono-irreducible.

2.2. Lemma. *Let X be an indecomposable module, let $X \subset Y$ be mono-irreducible. Then Y/X is indecomposable.*

Proof: First, we show the indecomposability of Y/X. If Y/X is decomposable, there are proper submodules X', X'' of Y containing X such that $X' + X'' = Y$ and $X' \cap X'' = X$. But the mono-irreducibility implies that the inclusions $X \to X'$ and $X \to X''$ split: there are submodules U', U'' with $X' = X \oplus U'$ and $X'' = X \oplus U''$. This implies $Y = X \oplus U' \oplus U''$, thus the inclusion $X \to Y$ also splits, a contradiction.

Corollary. *If Y is indecomposable and not simple, then there exists an exact sequence*
$$0 \to X \to Y \to Z \to 0$$
with X and Z indecomposable (just take as X a Gabriel-Roiter submodule of Y).

One may wonder about the possible modules which occur as factor modules Y/X, where Y is indecomposable and X is a Gabriel-Roiter submodule. For the path algebra of a quiver of type A_n, all these factors are serial and of length at most $\frac{n+1}{2}$. A factor of length $\frac{n+1}{2}$ occurs for the sincere representation of the quiver of type A_n (n odd) with a unique source



and with arms of equal length. More details about such factor modules Y/X will be found in [Ch].

2.3. If X is a Gabriel-Roiter submodule of Y, then we just have seen that the factor module Y/X is indecomposable. However something stronger holds: the cokernel of **any** monomorphism $X \to Y$ is indecomposable. The reason is trivial: the image of any monomorphism $X \to Y$ is a Gabriel-Roiter submodule again. This shows the first of the following two assertions.

Theorem. *Let X be a Gabriel-Roiter submodule of Y. Then the cokernel of any monomorphism $X \to Y$ is indecomposable and the set of non-monomorphisms $X \to Y$ is closed under addition.*

Let us denote by $\mathrm{Sing}(X, Y)$ the set of non-injective maps $X \to Y$. Then we claim that $\mathrm{Sing}(X, Y)$ *is a subgroup of* $\mathrm{Hom}(X, Y)$. For a short direct proof of the last assertion, we refer to [R6]. Here we will show a stronger assertion:

Proposition. *Let X be a Gabriel-Roiter submodule of Y. Let X/X' be the largest factor module of X with $\mu(X/X') < \mu(X)$. Then X' is contained in the kernel of any map in* $\mathrm{Sing}(X, Y)$.

Proof of Proposition: Let $f \in \mathrm{Sing}(X, Y)$. Then $|f(X)| < |X| < |Y|$ shows that $f(X)$ is a proper submodule of Y. By the definition of a Gabriel-Roiter submodule, we see that $\mu(f(X)) \le \mu(X)$. However, X and $f(X)$ have different lengths, thus $\mu(f(X)) = \mu(X)$ is impossible. Therefore $\mu(f(X)) < \mu(X)$. Let U be the kernel of f, thus $f(X)$ is isomorphic to X/U and $\mu(X/U) = \mu(f(X)) < \mu(X)$. The definition of X' implies that $X' \subseteq U$.

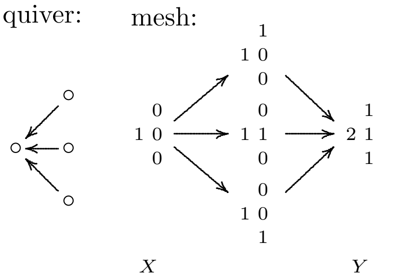
Proof of Theorem: If f, f' belong to $\mathrm{Sing}(X, Y)$, then X' is contained both in the kernel of f and in the kernel of f', thus in the kernel of $f + f'$. This shows that $\mathrm{Sing}(X, Y)$ is closed under addition. (Of course, if f is in $\mathrm{Sing}(X, Y)$, also $-f$ is in $\mathrm{Sing}(X, Y)$, thus $\mathrm{Sing}(X, Y)$ is a subgroup of $\mathrm{Hom}(X, Y)$.)

The fact that $\mathrm{Sing}(X, Y)$ is closed under addition is very helpful when making calculations. In order to look for the possible Gabriel-Roiter submodules of a given module Y, it is enough to deal with generating sets for the various groups $\mathrm{Hom}(X, Y)$. Thus, *if we deal with an artin algebra Λ of finite representation type, and if $X \subset Y$ is a Gabriel-Roiter inclusion, then there is a Gabriel-Roiter inclusion $u\colon X \to Y$ which is a composition of irreducible maps between indecomposable modules,* thus which is given by a path in the Auslander-Reiten quiver of Λ.

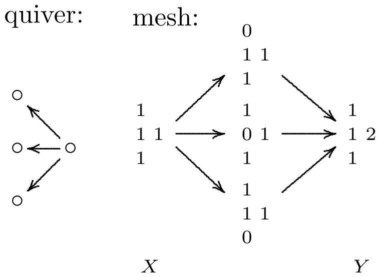
2.4. Let us stress that the properties encountered for the pair (X, Y), where X is a Gabriel-Roiter submodule of Y, are very special. Let us provide two typical examples of the usual behavior of pairs (X, Y), where X is a submodule of a module Y. We consider quivers of type D_4 (and many more examples can be produced by looking at quivers of type E_6, E_7, or E_8).

Both examples are rather similar and are obtained from each other using reflection functors (thus tilting functors). In both cases one deals with a mesh with three middle terms:

Example 1



Example 2



Example 1. Here we present a pair (X, Y) of modules with a family of embeddings $X \to Y$ such that for some of these embeddings, the cokernel will be indecomposable, for some other embeddings, the cokernel will be decomposable. Take the subspace orientation (one sink, three sources), let X be simple projective, and Y the indecomposable module of maximal length. Then $\dim \mathrm{Hom}(X, Y) = 2$. Any generic map $X \to Y$ will have an indecomposable cokernel (this is then the indecomposable injective module of length 4), but note that such homomorphism do not exist in case the base field k has only two elements. There are three different kinds of monomorphisms with decomposable cokernel: here the cokernel is the direct sum of a simple injective module and an indecomposable module of length 3.

Example 2. A pair (X, Y) of modules with a monomorphism $\phi\colon X \to Y$ which can be written as a sum $\phi = \phi_1 + \phi_2$ such that neither ϕ_1 nor ϕ_2 is a monomorphism. This time take D_4 with the factor space orientation (one source, three sinks). Again let Y be the indecomposable module of maximal length, and X the indecomposable projective module of length 4. Again, we have $\dim \mathrm{Hom}(X, Y) = 2$. Here, any generic map $X \to Y$ is a monomorphism (but such a monomorphism does not exist in case the base field k has only two elements). There are three different kinds of homomorphisms $X \to Y$ which are not monomorphisms: the corresponding images are the three indecomposable modules of length 3.

The examples show another (but related) complication when dealing with monomorphisms: *field extensions may create new types of monomorphisms.* Consider a field k' of characteristic 2, and let $k = \mathbb{Z}/2\mathbb{Z}$ be the corresponding prime field. When dealing with the subspace orientation, we see that for the prime field k, any monomorphism $X \to Y$ factorizes through an indecomposable module of length 2, whereas for $k' \neq k$, there are monomorphisms without such a factorization. On the other hand, if we look at the factorspace orientation and if k is the prime field, all the maps $X \to Y$ have non-zero kernel, thus belong to $\mathrm{Sing}(X,Y)$. This shows that for k the prime field $\mathrm{Sing}(X,Y) = \mathrm{Hom}(X,Y)$ and thus $\mathrm{Sing}(X,Y)$ is closed under addition, whereas for $k' \neq k$, the set $\mathrm{Sing}(X,Y)$ is a proper subset of $\mathrm{Hom}(X,Y)$. The subset $\mathrm{Sing}(X,Y)$ of $\mathrm{Hom}(X,Y)$ generates $\mathrm{Hom}(X,Y)$ as a vector space, thus it is not closed under addition.

Let us add a further example of a pair of modules X, Y with monomorphisms $f_1, f_2 \colon X \to Y$ such that the cokernel of f_1 is indecomposable whereas the cokernel of f_2 is not: let Λ be the path algebra of the Kronecker quiver and let X, Y be preprojective Λ-modules of length 1 and 5, respectively.

2.5. Given an indecomposable module X, we have seen in property 7 how to construct the largest factor module X/X' of X with Gabriel-Roiter measure smaller than $\mu(X)$.

Proposition. *Let $X_1 \subset X_2 \subset \cdots \subset X_t = X$ be a Gabriel-Roiter filtration. Let X/X' be the largest factor module of X with Gabriel-Roiter measure smaller than $\mu(X)$. Then $X_1 \subseteq X'$.*

Proof: Since $X' \neq 0$, there exists s minimal with $X_s \cap X' \neq 0$. If $s = 1$, then $X_1 \cap X' \neq 0$ implies $X_1 \subseteq X'$, since X_1 is simple. And this is what we want to show.

Now assume $s > 1$. Let $U = X_s \cap X'$. Then U is a submodule of X_s, and we can consider $Y = X_s/U$. Of course $Y = X_s/(X_s \cap X')$ is isomorphic to $(X_s + X')/X'$, thus it can be embedded into X/X' and therefore $\mu(Y) \leq \mu(X/X') < \mu(X)$.

By the minimality of s, we have $X_{s-1} \cap U = X_{s-1} \cap X' = 0$, thus X_{s-1} is mapped under the projection map $X_s \to Y$ injectively into Y, thus $\mu(X_{s-1}) \leq \mu(Y)$, by Gabriel's property (a). We even have $\mu(X_{s-1}) < \mu(Y)$, since otherwise the map $X_{s-1} \to X_s \to Y$ would be a split monomorphism by Gabriel's property (b), and then also the map $X_{s-1} \to X_s$ would be a split monomorphism, which is impossible. Altogether we have

$$\mu(X_{s-1}) < \mu(Y) < \mu(X),$$

thus also

$$I(X_{s-1}) < I(Y) < I(X) = I(X_{s-1}) \cup \{|X_s|, \dots, |X_t|\}$$

in the totally ordered set $(\mathcal{P}_f(\mathbb{N}_1), \leq)$. We can apply the following lemma for $I = I(X_{s-1}), J = I(Y)$ and $n_i = |X_{s+i}|$ for $0 \leq i \leq r = t - s$ and conclude that $J = I(Y)$ is not contained in the interval $[1, n_0]$. However, $Y = X_s/U$ is a proper factor module of X_s, thus $|Y| < n_0 = |X_s|$, a contradiction.

Lemma. *Let $n_0 < n_1 < \cdots < n_r$ be natural numbers. Let I, J be finite subsets of $\mathbb{N}_1$, with $I \subseteq [1, n_0 - 1]$. If*

$$I < J < I \cup \{n_0, \dots, n_r\}$$

in the total ordering $(\mathcal{P}_f(\mathbb{N}_1), \leq)$, then $J \not\subseteq [1, n_0]$.

Proof: Write $I' = I \cup \{n_0, \dots, n_r\}$. Consider the smallest element l in the symmetric difference $(J \setminus I') \cup (I' \setminus J)$. Since $J < I'$, we know that l belongs to $I' = I \cup \{n_0, \dots, n_r\}$. Assume l belongs to I. Then l belongs to $I \setminus J$, thus to the symmetric difference $(J \setminus I) \cup (I \setminus J)$, and is the smallest element in this set, but this means $J < I$, a contradiction. Thus $l = n_i$ for some $0 \le i \le r$. But then the elements of I all belong to J (since l is the smallest element in $I' \setminus J$). Since $I < J$, there is $l' \in J \setminus I'$. Since n_i is the smallest element in $(J \setminus I') \cup (I' \setminus J)$, we see that $n_0 \le n_i < l'$. This completes the proof.

2.6. Corollary 1. *Let $X_1 \subset X_2 \subset \cdots \subset X_t = X$ be a Gabriel-Roiter filtration. Let Z be a module with $\mu(Z) \le \mu(X)$ and let $f\colon X \to Z$ be a homomorphism. Then either f is a split monomorphism or else $f(X_1) = 0$.*

Proof. We can assume that Z is indecomposable, thus we have to show that f is an isomorphism or vanishes on X_1.

Consider first the case $\mu(Z) < \mu(X)$. According to property 7, the kernel of $f\colon X \to Z$ contains X', thus X_1. Second, let $\mu(Z) = \mu(X)$. If f is a monomorphism, then f is an isomorphism since X and Z have the same length. If f is not a monomorphism, then we consider instead the corresponding map $X \to f(X)$. Since $f(X)$ is a proper submodule of Z, we have $\mu(f(X)) < \mu(Z) = \mu(X)$, thus we are back in the first case and see that $f(X_1) = 0$.

Corollary 2. *Let $X_1 \subset X_2 \subset \cdots \subset X_t = X$ be a Gabriel-Roiter filtration. Let U be a nonzero submodule of X with $X_1 \cap U = 0$. Then $\mu(X) < \mu(X/U)$.*

Proof: Assume for the contrary $\mu(X/U) \le \mu(X)$. Since X is indecomposable and $|X/U| < |X|$, we have even $\mu(X/U) < \mu(X)$. According to Corollary 1, the projection map $X \to X/U$ vanishes on X_1, but this contradicts the assumption $X_1 \cap U = 0$.

3. Auslander-Reiten theory and the successor lemma

Assume from now on that $R = \Lambda$ is an artin algebra. This implies that the Auslander-Reiten translations τ (dual of transpose) and τ^{-1} (transpose of dual) are defined, see for example [ARS].

3.1. Lemma. *Let X be an indecomposable module, let $X \subset Y$ be mono-irreducible. Then Y/X is a factor module of $\tau^{-1}X$.*

Proof: We look at the exact sequence $0 \to X \to Y \to Y/X \to 0$. Since this sequence does not split, X cannot be injective. Since X is indecomposable, there is an Auslander-Reiten sequence starting with X, and we compare the two sequences: we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & \tau^{-1}X \arrow[r] \arrow[d, "f'"] & 0 \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Y/X \arrow[r] & 0 \end{tikzcd}$$

The existence of the map f so that the left square commutes is due to the fact that the upper sequence is an Auslander-Reiten sequence and that $X \to Y$ is not split mono. The commuting left square gives rise to the map f' so that the right square commutes.

We claim that f is surjective: otherwise the image of f would be a proper submodule of Y, but then fu would yield a split monomorphism $X \to f(Y)$, according to the mono-irreducibility of the inclusion map $X \to Y$. In particular, u itself would be a split monomorphism, this is impossible. With f also f' is surjective.

Corollary. *If $X \to Y$ is a Gabriel-Roiter inclusion, then $|Y| \le pq|X|$, where p is the maximal length of an indecomposable projective module and q is the maximal length of an indecomposable injective module.*

Proof: One knows that $|\tau^{-1}X| \le (pq-1)|X|$. Namely, take a minimal injective presentation $0 \to X \to Q_0 \to Q_1$ of X, then the socle of Q_0 is of length at most $|X|$, thus Q_0 is of length at most $q|X|$ and Q_0/X is of length at most $(q-1)|X|$. This shows that the socle of Q_1 is of length at most $(q-1)|X|$. Thus the length of the top of the projective cover P_0 of $\tau^{-1}X$ is at most $(q-1)|X|$ and therefore P_0 is of length at most $p(q-1)|X| \le (pq-1)|X|$. Consequently, $|\tau^{-1}X| \le (pq-1)|X|$. Of course, $|Y| = |X| + |Y/X| \le pq|X|$.

Remark*. Such a bound reflects an interesting finiteness condition. Example: Take a bimodule algebra with an indecomposable projective module P of length 2 and an indecomposable injective module of infinite length, for example $\begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}$. Then there is an indecomposable module M_n with top length n and socle length $n-1$, for any $n \ge 1$. Any Gabriel-Roiter submodule of M_n is of length 2, thus we have Gabriel-Roiter measures $(1,2,n)$ for any n. (A factor module M_n/P is a M_{n-1}, for $n \ge 3$.)

3.2. Successor Lemma. *Any Gabriel-Roiter measure I different from I^1 has a direct successor I'. If $I \subseteq [1,n]$, then $I' \subseteq [1,pqn]$, where p is the maximal length of an indecomposable projective module and q is the maximal length of an indecomposable injective module.*

Here, we call I' a *direct successor* of I provided, first, $I < I'$ and second, there does not exist a Gabriel-Roiter measure I'' with $I < I'' < I'$.

Proof: Let $I \ne I^1$ be a Gabriel-Roiter measure, let $I \subseteq [1,n]$, with n minimal (thus the modules in $\mathcal{A}(I)$ are of length n).

We claim that for any Gabriel-Roiter measure J with $I < J$, there is a Gabriel-Roiter measure J' with $I < J' \le J$ and such that $J' \subseteq [1,pqn]$.

Let t be minimal in $(I \setminus J) \cup (J \setminus I)$. Since $I < J$, we know that $t \in J$. Let $J' = J \cap [1,t]$. Then $I < J' \le J$. Since J is a Gabriel-Roiter measure for Λ and $J' == J \cap [1,t]$ for some t, also J' is a Gabriel-Roiter measure for Λ.

We claim that $t \le pqn$. If $t < n$, then clearly $t \le pqn$. Thus we can assume $t > n$. In this case, let Y be an indecomposable module with Gabriel-Roiter measure J', and let X be a Gabriel-Roiter submodule of Y. Then we have a Gabriel-Roiter inclusion $X \to Y$ with $|X| = n$ and $|Y| = t$. As we have seen above, this implies that $t \le pqn$.

But there are only finitely many possible Gabriel-Roiter measures in the set $[1,pqn]$. This shows that for any J with $I < J$ there is a Gabriel-Roiter measure J' with $I < J' \le J$ and such that $J' \subseteq [1,pqn]$.

In particular, for $J = I^1$ there is a Gabriel-Roiter measure $J' \subseteq [1,pqn]$ with $I < J'$. Now take the smallest Gabriel-Roiter measure $I' \subseteq [1,pqn]$ with $I < I'$.

Then there cannot be any other Gabriel-Roiter measure J between I and I', since otherwise we would obtain a Gabriel-Roiter measure J' with $I < J' \leq J < I'$, in contrast to the minimality of I'.

This completes the proof.

3.3. Remarks*. (a) We may draw a quiver as follows: its vertices are the Gabriel-Roiter measures, and we draw an arrow $\gamma \to \gamma'$ in case γ' is a direct successor of γ. Then the connectivity component containing I^1 is either finite or of the form $-\mathbb{N}$, the remaining connectivity components of this quiver are of the form $\mathbb{N}$ or $\mathbb{Z}$ (here we consider $\mathbb{Z}$ as a quiver with vertices the integers and with arrows $z \to z+1$, and we consider $\mathbb{N}$ and $-\mathbb{N} = \{z \mid -z \in \mathbb{N}\}$ as the corresponding full subquivers). The connectivity component containing I^1 is finite only in case there are only finitely many Gabriel-Roiter measures, see the discussion in part III.

(b) In case we deal with an arbitrary ring (not an artin algebra), or an arbitrary length category, then I_1 may not have a direct successor: Consider, for example, one of the following triangular matrix rings $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$ or $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}$, or the category of representations of the quiver with countable many vertices, say labeled by the natural numbers $\mathbb{N}_0$ and with one arrow $0 \to i$ for all $i \in \mathbb{N}_1$ (so that 0 is a source, whereas all the other vertices are sinks). Then one of the indecomposable projective modules is not of finite length; its non-zero factor modules of finite length are all local and have Loewy length at most 2, and we obtain in this way modules in $\mathcal{A}(\{1, r\})$, with r arbitrarily large. Thus we get a decreasing sequence of Gabriel-Roiter measures

$$... < \{1,5\} < \{1,4\} < \{1,3\} < \{1,2\}$$

and for any Gabriel-Roiter measure $I \neq I^1$, there is some $r \geq 2$ with $\{1, r\} \subseteq I$, then also $\{1, r\} < I$.

(c) There is no corresponding "predecessor lemma": Consider, for example, the Kronecker quiver. The measure $\{1, 2\}$ has no immediate predecessor.

4. Relative Σ-injectivity and direct sums of finite length modules

We are going to consider arbitrary, not necessarily finitely generated modules. Let M be an indecomposable module of finite length. Gabriel's main property (b) asserts that M is relative injective in $\operatorname{add}\mathcal{A}(\gamma)$. We want to extend this relative injectivity property to arbitrary modules.

4.1. For any real number γ, consider the full subcategory

$$\mathcal{D}(\gamma) = \varinjlim \operatorname{add}\mathcal{A}(\leq \gamma).$$

By definition, a module M belongs to $\mathcal{D}(\gamma)$ if and only if any indecomposable submodule M' of M of finite length satisfies $\mu(M') \leq \gamma$.

Note that the subcategories $\mathcal{D}(\gamma)$ are definable subcategories in the sense of [Cr] (this means that $\mathcal{D}(\gamma)$ is closed under products, direct limits and pure submodules). We only have to show the following:

Lemma. *$\mathcal{D}(\gamma)$ is closed under products.*

Proof: Let M_i be modules in $\mathcal{D}(\gamma)$, and consider $M = \prod_i M_i$. We have to show that any indecomposable submodule M' of M of finite length has Gabriel-Roiter measure at most γ. But if M' is a submodule of $M = \prod_{i \in S} M_i$, then M is also a submodule of some finite sum $\bigoplus_{i\in S'} M_i$, where S' is a finite subset of S (see Lemma 9.2). Thus Gabriel's main property (a) yields $\mu(M') \le \gamma$.

4.2. Theorem on relative Σ-injectivity. *Let X be an indecomposable module of finite length with $\mu(X) = \gamma$. Then X is relative Σ-injective in $\mathcal{D}(\gamma)$.*

This means: Let M be a module in $\mathcal{D}(\gamma)$. If M' is a submodule of M which is a direct sum of copies of X, then M' is a direct summand of M.

Proof: Here we need some knowledge about pure submodules, see [JL] and also [R2]. We show that M' is a pure submodule of M. Since X and therefore M' is endofinite, we know that M' is pure injective. Altogether this means that M' is a direct summand.

In order to show that M' is a pure submodule of M, consider a submodule M'' of M which contains M' and such that M''/M' is of finite length. We have to show that the embedding $M' \to M''$ splits. Take a finite length submodule U of M'' such that $M'+U = M''$. Since $M''\cap U$ is of finite length, there is a direct decomposition $M' = N\oplus N'$ such that N is a finite direct sum of copies of X and such that $M'\cap U$ is contained in N. We claim that M'' is the direct sum of $U+N$ and N'. Clearly, $U+N+N' = U+M' = M''$. In order to see that $(U+N)\cap N' = 0$, take elements $u \in U, x\in N$ such that $u+x \in N'$. Then $u = -x+(u+x)$ belongs to $U\cap M' \subseteq N$, thus in the decomposition $u = -x + (u + x)$ with $-x \in N$ and $u + x \in N'$ the second summand has to be zero. But this assertion, $u + x = 0$, is what we wanted to show.

Now we only have to observe that $U + N$ as a finite length submodule of M belongs to $\operatorname{add}\mathcal{A}(\le\gamma)$. Since N is relative injective in $\operatorname{add}\mathcal{A}(\le\gamma)$, we obtain a direct decomposition $U+N = U'\oplus N$, and now $M'' = (U+N)\oplus N' = U'\oplus N\oplus N'$.

Warning. The theorem is about direct sums of copies of a fixed module in $\mathcal{A}(\gamma)$, not about direct sums of modules in $\mathcal{A}(\gamma)$. Indeed, in case $\mathcal{A}(\gamma)$ is infinite, a direct sum of modules in $\mathcal{A}(\gamma)$ does not have to be relative injective in $\mathcal{D}(\gamma)$. As an example, take the Kronecker algebra $\Lambda = \begin{bmatrix} k & k^2 \\ 0 & k\end{bmatrix}$ with k an infinite field and $\gamma = \frac{3}{4}$, thus $\mathcal{A}(\gamma)$ consists of the indecomposable Λ-modules of length 2. Since k is infinite, there are infinitely many isomorphism classes of module M_i in $\mathcal{A}(\gamma)$. Then, according to the lemma above, $\prod M_i$ belongs to $\mathcal{D}(\gamma)$. However, the submodule $\bigoplus M_i$ of $\prod M_i$ is not a direct summand, see [R4].

As a consequence, there is the following result:

4.3. Direct sum theorem. *Assume M is a module with only finitely many isomorphism classes of indecomposable submodules of finite length. Then M is a direct sum of finite length modules.*

Of course, the conclusion may be strengthened by saying that M is a direct sum of indecomposable modules of finite length: just write the finite length modules as direct sums of indecomposable modules. Note that the proof gives a precise recipe in which order one may split off direct summands: one may use the ordering of the Gabriel-Roiter measures of the indecomposable submodules of M of finite length, starting with the maximal Gabriel-Roiter measure.

Proof: Let $M_1, \dots, M_t$ be the representatives of all the isomorphism classes of indecomposable submodules of M of finite length. The proof is by induction on t, the case $t = 0$ being trivial. Thus, let $t \geq 1$ and let $\mu(M_i) = \gamma_i$. We can assume that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_t$. By definition of $\mathcal{D}(\gamma_t)$, we know that M belongs to $\mathcal{D}(\gamma_t)$. Consider submodules of M which are direct sums of copies of M_t. Using Zorn's lemma, there exists such a submodule M' such that $M' \cap U \neq 0$ for any submodule U isomorphic to M_t. Since we know that M_t is relative Σ-injective in $\mathcal{D}(\gamma_t)$, we see that M' is a direct summand of M, say $M = M' \oplus M''$. Now consider M'': any indecomposable submodule of M'' of finite length is a submodule of M, thus of the form $M_1, \ldots, M_t$, however M_t does not occur, by the maximality property of M'. This shows that any indecomposable submodule of M of finite length is of the form $M_1, \ldots, M_{t-1}$, thus by induction M'' is a direct sum of finite length modules.

4.4. We recover here the result that for algebras of finite representation type, any module M is a direct sum of finite length modules ([RT], [T], and also [A1]), with a precise recipe in which order one should split off direct summands.

Recall that this result in the special case of serial algebras is due to Nakayama [N]. In this case one first splits off the indecomposables of longest possible length: if the module is faithful (which we can assume), then the indecomposables of longest possible length q are injective, thus indeed such submodules (and even submodules which are arbitrary direct sums of such modules) are direct summands. For a serial algebra, all the indecomposable modules are uniform, so that the Gabriel-Roiter measures are

$$[1,1] < [1,2] < [1,3] < \cdots < [1,q].$$

Thus the ordering of the indecomposables according to their length coincides with the ordering using the Gabriel-Roiter measure.

Remark. We have shown that if M is a module with only finitely many isomorphism classes of indecomposable submodules of finite length, then M is a direct sum of copies of finitely many indecomposable modules of finite length. We should stress, that this conclusion cannot be reversed: there are direct sums of copies of finitely many indecomposable modules N_i of finite length which have indecomposable submodules of arbitrarily large length, even if any of the modules N_i has only finitely many submodules. For example, consider any finite artin algebra Λ of infinite representation type. Let $N_1, \ldots, N_t$ be the indecomposable injective Λ-modules. Since we consider a finite artin algebra, the modules N_i are finite sets, thus they have only finitely many submodules. On the other hand, let M be the direct sum of countably many copies of the various N_i. Since $\bigoplus N_i$ is an injective cogenerator, any finite length module can be embedded into M, thus M has infinitely many isomorphism classes of indecomposable submodules of finite length.

II. The Take-Off Part.

Recall that for Λ non-zero, $I_1 = \frac{1}{2}$ is the smallest Gabriel-Roiter measure, wheras $I^1 = \sum_{i=1}^{q} 2^{-i}$ (with q the maximal length of an indecomposable injective Λ-module) is the largest Gabriel-Roiter measure. The successor lemma asserts that any Gabriel-Roiter measure $\gamma \neq I^1$ has a direct successor. In particular, starting with I_1, we may define I_{i+1} inductively as the direct successor of I_i, provided $I_i \neq I^1$. Thus there are either only finitely many Gabriel-Roiter measures for Λ, which then are labeled $I_1 < I_2 < \cdots < I_m$ and $I_m = I^1$, or else there is a countable sequence of Gabriel-Roiter measures

$$I_1 < I_2 < \cdots < I_i < \dots$$

such that any other Gabriel-Roiter measure γ satisfies $I_i < \gamma$ for all i. In both cases, we call the measures I_i displayed in this way the *take-off measures.* An indecomposable module M of finite length will be said to be a *take-off module* provided $\mu(M)$ is a take-off measure.

Let us describe I_2 explicitly. Of course, I_2 only exists in case Λ is not semisimple and then obviously $I_2 = \{1, s\}$ with s as large as possible. Now an indecomposable module M with Gabriel-Roiter measure of the form $\{1, r\}$ for some $r \geq 2$ is local and of Loewy length 2: it has a unique maximal submodule M' and M' is semisimple. Thus I_2 yields the information about the largest possible local modules with Loewy length 2.

Note that $I_2 = I^1$ iff Λ is a serial algebra with radical square zero., and then $I_2 = I^1 = \{1, 2\}$. (In general, Λ is left serial iff $I_2 = \{1, 2\}$.) If $I_2 < I^1$, then I_3 is either $\{1, r, s\}$ with $s > r$ as large as possible, or else $I_3 = \{1, r-1\}$. By the way, an explicit description of the take-off measures I_t with large t seems to be awkward.

5. The finiteness theorem for take-off measures

5.1. Finiteness theorem. *For any take-off measure I_i, there are only finitely many isomorphism classes of indecomposable modules M with $\mu(M) = I_i$.*

We will discuss several proofs of the finiteness theorem. But before we do this, let us mention the following consequences:

Corollary 1. *If Λ is of infinite representation type, there are infinitely many take-off measures.*

Proof: If there are only finitely many take-off measures, say $I_1 < I_2 < \cdots < I_t$, then these are all the possible measures. Thus there are only finitely many isomorphism classes of indecomposable modules.

Note that the corollary provides a proof of BTh 1. Actually, it strengthens the assertion of the first Brauer-Thrall conjecture. In contrast to the assertion of the first Brauer-Thrall conjecture itself, the statement is meaningfull even in case Λ is a finite ring (i.e. a ring with finitely many elements).

Corollary 2. *For any natural number n, there are only finitely many isomorphism classes of take-off modules of length n.*

Proof: If M is an indecomposable module of length n, then $I(M) \subseteq [1, n]$, thus there are only finitely many possible Gabriel-Roiter measures. The finiteness theorem asserts that for any fixed take-off measure γ, there are only finitely many isomorphism classes of modules with measure γ.

5.2. Three different proofs for the finiteness theorem are available.

First proof. The first one is essentially due to Roiter, and has been exhibited in [R5]. It uses the very interesting coamalgamation lemma.

Second proof. The argumentation is parallel to the Auslander-Yamagata proof of Brauer-Thrall I and uses the Harada-Sai lemma. Let us give an outline of this proof.

We need: $\operatorname{add}\mathcal{A}(\leq\gamma)$ has source maps. This comes from the fact that $\operatorname{add}\mathcal{A}(\gamma)$ is closed under cogeneration. Actually, we have:

Lemma. *The inclusion functor* $\operatorname{add}\mathcal{A}(\leq\gamma)\to\operatorname{mod}\Lambda$ *has a left adjoint which sends M to M_γ, where M_γ is the maximal factor module of M cogenerated by $\mathcal{A}(\leq\gamma)$.* (Thus, the canonical map $M\to M_\gamma$ is a minimal left $\operatorname{add}\mathcal{A}(\leq\gamma)$-approximation.)

Corollary. *The category* $\operatorname{add}\mathcal{A}(\leq\gamma)$ *has source maps.*

Proof: If X is in $\mathcal{A}(\leq\gamma)$, let $f\colon X\to X'$ be a source map in the module category. Now, let $p\colon X'\to X'_\gamma$ be the minimal left $\operatorname{add}\mathcal{A}(\leq\gamma)$-approximation of X'. Then pf is left almost split in $\operatorname{add}\mathcal{A}(\leq\gamma)$. Namely, given a map $h\colon X\to Y$ which is not split mono, with Y in $\operatorname{add}\mathcal{A}(\leq\gamma)$, then we can factorize h through f, say $h=h'f$. Since Y belongs to $\operatorname{add}\mathcal{A}(\leq\gamma)$, we can factorize h' through p, say $h'=h''p$. Thus $h=h'f=h''pf$. This shows that pf is left almost split. A minimal version of pf will be minimal left almost split (thus a source map).

Warning. Note that $\operatorname{add}\mathcal{A}(\leq\gamma)$ may not have sink maps: see $\gamma=\{1,2\}$ for the Kronecker quiver.

So here is the proof: The modules in $\mathcal{C}=\mathcal{A}(\leq I_r)$ are of bounded length, say bounded by b. From the Harada-Sai Lemma [R1] we know that the composition of 2^b-1 non-invertible maps between modules in $\mathcal{C}$ is zero.

Now take any module M in $\mathcal{C}$, and a simple submodule X_0 of M, thus there is given a non-zero map $\psi_0\colon X_0\to M$.

We note the following: Assume there is given a sequence of non-invertible maps $\phi_i\colon X_{i-1}\to X_i$ with $1\leq i\leq t$ and a non-invertible map $\psi_t\colon X_t\to M$ with all the modules $X_i\in\mathcal{C}$ and such that the composition $\psi_t\phi_t\cdots\phi_1\neq 0$. Then we can find a module X_{t+1} in $\mathcal{C}$, a non-invertible map $\phi_{t+1}\colon X_t\to X_{t+1}$ and a map $\psi_{t+1}\colon X_{t+1}\to M$ such that $\psi_{t+1}\phi_{t+1}\phi_t\cdots\phi_1\neq 0$.

For the proof, just factorize ψ_t through a source map $f\colon X_t\to X'_t$ of X_t in $\mathcal{C}$, decompose $X'_t=\bigoplus_i Y_i$ with Y_i indecomposable, and let $f=(f_i)_i$ with $f_i\colon X'_t\to Y_i$. We obtain $\psi_t=\sum_i g_if_i$ with $f_i\colon X_t\to Y_i$ and $g_i\colon Y_i\to M$. Then

$$\psi_t\phi_t\cdots\phi_1=\sum_i g_if_i\phi_t\cdots\phi_1$$

is non-zero, thus one of the summands must be non-zero, say the summand with index $i=1$. Let $\phi_{t+1}=f_1$ and $\psi_{t+1}=g_1$. Thus $\psi_{t+1}\phi_{t+1}\phi_t\cdots\phi_1\neq 0$. Also, since $\phi_{t+1}=f_1$ is part of a source map, it is non-invertible.

Applying inductively this procedure, it has to stop after at most 2^b steps, since otherwise we would obtain a contradiction to the Harada-Sai Lemma. However, the procedure stops only when we obtain a map ψ' which is invertible. But this then means that M is isomorphic to X_{t+1}.

Altogether we see that we obtain all the objects in $\mathcal{C}$ by inductively forming indecomposable direct summands of the target of a relative source map for a module already constructed.

REMARK. One sees in this way that the take-off part has an Auslander-Reiten quiver, which can be constructed step by step looking at the various $\mathcal{A}(\leq I_r)$ (the process stabilizes with increasing r, since the neighbors of a module of length n are of length at most pqn).

The third proof will be presented in the next section.

6. Proof of the finiteness theorem, using infinite length modules

6.1. It is of interest that the theorem on relative Σ-injectivity can be used in order to provide a third proof for the finiteness theorem for take-off measures: *Let I_r be a take-off measure. Then $\mathcal{A}(\leq I_r)$ is finite for all r.*

Proof. If not, choose pairwise non-isomorphic modules M_i ($i \in \mathbb{N}$) in $\mathcal{A}(\leq I_r)$. Consider $W = \prod M_i / \bigoplus M_i$. Any finite length submodule of W is isomorphic to a submodule of $\prod M_i$ (see lemma 9.1), thus with $\prod M_i$ also W belongs to $\mathcal{A}(\leq I_r)$. Note that $W \neq 0$. Choose an indecomposable submodule U of finite length with maximal possible Gabriel-Roiter measure. We can assume that U is not isomorphic to any M_i (otherwise delete that index i, this does not change W.) By the theorem, U is a direct summand of W. This shows that there are submodules U', V' of $\prod M_i$ such that $U' + V' = \prod M_i$, $U' \cap V' = \bigoplus M_i$ with $U'/(\bigoplus M_i) = U$. Since $\bigoplus M_i$ is pure in $\prod M_i$, it follows that the embedding $\bigoplus M_i \subseteq U'$ splits: there is a submodule U'' of U' such that $U' = U'' \oplus \bigoplus M_i$. It follows that $\prod M_i = U'' \oplus V'$. Since U'' is isomorphic to U, we see that $\prod M_i$ splits off a copy of U, thus U is a direct summand of some M_i, according to the Auslander lemma (see 9.3). This yields a contradiction.

6.2. SPECIAL CASE OF THE DIRECT SUM THEOREM. *Let I_r be a take-off measure. Then any module which belongs to $\mathcal{D}(I_r)$ is a direct sum of finite length modules.*

Proof: Let M be in $\mathcal{D}(I_r)$. The indecomposable submodules of M of finite length belong to $\bigcup_{i=1}^r \mathcal{A}(I_i)$, thus there are only finitely many isomorphism classes, and we can apply the direct sum theorem 4.3.

7. Indecomposable infinite length modules

7.1. Let us extend the definition of the Gabriel-Roiter measure $\mu(M)$ to modules M of infinite length. If M is not of finite length, let $\mu(M)$ be the supremum of the numbers $\mu(M')$ taken over all submodules M' of M of finite length (or just the indecomposable ones).

Also, we extend the notion of a Gabriel-Roiter filtration as follows: Let M be a Λ-module which is not finitely generated. A sequence

$$U_\bullet = (U_1 \subset U_2 \subset \cdots \subset U_t \cdots)$$

is called a *Gabriel-Roiter filtration* of M provided the following three conditions are satisfied:

(i) U_1 is a simple module.

(ii) U_{i-1} is a Gabriel-Roiter submodule of U_i, for all $2 \leq i$.

(iii) $M = \bigcup_i U_i$.

Note that by definition all the modules with a Gabriel-Roiter filtration are countably generated. And there is the following pleasant result:

7.2. THEOREM. *Any module M with a Gabriel-Roiter filtration is indecomposable.*

Of course, only the case when M is not finitely generated is of interest, since any finite length module with a Gabriel-Roiter filtration is indecomposable by definition. For the proof of the theorem we refer to [R5].

It remains to construct infinite sequences of Gabriel-Roiter inclusions.

7.3. THEOREM *Assume Λ is of infinite representation type. Then there exists an infinite sequence of Gabriel-Roiter inclusions*

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots$$

where all the modules M_i are take-off modules.

Proof: Consider the following graph (or quiver): its vertices are the isomorphism classes $[X]$, where X is an indecomposable take-off module, and draw an arrow $[X] \to [Y]$ provided there is a Gabriel-Roiter inclusion $X \to Y$. There are only finitely many sources, namely the isomorphism classes of the simple modules (since any non-simple indecomposable module Y has a Gabriel-Roiter submodule X, and with Y also X is a take-off module). Also, for any vertex $[X]$, there are only finitely many isomorphism classes $[Y]$ with an arrow $[X] \to [Y]$. Namely, if X is of length n, then we know that Y is of length at most pqn, where where p is the maximal length of an indecomposable projective module and q is the maximal length of an indecomposable injective module. Now by the finiteness theorem for take-off measures we know that there are only finitely many isomorphism classes of take-off modules of length at most pqn. But there are infinitely many isomorphism classes of take-off modules, thus the König graph theorem asserts that there are paths of infinite length. But this is what we were looking for.

In particular, we get the Gabriel-Roiter measure $I_\omega = \sup I_r$.

COROLLARY. *Assume Λ is of infinite representation type. Then there is an indecomposable module M with a Gabriel-Roiter filtration*

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots$$

where all the modules M_i are take-off modules. In particular, $\mu(M) = I_\omega$.

Note that for such a module M any finitely generated submodule M' of M is contained in some M_t, thus belongs to the take-off part. In particular, for any natural number n, M has only finitely many isomorphism classes of submodules of length n. Using the notation introduced above, we see that such a module M belongs to $\mathcal{D}(I_\omega)$, If we call the modules in $\mathcal{D}(I_\omega)$ *take-off modules,* we can write: *there do exist indecomposable take-off modules of infinite length.* One should compare this result with a corresponding result in the last section: there we have seen that all the modules in $\mathcal{D}(I_r)$ (with $r \in \mathbb{N}_1$) are direct sums of finite length modules. To repeat: We have the chain of inclusion of subcategories

$$\mathcal{D}(I_1) \subset \mathcal{D}(I_2) \subset \cdots \subset \mathcal{D}(I_r) \subset \cdots$$

and all the modules in the various subcategories $\mathcal{D}(I_r)$ are direct sums of finite length modules. However, as soon as we go over to $\mathcal{D}(I_\omega)$, the behaviour changes completely: there are infinite length modules in $\mathcal{D}(I_\omega)$ which are indecomposable.

The existence of infinitely generated indecomposables for any artin algebra of infinite representation type was first shown by Auslander [A2]. The Gabriel-Roiter approach allows to locate some of these modules very well: one finds such modules already in the take-off part of the module category.

7.4. Using different infinite sequences of Gabriel-Roiter inclusions as asserted in theorem 7.3, we may obtain a large number of pairwise non-isomorphic indecomposable Λ-modules M, however all these modules have the same Gabriel-Roiter measure!

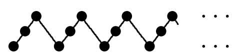
Example 1*. If Δ is the Kronecker quiver and k is a countable and algebraically closed field, then all the "torsionfree $k\Delta$-modules of rank 1" (see [R2]) occur in this way, and $I(M) = \{1, 2, 4, 6, 8, \dots\}$.

Example 2*. Consider the tame hereditary algebra Λ of type $\widetilde{A}_{21}$

$$\begin{tikzcd} & b \arrow[dl] & \\ a & & c \arrow[ul] \arrow[ll] \end{tikzcd} \quad .$$

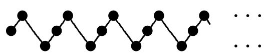
We will use this example in order to exhibit some typical phenomena which occur for indecomposable infinite length modules M in $\mathcal{D}(I_\omega)$. Note that Λ is special biserial, thus the indecomposable Λ-modules of finite length are strings and bands and can be exhibited by using finite "words". Similarly, one may consider "$\mathbb{N}$-words" in order to obtain suitable indecomposable Λ-modules of infinite length, see [R3]. As in [R3], these $\mathbb{N}$-words are depicted below by bullets and arrows (all the arrows point downwards, thus we will delete the arrowheads). As a further explanation of these pictures, we note that the upper row of bullets refers to composition factors of the form $S(c)$, those of the middle row to $S(b)$, and the lower row to $S(a)$. Further examples of Λ-modules will be presented below in part III, see also [R5].

- First, consider the string module corresponding to



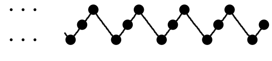
its Gabriel-Roiter measure is $\{1, 2, 4, 5, 7, 8, \dots\}$; instead of $\{1, 2, 4, 5, 7, 8, \dots\}$ let us write $124578\cdots$, or better $12|45|78|\cdots$ (in order to mark the gaps).

- Here is an indecomposable module M in $\mathcal{D}(I_\omega)$ which has a proper submodule of finite index with a Gabriel-Roiter filtration. It is the Prüfer module for the simple module $S(b)$:



its Gabriel-Roiter measure is again $12|45|78|\cdots$, but there is no corresponding sequence of submodules which exhaust all of M.

- But there is also an indecomposable modules M in $\mathcal{D}(I_\omega)$ which has no submodules with an infinite Gabriel-Roiter filtration. Take the following string module



its Gabriel-Roiter measure is again $12|45|78|....$ There are indecomposable submodules with measure $12|45|\cdots|3n{+}1, 3n{+}2$ for any n, but there is no infinite chain of such submodules.

7.5. We have introduced above (see section 1) an embedding of $\mathcal{P}_f(\mathbb{N}_1)$ into $\mathbb{Q}$. In order to deal also with modules which are not finitely generated, we consider the set $\mathcal{P}_l(\mathbb{N}_1)$ of all subsets I of $\mathbb{N}_1$ such that for any $n \in \mathbb{N}_1$, there is $n' \geq n$ with $n' \notin I$.

Lemma. *The Gabriel-Roiter measure $\mu(M)$ of any module M belongs to $\mathcal{P}_l(\mathbb{N}_1)$.*

Proof. There is $q \in \mathbb{N}_1$ such that any indecomposable injective Λ-module has length at most q. Let $\mu(M) = \{a_1 < a_2 < \cdots < a_i < \cdots\}$ and assume that for some n we have $a_{n+t} = a_n + t$ for all $t \in \mathbb{N}_1$. Let $s = q \cdot a_n$. There is a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_{n+s}$ with $|M_i| = a_i$ for $1 \leq i \leq n+s$. Since $|M_{n+t}| = a_{n+t} = a_{n+t-1} + 1 = |M_{n+t-1}| + 1$, we see that M_{n+t-1} is a maximal submodule of M_{n+t}. Since M_{n+t} is indecomposable, the socle of M_{n+t} has to be contained in M_{n+t-1}. Inductively, we see that the socle of M_{n+t} is contained in M_n, for any $t \geq 1$, in particular, the socle of M_{n+s} is contained in M_n, thus M_{n+s} can be embedded into the injective envelope of M_n. Since any indecomposable injective module is of length at most q, the injective envelope of M_n has length at most $q \cdot a_n$, thus $|M_{n+s}| \leq q \cdot a_n$. But $|M_{n+s}| = |M_n| + s = (q+1)a_n > q \cdot a_n$, a contradiction.

7.6. The embedding of $\mathcal{P}_f(\mathbb{N}_1)$ into $\mathbb{Q}$ (thus into $\mathbb{R}$) extends to an embedding of $\mathcal{P}_l(\mathbb{N}_1)$ into the real interval $[0,1]$:

Lemma. *The map $r\colon \mathcal{P}_l(\mathbb{N}_1) \to \mathbb{R}$ given by $r(I) = \sum_{i \in I} \frac{1}{2^i}$ for $I \in \mathcal{P}_l(\mathbb{N}_1)$ is injective, its image is contained in the interval $[0,1]$ and it preserves and reflects the ordering.*

Remark. The map r can be defined not just on $\mathcal{P}_l(\mathbb{N}_1)$, but on all of $\mathcal{P}(\mathbb{N}_1)$, however it will no longer be injective (indeed, for any element I in $\mathcal{P}(\mathbb{N}_1) \setminus \mathcal{P}_l(\mathbb{N}_1)$, there is a unique finite set I' with $r(I) = r(I')$). Of course, one may easily change the definition of r in order to be able to embed all of $\mathcal{P}(\mathbb{N}_1)$ into $\mathbb{R}$: just use say 3 instead of 2 in the denominator. However, our interest lies in the Gabriel-Roiter measures which occur for finite dimensional algebras and the previous lemma assures us that the definition of r as proposed is sufficient for these considerations.

III. The Landing Part.

Let us recall that (for Λ non-zero) there exists a largest Gabriel-Roiter measure $I^1 = [1,q]$ with $\mathcal{A}(I^1)$ being the indecomposable injective Λ-modules of largest length. Now in general, there will be a lot of Gabriel-Roiter measures without a direct predecessor. However, there is the following result:

8.1. Theorem. *Let Λ be of infinite representation type. Then there are Gabriel-Roiter measures I^t for Λ with*

$$\cdots \quad < I^3 < I^2 < I^1$$

such that any other Gabriel-Roiter measure I for Λ satisfies $I < I^t$ for all $t \in \mathbb{N}_1$. For any t, there are only finitely many isomorphism classes in $\mathcal{A}(I^t)$ and all the modules in $\mathcal{A}(I^t)$ are preinjective (in the sense of Auslander-Smalø.)

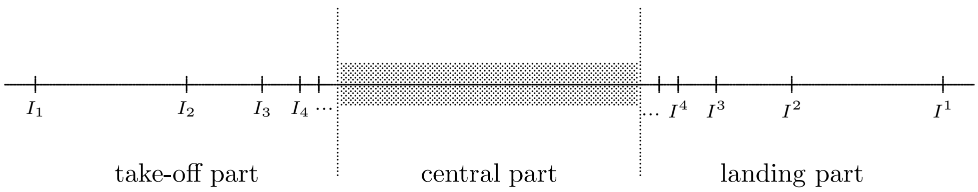
Recall that Auslander-Smalø have introduced in [AS] the classes of preprojective and preinjective modules (actually with reference to the work of Roiter and Gabriel), and this is the notion we refer to. For a proof of the theorem see [R5].

The modules in $\bigcup_t \mathcal{A}(I^t)$ have been called the *landing part* of the category $\operatorname{mod}\Lambda$. Note that *for any n, there are only finitely many isomorphism classes of indecomposable modules of length n which belong to the landing part.* Of course, the landing part again provides a proof of BTh 1.

We have noted above that the existence of the take-off measures I_t provides a proof of BTh 1. The same is true for the existence of the landing measures I^t, and again we obtain a strengthening, since also this statement is of interest for finite artin algebras Λ.

As we have mentioned, the modules in $\mathcal{A}(I^1)$ are the indecomposable injective modules of largest possible length. For general t, it seems to be difficult to characterize the modules in $\mathcal{A}(I_t)$ or $\mathcal{A}(I^t)$ in a direct way.

8.2. The indecomposable modules of finite length which belong neither to the take-off part nor to the landing part are said to form the *central part.* It is the central part which should be of particular interest in future (by the way, one also should be concerned about the military involvement of the publisher of this volume):



8.3. Usually there will exist preinjective indecomposables which do not belong to the landing part. For example, any simple module belongs to $\mathcal{A}(I_1)$, thus a simple injective module is preinjective and in the take-off part, thus not in the landing part. Also, there may exist preinjective modules Q such that $\mathcal{A}(\mu(Q))$ is infinite, as the example of the radical-square-zero algebra with quiver

$$\circ \longleftarrow \circ \leftleftarrows \circ$$

shows: take for Q the indecomposable injective module of length 2. But there may be even infinitely many isomorphism classes of preinjective indecomposables which do not belong to the landing part:

Example. Consider again the tame hereditary algebra Λ of type $\widetilde{A}_{21}$

$$\begin{tikzcd} & b \arrow[dl] & \\ a & & c \arrow[ul] \arrow[ll] \end{tikzcd}$$

Note that for the Auslander-Smalø preinjective Λ-modules are just those modules which belong to the preinjective component.

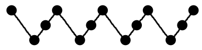
There are two kinds of such modules. First, let us consider those preinjective indecomposable modules which have projective socle (thus the socle is a direct sum of copies of $S(a)$). Then the Gabriel-Roiter-measures are as follows:

$$\cdots > 1235689, 10 > 123567 > 1234,$$

the general form is

$$123|56|89|\cdots|3i-1, 3i|\cdots|3n-1, 3n|3n+1,$$

with $n \geq 0$. For $n = 4$, it looks as follows



and for $n \geq 1$, the Gabriel-Roiter filtration starts with $M_1 \subset M_2 \subset M_3$, where M_3 is the indecomposable length 3 module seen left: it is uniform, but not serial.

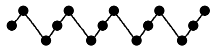
On the other hand, those preinjectives with $S(b)$ in the socle have Gabriel-Roiter measure

$$123|6|9|\cdots|3i|\cdots|3n|3n+2,$$

with $n \geq 0$. For small $n \geq 1$, we obtain the values

$$\cdots > 12369, 11 > 12368 > 1235.$$

Here is the picture for $n = 4$



now, for $n \geq 1$, the Gabriel-Roiter-filtration starts with $M_1 \subset M_2 \subset M_3$, where M_3 is the serial length 3 module seen right.

It follows that all the preinjective modules with $S(b)$ in the socle belong to the central part.

8.4. In contrast to the landing part, the modules in the take-off part are usually **not** preprojective. Here is an example: Let $\Lambda = k[X, Y]/\langle XY, X^3, Y^3\rangle$ and J the ideal generated by X^2 and Y^2. The take-off part for Λ is the same as the take-off part for Λ/J and these modules are the preprojective Λ/J-modules, but none of them is preprojective as a Λ-module.

Note that there is no dualization principle concerning the take-off and the landing part (whereas the notions of preprojectivity and the preinjectivity are dual ones)! If we want to invoke dual considerations, then we have to work with a corresponding Gabriel-Roiter **comeasure** which is based on looking at indecomposable **factor** modules in contrast to the Gabriel-Roiter measure which is based on indecomposable **sub**modules, see [R5].

8.5. It is usually difficult to specify the position of the possible Gabriel-Roiter measures. But here is such an assertions, dealing with uniform modules:

PROPOSITION. *Let $I^1 = [1, q]$ and $1 \leq s < q$. Assume the following: for any simple Λ-module with injective envelope $Q(S)$ of length greater than s, there are only finitely many indecomposable Λ-modules with a submodule of the form S. Then $[1, s]$ is a landing measure.*

Proof: As a start, let us note the following observation: for any subset $J \in \mathcal{P}_f(\mathbb{N}_1)$, the relation $[1, s] < J$ with respect to the total ordering $\leq$ is equivalent to the subset relation $[1, s] \subset J$.

We want to show that there are only finitely many indecomposable modules M with $[1, s] < \mu(M)$. Take such a module M, say with Gabriel-Roiter filtration $M_1 \subset M_2 \subset \cdots \subset M_t = M$. Since $[1, s] < \mu(M)$, we have $[1, s] \subset \mu(M)$, thus $|M_i| = i$ for $1 \leq i \leq s$. In particular, M_s is uniform. It follows that the injective envelopes $Q(M_s) = Q(M_1)$ of M_1 and M_s coincide. Since $Q(M_s)$ is injective, there is a map $f : M_t \to Q(M_s)$ whose restriction to M_s is the identity. The map f cannot be surjective, since otherwise the embedding $M_s \subset M_t$ would be a split monomorphism. Thus $Q(M_s) = Q(M_1)$ has length at least $s+1$. By assumption, there are only finitely many indecomposable modules with submodule M_1 (and there are only finitely many possibilties for M_1). It follows that there are only finitely many indecomposable modules M with $\mu(M) > [1,s]$. Thus $[1,s]$ belongs to the landing part.

9. Appendix. Products of modules and finite length modules

9.1. Lemma. *Consider* $W = \prod M_{i\in S}/\bigoplus M_{i\in S}$ *with arbitrary modules* M_i. *Any finite length submodule of* W *is isomorphic to a submodule of* $\prod M_i$.

Proof: Let U be a finite length submodule of W, let $P(U)$ be its projective cover and ΩU the kernel of $P(U) \to U$. The inclusion map $u\colon U \to W$ can be lifted to a map $u'\colon P(U) \to \prod_i M_i$, thus we get also a map $u''\colon \Omega U \to \bigoplus_i M_i$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \Omega U \arrow[r] \arrow[d, "u''"] & P(U) \arrow[r] \arrow[d, "u'"] & U \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & \bigoplus_i M_i \arrow[r] & \prod_i M_i \arrow[r, "p"] & W \arrow[r] & 0 \end{tikzcd}$$

Since U is of finite length, u'' maps into a finite sum, say into $\bigoplus_{i\in S'} M_i$ with $S' \subseteq S$ a finite subset. Let as factorize the projection $p\colon \prod_i M_i \to W$ through $\prod_{i\in S} M_i/\bigoplus_{i\in S'} M_i$:

$$\prod_{i\in S} M_i \xrightarrow{p'} \prod_{i\in S} M_i/\bigoplus_{i\in S'} M_i \xrightarrow{p''} \prod_{i\in S} M_i/\bigoplus_{i\in S} M_i.$$

By the definition of S', the kernel of $p'u'$ contains ΩU. Since the kernel of $pu' = p''p'u'$ is equal to ΩU, we see that also the kernel of $p'u$ is equal to ΩU. Thus the image of $p'u'$ is isomorphic to U. However, $\prod_{i\in S} M_i/\bigoplus_{i\in S'} M_i = \prod_{i\in S} M_i/\prod_{i\in S'} M_i$ is a direct summand of $\prod_{i\in S} M_i$, thus U is isomorphic to a submodule of $\prod_{i\in S} M_i$.

9.2. Lemma. *Let* U *be of finite length, let* M_i *be arbitrary modules with* $i \in S$. *If* U *embeds into* $\prod_{i\in S} M_i$, *then also into* $\bigoplus_{i\in S'} M_i$, *for some finite subset* S' *of* S.

Proof: Denote by u the embedding $u\colon U \to M = \prod_{i\in S} M_i$, thus there are given maps $u_i\colon U \to M_i$ such that the intersection of their kernels is 0. However, then there is a finite subset S' of the index set S such that the intersection of the kernels is 0, say $\bigcap_{i\in S'} \mathrm{Ker}(u_i) = 0$ (since a finite length module is artinian). The maps u_i with $i \in S'$ combine to a monomorphism $U \to \bigoplus_{i\in S'} M_i$.

9.3. Lemma (Auslander). *Let* Λ *be an artin algebra. Let* M_i *be arbitrary modules with* $i \in S$. *Let* U *be an indecomposable module of finite length. If* U *is a direct summand of* $\prod_{i\in S} M_i$, *then* U *is a direct summand of at least one of the modules* M_i.

Proof: There are given maps $u\colon U \to \prod M_i$ and $p\colon \prod M_i \to U$ with $pu = 1$. Write $u = (u_i)_i$ with $u_i\colon U \to M_i$. If no u_i is split mono, factorize $u_i = u_i'f$ where f is the minimal almost split map starting in U. However, then $1 = pu = pu'f$ with $u' = (u_i')_i$ implies that f is a split monomorphism, which is impossible.

Acknowledgment

Parts of our considerations have been influenced by fruitful discussions with Bo Chen [Ch,R6]. In addition, the author is indebted to Liu Gongxiang and Philipp Fahr for supplying detailed notes of the lectures at Hangzhou and Queretaro, respectively, and to Philipp Fahr, Angela Holtmann and Andrew Hubery for checking the final version for misprints and inaccuracies.

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Added in proof (22.04.2006)

For the benefit of the reader, some additional developments concerning the Gabriel-Roiter measure and variations of it should be mentioned.

1. Bo Chen [Ch] has pointed out that the discussions in section 1.8 should be completed by the following lemma:

LEMMA. *Let M be an indecomposable module. Then there is at most one simple submodule S of M with $\mu(M/S) < \mu(M)$.* In particular, if M is indecomposable and not uniform, then there is a simple submodule S' such that $\mu(M) < \mu(M/S')$.

The second assertion yields the implication (5) $\implies$ (1) in 1.8. Also, as an immediate consequence of the lemma, one obtains Property 6: Assume that M is indecomposable and not uniform. Then there is a simple submodule S' with $\mu(M) < \mu(M/S')$. Let M' be an indecomposable direct summand of M/S' with $\mu(M/S') = \mu(M')$. Either M' is uniform, or by induction there is a uniform factor module M'' of M' with $\mu(M') < \mu(M'')$.

2. The crucial lemma 3.1 shows that given a Gabriel-Roiter inclusion $X \subset Y$, the module Y/X is an epimorphic image of $\tau^{-1}X$. This can be strengthened as follows:

LEMMA. *Let $X \subset Y$ be a Gabriel-Roiter inclusion. Then there is an irreducible homomorphism $\mu\colon X \to Y'$ with Y' indecomposable and an epimorphism $\pi\colon Y' \to Y$ such that $\pi\mu$ is a monomorphism.* (In particular, μ itself has to be a monomorphism and $\pi\mu$ is again a Gabriel-Roiter inclusion. The main point is the fact that the cokernel of $\pi\mu$ is a factor module of the cokernel of μ.).

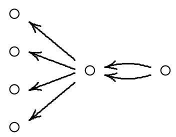
Note that the cokernels of irreducible monomorphisms $\mu\colon X \to Y'$ with X, Y' indecomposable have attracted a lot of interest. In particular, there do exist interesting classes of representation-infinite algebras where these modules are of bounded length (for example the domestic string algebras over an algebraically closed base field). Also, we see that an indecomposable module X can occur as a Gabriel-Roiter submodule of some other module only in case there is an irreducible monomorphism $X \to Y'$ with Y' indecomposable.

3. It should be stressed that there is a whole family of functions from the set of finite length R-modules to the rational numbers which behave similar to the Gabriel-Roiter measure: they are obtained by asserting weights to the simple R-modules.

For example, when dealing with an artin k-algebra Λ', we may replace in the definition 1.1 the use of the length of the module M by the use of its k-length. The function μ' obtained in this way will have essentially the same properties as the ordinary Gabriel-Roiter measure (in particular, Gabriel's main property will still hold). However such a function will distinguish some of the simple Λ-modules. Starting with a basic artin algebra Λ, we may look at all the artin algebras Λ' which are Morita equivalent to Λ and obtain in this way "weighted Gabriel-Roiter measures" on the category $\operatorname{mod}\Lambda$ (since the categories $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$ are equivalent).

Note that the weighted Gabriel-Roiter measures may yield different take-off parts and different landing parts. As an example, consider a tubular algebra Λ'

with quiver as shown on the left



$$\begin{array}{ccc} 1 & & \\ 1 & & \\ & 2 & 1 \\ 1 & & \\ 1 & & \end{array}$$

such that the dimension $d(S(i))$ of the simple modules $S(i)$ is as exhibited on the right. Let M be an indecomposable Λ'-module without a simple submodule of dimension 1. Then M lives on the Kronecker subquiver and is not simple injective. Since any simple submodule of M has dimension 2, its weighted Gabriel-Roiter measure as an element of $\mathcal{P}_f(\mathbb{N}_1)$ is a subset of $[2, n]$, thus it corresponds to a rational number $\mu'(M) < \frac{1}{2}$. On the other hand, the remaining indecomposable Λ'-modules N have a one-dimensional submodule, thus $\mu'(N) \geq \frac{1}{2}$. It follows that the preprojective Kronecker modules form the take-off part with respect to this weighted Gabriel-Roiter measure, whereas for the usual Gabriel-Roiter measure μ, the take-off modules are just those Λ'-modules which belong to the preprojective component of $\operatorname{mod}\Lambda'$.

4. An aximomatic characterization of the Gabriel-Roiter measure has been given by H. Krause in his Trieste lectures 2006, see his *Notes on the Gabriel-Roiter measure.* In addition, he discusses possible extensions of the Gabriel-Roiter measure to the corresponding derived categories.

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