Linear Nakayama algebras which are higher Auslander algebras.

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Abstract: An artin algebra A is said to be a higher Auslander algebra provided the global dimension is finite and bounded by the dominant dimension. We say that a linear Nakayama algebra is concave, provided its Kupisch series first increases, then decreases. We are going to classify the concave Nakayama algebras which are higher Auslander algebras. Let us stress that the classification strongly depends on the parity of the global dimension of A.

Key words. Nakayama algebra. Higher Auslander algebra.

2020 Mathematics Subject classification. Primary 16G10, Secondary 16G20, 16G70, 16E05, 16E10, 16E65.

0. Introduction.

Following Iyama [2], an artin algebra A is said to be a *higher Auslander algebra* provided the global dimension of A is finite and bounded by the dominant dimension of A. It is well-known that an artin algebra which is a higher Auslander algebra is either semisimple (thus the global dimension is zero, the dominant dimension is infinite), or else the global dimension and the dominant dimension are non-zero, finite and equal (see 5.1).

In the present note, A usually will be a connected Nakayama algebra with $n = n(A)$ simple modules and with global dimension $d(A) \geq 1$. Since for a Nakayama algebra A the injective envelope of an indecomposable projective module is an indecomposable projective module, the dominant dimension of A is at least 1.

For simplicity, we assume that A is a k-algebra given by a quiver with relations, where k is a field. The modules M to be considered are usually left A-modules with finite length $|M|$; we denote by PM and IM a projective cover and an injective envelope of M, respectively, and $\operatorname{soc} M$, $\operatorname{rad} M$, and $\operatorname{top} M$ denote the socle, the radical and the top of M, respectively. We write ΩM for the first syzygy module of M; this is the kernel of a projective cover $PM \to M$ and $\operatorname{pd} M$ denotes the projective dimension of M; for the zero module, we use the convention $\operatorname{pd} 0 = -\infty$. An indecomposable module M is said to be *even*, or *odd*, if $\operatorname{pd} M$ is even, or odd, respectively. We write ΣM for the first suspension module of M: it is the cokernel of an injective envelope $M \to IM$. If A is a linear Nakayama algebra, we denote by ω_A its simple injective module.

It is well-known that if A has finite global dimension, then $d(A) \leq 2n(A) - 2$ (of course, if A is a linear Nakayama algebra, then we even have $d(A) < n(A)$). A quite surprising recent paper [5] by Madsen, Marczinzik and Zaimi shows that for any numbers $n \leq d \leq 2n - 2$, there is a unique (necessarily cyclic) Nakayama algebra A with $n = n(A)$ and $d = d(A)$ which is a higher Auslander algebra. An explanation of this fact has been given by Sen [7] using the powerful ϵ-reduction which he has analysed in previous papers. Sen's reduction shows: in order to classify the Nakayama algebras which are higher Auslander algebras, it remains to consider the linear ones.

This is the purpose of the present note: to look at linear Nakayama algebras A which are higher Auslander algebras of global dimension d. Some of them have been exhibited already in [8] and [1]. As we will see, the parity of d is a decisive datum.

The maximal length of indecomposable modules will be called the *height* $h(A)$ of A, and the indecomposable modules of length $h(A)$ are called *summits* (they have to be both projective and injective). A linear Nakayama algebra is said to be *concave* provided its Kupisch series is first increasing, then decreasing. If A is a concave algebra, there is a unique indecomposable module P of length $h(A)$ such that $\operatorname{rad} P$ is projective. We say that P is the *first summit* of A; since we assume that A is not simple, we have $|P| \geq 2$.

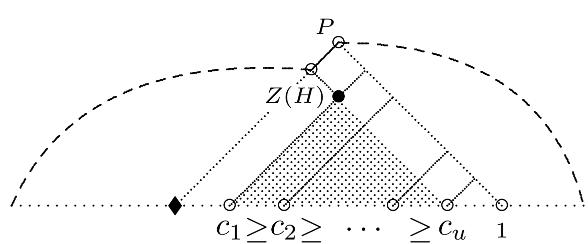
We are going to provide a complete system of invariants for the concave Nakayama algebras A which are higher Auslander algebras of global dimension d, say with first summit P. For d odd, we will look at the module $Z(A) = \operatorname{rad} P/\operatorname{soc} P$ (it is an odd module), for d even at the module $Z'(A) = \operatorname{rad}^2 P$ (again, this is an odd module).

If A is a Nakayama algebra and M is an odd module, let $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ be the composition series of M. We call $\operatorname{char} M = (\operatorname{pd} M_1/M_0, \ldots, \operatorname{pd} M_m/M_{m-1})$ the *characteristic* of M. In addition, we need the characteristic of the zero module: by definition, $\operatorname{char} 0$ is the empty sequence (later, we will introduce also the characteristic of an even module, but this is more subtle).

Theorem 1. *Let A be a concave Nakayama algebra with first summit P. If A is a higher Auslander algebra of* **odd** *global dimension d, let $Z(A) = \operatorname{rad} P/\operatorname{soc} P$. Then $Z(A)$ is either the zero module or else an odd module and $\operatorname{char} Z(A)$ is a decreasing sequence of odd numbers which are bounded by d.*

The assignment $A \mapsto \operatorname{char} Z(A)$ provides a bijection between the isomorphism classes of the concave higher Auslander Nakayama algebras A of odd global dimension d and the decreasing sequences of odd numbers bounded by d.

Theorem 1 asserts that for any decreasing sequence $d \geq c_1 \geq c_2 \geq \cdots \geq c_u$ of odd numbers, there is a (unique) concave higher Auslander algebra $H = H_d(c_1, \ldots, c_u)$ of global dimension d with $\operatorname{char} Z(H) = (c_1, \ldots, c_u)$. The algebra $H = H_d(\emptyset)$ is the linear Nakayama algebra H with $h(H) = 2$ and $n(H) = d+1$. Here is a sketch of $\operatorname{mod} H_d(c_1, \ldots, c_u)$ in case $u \geq 1$:



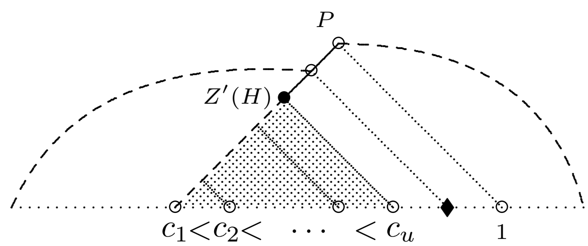
The bullet is $Z(H) = \operatorname{rad} P/\operatorname{soc} P$; it is an odd module with $\operatorname{char} Z(H) = (c_1, \ldots, c_u)$. The shaded part are the subfactors of $Z(H)$. We have added below any odd composition factor of P its projective dimension (since P is the first summit of H, the module $\operatorname{rad} P$ is projective, thus $\operatorname{pd} \operatorname{top} P = 1$). The socle of P has even projective dimension, namely $c_1 - 1$, and is marked by a black lozenge ♦.

Theorem 1′. *Let A be a concave Nakayama algebra with first summit P. If A is a higher Auslander algebra of* **even** *global dimension $d > 0$, let $Z'(A) = \operatorname{rad}^2 P$. Then $Z'(A)$*

is either the zero module or else an odd module and char $Z'(A)$ *is a strictly increasing sequence of odd numbers bounded by* d.

The assignment $A \mapsto \operatorname{char} Z'(A)$ *provides a bijection between the isomorphism classes of the concave higher Auslander Nakayama algebras* A *of even global dimension* $d > 0$ *and the strictly increasing sequences of odd numbers bounded by* d.

As in the case of d being odd, Theorem 1′ asserts that for d even and any strictly increasing sequence $c_1 < c_2 < \cdots < c_u$ of odd numbers bounded by d, there is a (unique) concave higher Auslander algebra $H = H_d(c_1, \dots, c_u)$ of global dimension d now with $\operatorname{char} Z'(H) = (c_1, \dots, c_u)$. The algebra $H = H_d(\emptyset)$ is the linear Nakayama algebra H with $h(H) = 2$ and $n(H) = d+1$. Here is a sketch of $\operatorname{mod} H_d(c_1, \dots, c_u)$ in case $u \geq 1$:



The bullet is $Z'(H) = \operatorname{rad}^2 P$; it is an odd module with $\operatorname{char} Z'(H) = (c_1, \dots, c_u)$. The shaded part are the subfactors of $Z'(H)$. Again, we have added below any odd composition factor of P its projective dimension (since P is the first summit of H, the radical $\operatorname{rad} P$ of P is projective, thus $\operatorname{pd} \operatorname{top} P = 1$). The composition factor $\operatorname{top} \operatorname{rad} P$ has even projective dimension, namely $c_u + 1$, and has been marked by a black lozenge ♦.

The algebras $H_d(c_1, \dots, c_u)$ can be constructed directly quite easily, as the following Theorems 2 and 2′ show. In order to describe these constructions, we need some further ingredients.

If A is a Nakayama algebra of finite global dimension, we attach to **every** indecomposable module M (not only the odd ones), its characteristic sequence $\operatorname{char} M$, see 1.3. If M has length m, then $\operatorname{char} M = (z_1, \dots, z_m)$ is a sequence of m non-negative numbers, with all but at most one of the numbers z_i being odd (the numbers z_i exhibit the projective dimension of M itself or of composition factors of M).

Starting with a sequence $\mathbf{z} = (z_1, \dots, z_m)$ of non-negative numbers, with one of the numbers zero, whereas the remaining numbers are odd, there is a concave Nakayama algebra $A(\mathbf{z})$ of height m with $\operatorname{char} P(\omega_A) = \mathbf{z}$; the algebra $A(\mathbf{z})$ will be called the ascent algebra of $\mathbf{z}$, see Proposition 2.1.

For any linear Nakayama algebra A, and d a positive integer, the partial d-closure $C_d(A)$ of A will be defined in section 3.

Theorem 2. *Let* d *be odd. Let* $c_1 \geq c_2 \geq \cdots \geq c_u$ *be odd numbers bounded by* d. *Let* $H = H_d(c_1, \dots, c_u)$ *be the partial* d*-closure of* $A(0, c_1, \dots, c_u, 1)$. *Then* H *is a concave higher Auslander algebra of global dimension* d *with* $\operatorname{char} Z(H) = (c_1, \dots, c_u)$.

Theorem 2′. *Let* $d > 0$ *be even. Let* $c_1 < c_2 < \cdots < c_u$ *be odd numbers bounded by* d. *Let* $H = H_d(c_1, \dots, c_u)$ *be the partial* d*-closure of* $A(c_1, \dots, c_u, 0, 1)$. *Then* H *is a concave higher Auslander algebra of global dimension* d *with* $\operatorname{char} Z'(H) = (c_1, \dots, c_u)$.

Until now, we have considered only the first summit of the algebras $H_d(c_1,\dots,c_u)$. Given a concave Nakayama algebra, there is a unique indecomposable module Q of length $h(A)$ such that $Q/\operatorname{soc} Q$ is injective. We say that Q is the *last summit* of A and call $Q/\operatorname{soc} Q$ the *principal cliff module* of A.

Theorem 3. *Let d be odd. Let $c_1 \ge c_2 \ge \cdots \ge c_u$ be odd numbers bounded by d. Let $H = H_d(c_1,\dots,c_u)$. Let P be the first summit of H and Q the last summit.*

If $u = 0$, then H has d summits, $\operatorname{char} P = (0,1)$, *and* $\operatorname{char} Q = (0,d)$.

If $u \ge 1$, let $t = \frac{1}{2}(d - c_1)$. Then $t \ge 0$, the algebra H has $(u+2)t+1$ summits, and

$$\operatorname{char} P = (0, c_1, \dots, c_u, 1), \quad \operatorname{char} Q = (0, c_1{+}2t, \dots, c_u{+}2t, 1{+}2t).$$

Theorem 3′. *Let $d > 0$ be even. Let $c_1 < c_2 < \cdots < c_u$ be odd numbers bounded by d. Let $H = H_d(c_1,\dots,c_u)$. Let P be the first summit of H and Q the last summit.*

If $u = 0$, then H has d summits, $\operatorname{char} P = (0,1)$, *and* $\operatorname{char} Q = (d{-}1,0)$.

If $u \ge 1$, let $t = \frac{1}{2}(d - c_u - 1)$. Then $t \ge 0$, H has $(u+2)t+u$ summits, and

$$\operatorname{char} P = (c_1,\dots,c_u,0,1), \quad \operatorname{char} Q = (d{-}1, 0, 1{+}2t, c_1{+}2t+2, \dots, c_{u-1}{+}2t+2).$$

Outline of the paper.

If $M_0,\dots,M_m$ are indecomposable modules with $\operatorname{Hom}(M_{i-1},M_i) \ne 0$ for $1 \le i \le m$, we say that M_0 is a *predecessor* of M_m and M_m a *successor* of M_0. As usual, we denote by τ the Auslander-Reiten translation (with $\tau P = 0$ for P projective). If M is an indecomposable module, let $\mathcal{F}(M)$ be the set of the (isomorphism classes of) non-zero subfactors of M. A *Serre subcategory* of $\operatorname{mod} A$ is the full subcategory given by the extension closure of a set of simple modules. The Serre subcategory *generated by a set of modules* is the Serre subcategory given by the composition factors of these modules. If A is a Nakayama algebra, the modules with a fixed socle are said to form a *ray*; those with a fixed top are said to form a *coray*.

In section 1, we introduce and analyze the characteristic sequence $\operatorname{char} M$ of an indecomposable module M. This seems to be of independent interest. As we will see in 1.6 (see also 4.8), this sequence determines the projective dimension of all subfactors of M.

A linear Nakayama algebra A is called *ascending* provided for indecomposable projective modules P, P' with $\operatorname{Hom}(P,P') \ne 0$, we have $|P| \le |P'|$, or, equivalently, provided $|P(\tau S)| \le |PS|$ for any simple module S. Note that a linear Nakayama algebra A is ascending iff A is concave and $P(\omega_A)$ is a summit. In section 2, we construct for any projective characteristic sequence $\mathbf{z}$ its ascent algebra $A = A(\mathbf{z})$; it is an ascending Nakayama algebra A with $\operatorname{char} P(\omega_A) = \mathbf{z}$. In section 3, we construct the partial d-closure of a Nakayama algebra.

Section 5 is devoted to Nakayama algebras which are higher Auslander algebras. We show that if A is a higher Auslander Nakayama algebra of global dimension d and S is a simple module, then S is torsionless or else $\operatorname{pd} S = \operatorname{pd} IS = d$.

Section 7 contains the proof of Theorems 1, 2, and 3; section 8 the proof of Theorems 1′, 2′, and 3′. In both sections, we provide various characterizations of the concave higher Auslander algebras A of odd or even global dimension, respectively, with reference to the existence of suitable modules M of length $h(A) - 1$. The modules M which we single out are, on the one hand, the radical of the first summit (and certain τ^--shifts), and, on the other hand, the principal cliff module (and certain τ-shifts).

Left rotations, right rotations. The main tools used in the paper are the left rotations (see section 2) and the right rotations (see section 3), both are defined for characteristic sequences. If A is a concave Nakayama algebra and P a summit, left rotations λ are used in order to reconstruct all predecessors of P: here, using λ, we obtain from PS, where S is a simple module, the module $P(\tau S)$. The right rotations are used in order to construct successors of a given indecomposable module. If I is an indecomposable module which is both projective and injective, then the right rotation ρ sends $\operatorname{rad} I$ to $I/\operatorname{soc} I$.

Let A be concave and P a summit. Whereas the predecessors of P can be reconstructed from P using λ, it is not possible in general, to reconstruct the successors of $\operatorname{char} P$. A linear Nakayama algebra is said to be *d-bound* provided its global dimension is bounded by d and any indecomposable module I which is injective, but not projective satisfies $\operatorname{pd} I = d$. In case A is d-bound, then the characteristic sequence of any summit P determines uniquely all the successors of P. Here we use the right rotation ρ, see section 6. All algebras which we are interested in, are d-bound: either by construction (the algebras $H_d(c_1, \dots, c_u)$), or by assumption (the higher Auslander algebras).

Memory quivers. Throughout the paper, we work with "memory functions" on translation quivers. Special memory quivers will be discussed in section 4. The use of characteristic sequences in section 1 is already a kind of shadow.

Let Γ be a translation quiver. A function $\mu\colon \Gamma_0 \to \mathbb{N} \cup \{\infty\}$ is called a *memory function* provided there is an artin algebra C with a Serre subcategory $\mathcal{C} \subseteq \operatorname{mod} C$ such that Γ is the Auslander-Reiten quiver of $\mathcal{C}$ and we have $\mu(X) = \operatorname{pd}_C X$ for any indecomposable object $X \in \mathcal{C}$. The pair (Γ, μ) will be called a *memory translation quiver* or just a *memory quiver*. In this paper, we only will deal with memory functions with values in $\mathbb{N}$.

Module categories of artin algebras, and more generally, memory quivers, are sometimes exhibited by showing the corresponding (Auslander-Reiten) quiver. Note that a vertex of an Auslander-Reiten quiver is the isomorphism class $x = [M]$ of an indecomposable module M. In the picures which we provide, it often will be convenient to replace x by the projective dimension $\operatorname{pd} M$ of M, or by the value $\mu(x)$ of the memory function in question.

The parameter d. Note that all semisimple algebras are higher Auslander algebras (the global dimension is 0, the dominant dimension is ∞). It is easy to see that a connected higher Auslander algebra of global dimension 1 is the path algebra of a linearly oriented quiver of type $\mathbb{A}_n$ (without relations). Thus, we usually may assume that $d \geq 2$.

Iyama, who has introduced the higher Auslander algebras, called a higher Auslander algebra A of global dimension d a $(d-1)$-Auslander algebra, see [3]. Of course, this index shift is well-thought since in this way the Auslander algebras themselves are just the 1-Auslander algebras. (There is the similar, and of course related, problem concerning the calibration of the representation dimension: a shift by 1 has the advantage that then the value 1 (instead of 2) is attached to the non-semisimple representation-finite algebras.)

However, we have refrained from following this convention (as also several other authors do), since in our discussion of the higher Auslander algebras of global dimension d, we want to stress that it is the number d (and not $d-1$) and its parity, which are decisive.

1. The characteristic sequence of an indecomposable module.

For any indecomposable module M, we are going to introduce its characteristic sequence $\operatorname{char} M$; it is a sequence $(z_1, \dots, z_m)$ of non-negative integers with at most one even entry, where $m = |M|$. In addition, let $\operatorname{char} 0$ be the empty sequence. The relevance of $\operatorname{char} M$ relies on Madsen's maximum principle 1.1 and the subfactor formula 1.4 which shows that $\operatorname{char} M$ determines the projective dimension of all subfactors of M.

1.1. Proposition (Madsen's maximum principle). *Let A be a Nakayama algebra. Let M be an indecomposable module.*

(1) *At most one of the composition factors of M is even.*

(2) *If all composition factors F_iM of M are odd, then $\operatorname{pd} M = \max_i \operatorname{pd} F_iM$; thus M is odd. Conversely, if M is odd, then all its composition factors are odd.*

(3) *If at least one composition factor of M is even, then M is even.*

(4) *If S is simple, with $\tau^i S$ being odd for $0 \le i \le m$, then $|PS| \ge m+2$.*

In case M is even, we should stress that the projective dimension of the composition factors of M do not determine the projective dimension of M, see 1.7. For completeness, we recall from [4] that *the existence of an even simple module implies that A has finite global dimension;* this is the reason that (2) implies (3).

Proof of proposition. For (2) and (3), see [4], or also [6].

Thus, let us look at (1). Assume that M has a submodule U with $\operatorname{top} U$ even and that M/U has only odd composition factors. Since $S = U/\operatorname{rad} U$ is even, $\Omega S = \operatorname{rad} PS$ is odd, thus, according to (3), all composition factors of $\operatorname{rad} PS$ are odd. We have $PU = PS$ and $U = PS/\Omega U$, thus $\operatorname{rad} U = \operatorname{rad} PS/\Omega U$. Since all composition factors of $\operatorname{rad} PS$ are odd, all composition factors of $\operatorname{rad} U$ are odd. We see that $U/\operatorname{rad} U$ is the only even composition factor of M.

Proof of (4). Assume that $|PS| \le m+1$. The composition factors of PS are the modules $\tau^i S$ with $0 \le i \le |PS|-1$, thus they are odd. According to the maximum principle, it follows that PS is odd. But $\operatorname{pd} PS = 0$. □

1.2. Further consequences. Let us add some consequences of 1.1 which one should keep in mind. Recall that an indecomposable module M is said to be *multiplicity-free* provided any simple module occurs at most once as a composition factor of M.

Let A be a Nakayama algebra.

(1) *If S is simple and not projective, at most one of S, τS can be even.*

(2) *If $A \neq 0$, then not all simple modules are odd.*

(3) *An indecomposable odd module is multiplicity-free.*

(4) *If A has finite global dimension and rank n, then the height of A is bounded by $2n-1$.*

Proofs. (1) If $S, \tau S$ both would be even, the middle term of the corresponding Auslander-Reiten sequence would be an indecomposable module with two even composition factors, in contrast to 1.1 (1).

(2) If $A \neq 0$, there is an indecomposable projective module P. Since $\operatorname{pd} P = 0$, we see that P is even. According to 1.1.(2), not all composition factors of M can be odd.

(3) If M is indecomposable and not multiplicity-free, then all simple modules occur as composition factors. But if M is odd, then $A \neq 0$, thus according to (2), there are simple modules which are not odd.

(4) If an indecomposable module M has length at least $2n$, then every simple module occurs in M with multiplicity at least 2. According to (2), there is an even simple module S, and 1.1 (1) asserts that S occurs at most once as a composition factor of M. This contradiction shows that any indecomposable module has length at most $2n-1$. □

1.3. Characteristic sequences. A sequence $(z_1, \dots, z_m)$ of non-negative numbers will be said to be a *characteristic sequence* provided at most one of the entries is even. A characteristic sequence $(z_1, \dots, z_m)$ with one entry being zero is said to be *projective.*

1.4. The characteristic of an indecomposable module. Assume that A has finite global dimension. To every indecomposable module M we attach a sequence of numbers, called its characteristic. On a first reading, the choice and the ordering of the numbers may seem to be rather curious, but we hope to convince the reader that the definition is natural. The characteristic $\operatorname{char} M$ of an indecomposable module of length m will be a sequence of m non-negative numbers, with all but at most one being odd (and all numbers being positive iff M is not projective).

Let M be an indecomposable module of length m, say with composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M,$$

thus $F_i(M) = M_i/M_{i-1}$ (with $1 \le i \le m$) are the composition factors of M.

We define $\operatorname{char} M$ as the following sequence of n non-negative integers

$$(\operatorname{char} M)_i = \begin{cases} \operatorname{pd} F_i(M) & \text{if} \quad F_i(M) \text{ is odd,} \\ \operatorname{pd} M & \text{if} \quad F_i(M) \text{ is even,} \end{cases}$$

where $1 \le i \le m$. In addition, we look also at the zero module and define $\operatorname{char} 0 = (\emptyset)$ (the empty sequence). The sequence $\operatorname{char} M$ will be called the *characteristic of* M.

Lemma. *Let A be a Nakayama algebra of finite global dimension.*

(1) *The characteristic of an indecomposable module is a characteristic sequence.*
(2) *An indecomposable module is projective iff its characteristic is projective.*
(3) *An indecomposable module is odd iff its characteristic has only odd entries.*
(4) *The characteristic* $\operatorname{char} M = (z_1, \dots, z_m)$ *of an indecomposable module M determines* $\operatorname{pd} M$ *(if all z_i are odd, then* $\operatorname{pd} M = \max z_i$*; if some z_i is even, then* $\operatorname{pd} M = z_i$*).*
(5) *Let M be an indecomposable module of length m.*
Then $\operatorname{char} M = (\operatorname{pd} F_1(M), \dots, \operatorname{pd} F_m(M))$ *iff M has a composition factor S with* $\operatorname{pd} S = \operatorname{pd} M$.
(6) *Let M be an indecomposable module of length m with* $\operatorname{pd} M = d(A)$.
Then $\operatorname{char} M = (\operatorname{pd} F_1(M), \dots, \operatorname{pd} F_m(M))$.

Proof. (1) At most one of the coefficients of $\operatorname{char} M$ is even, see 1.1 (1).

(2) If $(\operatorname{char} M)_i$ is even, then by definition $(\operatorname{char} M)_i = \operatorname{pd} M$. If all coefficients of $\operatorname{char} M$ are odd, then the maximum principle for odd modules asserts that $\operatorname{pd} M = \max_i \operatorname{char} M_i$. As a consequence, if M is non-zero and projective, not all coefficients of $\operatorname{char} M$ can be odd, since otherwise $\operatorname{pd} M = 0$, but $\max_i(\operatorname{char} M_i) > 0$. Thus $(\operatorname{char} M)_i$ is even, for some i, and then $(\operatorname{char} M)_i = \operatorname{pd} M = 0$. Conversely, if $(\operatorname{char} M)_i = 0$, for some i, then $\operatorname{pd} M = (\operatorname{char} M)_i = 0$, thus M is projective (and non-zero).

(3) See 1.1 (2).

(4) Again, see 1.1 (2).

(5) We can assume that M is even. Let $\operatorname{char} M = (z_1, \dots, z_m)$. Let S be a composition factor of M with $\operatorname{pd} S = \operatorname{pd} M$, say $S = F_i(M)$. Since $F_i(M)$ is even, we have $z_i = \operatorname{pd} M$, thus $z_i = \operatorname{pd} F_i(M)$. Of course, also $z_j = \operatorname{pd} F_j(M)$ for $j \neq 0$. Conversely, assume that $z_j = \operatorname{pd} F_j(M)$ for all $1 \le j \le m$. Since M is even, one of the composition factors is even, say $S = F_i(M)$. By the definition of $\operatorname{char} M$, we have $z_i = \operatorname{pd} M$. It follows that $\operatorname{pd} S = \operatorname{pd} F_i(M) = z_i = \operatorname{pd} M$.

(6) Let $d = d(A)$. Then $\operatorname{pd} F_j(M) \le d$ for all j. We cannot have $\operatorname{pd} F_j(M) < d$ for all j, since otherwise also $\operatorname{pd} M < d$, contrary to the assumption $\operatorname{pd} M = d$. Thus, there is some i with $\operatorname{pd} F_i(M) = d$. This shows that the composition factor $S = F_i(M)$ satisfies $\operatorname{pd} S = \operatorname{pd} M$. The assertion follows now from (4). $\square$

1.5. Given a projective characteristic sequence $(z_0, \dots, z_m)$ of length $m+1$, let

$$Y(z_0, \dots, z_m) = \begin{cases} (z_1, \dots, z_m) & \textit{if} \quad z_0 = 0, \\ (z_1, \dots, z_{v-1}, z_0 + 1, z_{v+1}, \dots, z_m) & \textit{if} \quad z_v = 0, \quad v \ge 1. \end{cases}$$

This is a non-projective characteristic sequence of length m.

Given a non-projective characteristic sequence $(z_1, \dots, z_m)$ of length m, let

$$P(z_1, \dots, z_m) = \begin{cases} (0, z_1, \dots, z_m) & \textit{if all } z_i \textit{ are odd,} \\ (z_v - 1, z_1, \dots, z_{v-1}, 0, z_{v+1}, \dots, z_m) & \textit{if } z_v \textit{ is even.} \end{cases}$$

This is a projective characteristic sequence of length $m+1$.

The maps $Y(-)$ and $P(-)$ provide bijections between the set of projective characteristic sequences and the set of non-projective characteristic sequences which are inverse to each other. $\square$

Lemma. *Let A be a Nakayama algebra of finite global dimension. Let P be indecomposable projective and $Y = P/\operatorname{soc} P$. If $\operatorname{char} P = (z_0, \dots, z_m)$, then $\operatorname{char} Y = Y(z_0, \dots, z_m)$.*

Proof. First, let $z_0 = 0$. Then the numbers z_i with $1 \le i \le m$ are odd, and are the projective dimension of the composition factors of Y.

Second, let $z_v = 0$ for some $v \ge 1$. Then $P/\operatorname{soc} P$ has an even composition factor, thus $\operatorname{soc} P$ is an odd composition factor, therefore the first entry of $\operatorname{char} P$ is $z_0 = \operatorname{pd} \operatorname{soc} P$. Since $\Omega Y = \operatorname{soc} P$, we have $\operatorname{pd} Y = z_0 + 1$. It follows that for $i \neq v$, we have $(\operatorname{char} Y)_i = \operatorname{pd} F_i Y = \operatorname{pd} F_{i+1} P = z_i$ and $(\operatorname{char} Y)_v = \operatorname{pd} Y = z_0 + 1$. $\square$

1.6. The subfactor formula. We consider subfactors X of an indecomposable module M. In case X is non-zero and even, there is the following important formula for $\operatorname{pd} X$.

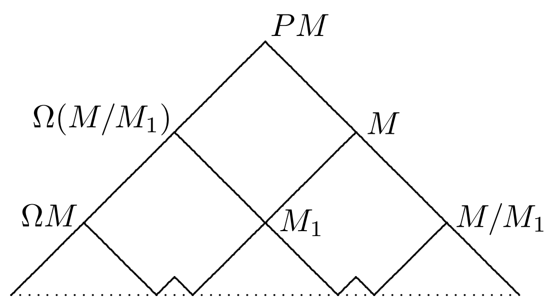
Proposition. *Let A be a linear Nakayama algebra. Let M be an indecomposable module. Let $M_1 \subset M_2 \subseteq M$ be submodules and assume that M_2/M_1 is even. Then M_1 and M/M_2 are odd or zero, and*

$$\operatorname{pd} M_2/M_1 = \max\{\operatorname{pd} M_1 + 1, \operatorname{pd} M, \operatorname{pd} M/M_2 - 1\}.$$

Proof. According to 1.1, M has precisely one even composition factor, and this has to be a subfactor of M_2/M_1. In particular, we see that M_1 and M/M_2 are odd or zero.

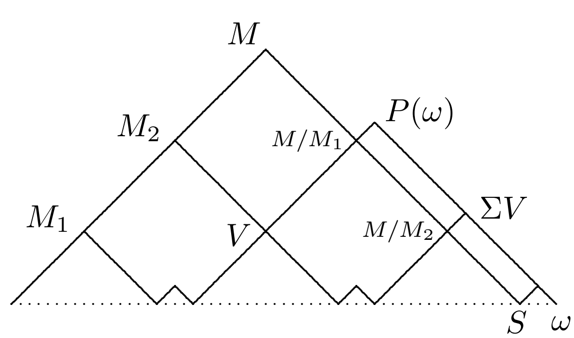
Case 1: $M_2 = M$. If, in addition, $M_1 = 0$, then the assertion is obvious. Thus, we can assume that $M_1 \neq 0$. If M is projective, then $\Omega(M/M_1) = M_1$ shows that $\operatorname{pd} M/M_1 = \operatorname{pd} M_1 + 1$.

Thus, we assume that $M_2 = M$, $M_1 \neq 0$ and that M is non-projective. Let PM be a projective cover of M.



Then $\operatorname{pd} M/M_1 = \operatorname{pd} \Omega(M/M_1)+1$ and there is an exact sequence $0 \to \Omega M \to \Omega(M/M_1) \to M_1 \to 0$. Since both modules ΩM and M_1 are odd, $\operatorname{pd} \Omega(M/M_1) = \max\{\operatorname{pd} M_1, \operatorname{pd} \Omega M\} = \max\{\operatorname{pd} M_1, \operatorname{pd} M - 1\}$, therefore $\operatorname{pd} M/M_1 = \operatorname{pd} \Omega(M/M_1) + 1 = \max\{\operatorname{pd} M_1 + 1, \operatorname{pd} M\}$.

Case 2: M_2 is a proper submodule of M. We first may assume that $S = \operatorname{top} M$ is the simple injective module (we just delete the simple modules $\tau^{-i}S$ with $i \geq 1$), and consider the one-point extension $B = A[M/M_1]$, say with extension vertex ω (which now is the simple injective module), thus $\operatorname{rad} P(\omega) = M/M_1$, and therefore $\Omega(\omega) = M/M_1$. Of course, B is again a linear Nakayama algebra with $\tau(\omega) = S$. Since M/M_1 has the even factor module $V = M_2/M_1$, we see that M/M_1 is even, thus $\Omega(\omega) = M/M_1$ shows that ω is odd.



The module V is a submodule of the projective-injective module $P(\omega)$ and $\Sigma V = P(\omega)/V$ is an extension of M/M_2 by ω and therefore an odd module with

$$\operatorname{pd} \Sigma V = \max\{\operatorname{pd} M/M_2, \operatorname{pd} \omega\}.$$

Since $P(\omega)$ is projective-injective, we have $\Omega\Sigma V = V$, thus

$$\operatorname{pd} V = -1 + \operatorname{pd} \Sigma V = \max\{-1 + \operatorname{pd} M/M_2, -1 + \operatorname{pd} \omega\}.$$

Now, $-1 + \operatorname{pd} \omega = \operatorname{pd} \Omega(\omega) = \operatorname{pd} M/M_1$. Thus, it remains to observe that

$$\operatorname{pd} M/M_1 = \max\{\operatorname{pd} M_1 + 1, \operatorname{pd} M\},$$

but this we know already, due to case (1). □

Corollary 1. *Let A be a linear Nakayama algebra. Let M be an indecomposable module. Let X be a non-zero subfactor of M. If X is even, then* $\operatorname{pd} X \geq \operatorname{pd} M$.

Proof. Let $X = M_2/M_1$, where $M_1 \subset M_2$ are submodules of M. According to the Proposition, $\operatorname{pd} M_2/M_1 = \max\{\operatorname{pd} M_1 + 1, \operatorname{pd} M, \operatorname{pd} M/M_2 - 1\}$, thus $\operatorname{pd} X = \operatorname{pd} M_2/M_1 \geq \operatorname{pd} M$. □

As a special case of the Proposition, we see that $\operatorname{char} M$ determines $\operatorname{pd} F_i(M)$ for all i. This is of course trivial if $F_i(M)$ is odd. In case $F_i(M)$ is even, Proposition provides a formula for $\operatorname{pd} F_i(M)$ in terms of the entries of $\operatorname{char} M$:

Corollary 2. *Let A be a linear Nakayama algebra. Let M be an indecomposable module with* $\operatorname{char} M = (z_1, \dots, z_m)$. *If $F_i(M)$ is even, then*

$$\operatorname{pd} F_i(M) = \max\{z_1 + 1, \dots, z_{i-1} + 1, z_i, z_{i+1} - 1, \dots, z_m - 1\}.$$

Proof. Take the composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of M, thus $F_i(M) = M_i/M_{i-1}$. Proposition provides the formula

$$(*) \qquad \operatorname{pd} F_i(M) = \max\{\operatorname{pd} M_{i-1} + 1, \operatorname{pd} M, \operatorname{pd} M/M_i - 1\}.$$

The modules M_{i-1} and M/M_i are odd or zero. We have $\operatorname{char} M_{i-1} = (z_1, \dots, z_{i-1})$ and $\operatorname{char} M/M_i = (z_{i+1}, \dots, z_m)$. If M_{i-1} is non-zero, then $\operatorname{pd} M_{i-1} = \max\{z_1, \dots, z_{i-1}\}$. If M/M_{i-1} is non-zero, then $\operatorname{pd} M/M_i = \max\{z_{i+1}, \dots, z_m\}$. If $M_{i-1} = 0$, then the term $\operatorname{pd} M_{i-1} + 1 = -\infty$ can be omitted in $(*)$. Similarly, if $M/M_i = 0$, the term $\operatorname{pd} M/M_i - 1 = -\infty$ can be omitted in $(*)$. □

Corollary 3. *Let A be a linear Nakayama algebra. Let M be an indecomposable module. If S is an even composition factor of M, then* $\operatorname{pd} M \leq \operatorname{pd} S$. □

Thus, we see: *If A is a linear Nakayama algebra and M is an indecomposable module with* $\operatorname{char} M = (z_1, \dots, z_m)$, *then $z_i \leq \operatorname{pd} F_i(M)$ for all $1 \leq i \leq m$.*

Corollary 4. *Let M be an indecomposable module and X a subfactor of M. Let* $\operatorname{char} M = (z_1, \dots, z_m)$. *Then* $\operatorname{pd} X \le 1 + \max z_i$.

Proof. We can assume that $X \neq 0$. Let $g(M) = 1+\max_i z_i$, and $f(M) = \max_i \operatorname{pd} F_i(M)$. According to Corollary 2, we have $f(M) \le g(M)$. If X is a subfactor of M, then the composition factors of X are composition factors of M, therefore $f(X) \le f(M)$. On the other hand, for any module N we clearly have $\operatorname{pd} N \le f(N)$ (for N even, see Corollary 3). Altogether, we have $\operatorname{pd} X \le f(X) \le f(M) \le g(M)$. □

1.7. The pd-controlled modules. An indecomposable module M with $\operatorname{pd} Z \le \operatorname{pd} M$ for all subfactors Z of M will be said to be pd-*controlled.* According to the maximum principle for odd modules, *all odd modules are* pd-*controlled.* Even modules are usually not pd-controlled: for example, an indecomposable projective module P is pd-controlled only in case P is simple.

In sections 7 and 8, we are going to work a lot with pd-controlled modules of projective dimension $d(A)$. Namely, Let A be a linear Nakayama algebra which is a higher Auslander algebra. If M is an indecomposable module which is not torsionless, then $\operatorname{pd} M = d(A)$ (see 5.5), thus M is pd-controlled. In particular, the indecomposable injective modules I which are not projective are pd-controlled. The aim of sections 7 and 8 is to characterize the concave linear Nakayama algebra which are higher Auslander algebras by properties of the principal cliff module.

There is the following Lemma.

Lemma. *Let M be an indecomposable module. If there is a submodule $U \subseteq M$ with* $\operatorname{pd} U \ge \operatorname{pd} Z$ *for all subfactors Z of M, then M is* pd-*controlled.*

Proof of Lemma. Since all odd modules are pd-controlled, we can assume that M is even. Let U be a submodule of M with $e = \operatorname{pd} U \ge \operatorname{pd} Z$ for all subfactors Z of M.

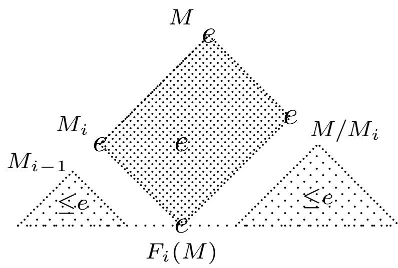
We claim that e is even. Assume for the contrary that e is odd. If $U \subset U' \subseteq M$ and U' is odd, then there is the inequality $\operatorname{pd} U \le \operatorname{pd} U'$ (since U' is odd), but also the inequality $\operatorname{pd} U \ge \operatorname{pd} U'$. Therefore $\operatorname{pd} U = \operatorname{pd} U'$. Thus, we can assume that U is not a proper submodule of an odd submodule of M. Therefore $U = \operatorname{rad} V$ for some even submodule V of M. Let $T = V/U$; this is a simple module which has to be even. Consider the pile Γ with radical M and one summit. There is the vertex U' such that $\Omega_\Gamma U' = U$, thus $\operatorname{pd} U' = 1 + e$. The socle of U' is equal to T. Since $T \subseteq U'$ are even modules, we have $\operatorname{pd} T \ge \operatorname{pd} U'$. Therefore $\operatorname{pd} T \ge \operatorname{pd} U' = 1 + e > e$. On the other hand, T is a subfactor of M, thus $\operatorname{pd} T \le \operatorname{pd} U = e$. This contradiction shows that U cannot be odd.

If $U = M$, nothing has to be shown. Thus, assume that U is a proper submodule of M. There is a submodule V of M such that $U = \operatorname{rad} V$. Again, consider the pile Γ with radical M and one summit. Let U' be the vertex of Γ with $\Omega_\Gamma U' = U$. Thus $\operatorname{pd} U' = e+1$ is odd. Let $Z = \operatorname{rad} U'$. This is a subfactor of M, and therefore $\operatorname{pd} Z \le e$. Since $\operatorname{pd} U' = e + 1$ and $\operatorname{pd} \operatorname{rad} U' \le e$, the maximum principle asserts that $\operatorname{pd} U'/\operatorname{rad} U' = e+1$. But we have $\Omega_\Gamma(U'/\operatorname{rad} U') = M$. Thus, we see that $\operatorname{pd} M = e$. □

Corollary. *Let M be an indecomposable module. Assume that* $\operatorname{pd} \operatorname{soc} M \ge \operatorname{pd} F_i(M)$ *for all i. Then M is* pd-*controlled and* $\operatorname{pd} M = \operatorname{pd} \operatorname{soc} M$. *As a consequence,* $\operatorname{char} M = (\operatorname{pd} F_1(M), \dots, \operatorname{pd} F_m(M))$.

Proof. Let $f(M) = \max_i \operatorname{pd} F_i(M)$. If X is a subfactor of M, then $f(X) \le f(M)$. Also, we have $\operatorname{pd} X \le f(X)$. Thus we see that $f(X) \le f(M) \le \operatorname{pd}\operatorname{soc} M$. This shows that we can apply the Lemma for $U = \operatorname{soc} M$ and conclude that M is pd-controlled. But if M is pd-controlled, then $\operatorname{pd}\operatorname{soc} M \le \operatorname{pd} M$. On the other hand, we have $\operatorname{pd} M \le f(M)$. Thus $\operatorname{pd}\operatorname{soc} M \le \operatorname{pd} M \le f(M) \le \operatorname{pd}\operatorname{soc} M$. According to 1.4 (5), it follows from $\operatorname{pd} M = \operatorname{pd}\operatorname{soc} M$ that $\operatorname{char} M = (\operatorname{pd} F_1(M), \dots, \operatorname{pd} F_m(M))$. □

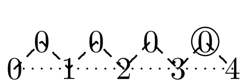
Remark. Let M be an indecomposable module which is pd-controlled, say with $\operatorname{pd} M = e$. There is a composition factor $F_i(M) = M_i/M_{i-1}$ of M with $\operatorname{pd} F_i(M) = e$ and the subfactors of M can be separated into three parts:



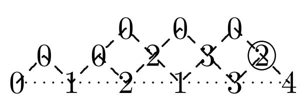
There is the lozenge between $F_i(M)$ and M: all subfactors $X = U/V$ of M with $V \subseteq M_{i-1} \subset M_i \subseteq U$ have $\operatorname{pd} X = e$.

The modules M_{i-1} and M/M_i have odd projective dimension at most e, thus there are the two triangles (on the left and on the right) such that all modules in the triangles have projective dimension at most e (even at most $e-1$ in case e is even).

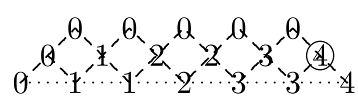
1.8. Remarks. (1) *The sequence of numbers* $\operatorname{pd} F_i(M)$ *does* **not** *determine* $\operatorname{pd} M$, *thus not* $\operatorname{char} M$. Examples: Here are three modules M of length 2 (they are encircled), all with $\operatorname{pd} F_1(M) = 3$, $\operatorname{pd} F_2(M) = 4$. But we have $\operatorname{pd} M$ equal to $0, 2, 4$, respectively:



$\operatorname{char} M = (3,0)$

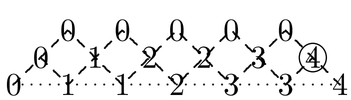


$\operatorname{char} M = (3,2)$

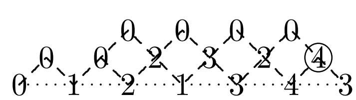


$\operatorname{char} M = (3,4)$

(2) *Also,* $\operatorname{char} M$ *is not determined by* $\operatorname{pd} M$ *and the sequence of the odd values* $\operatorname{pd} F_i(M)$ (we have to know in addition the index i with $\operatorname{pd} F_i(M)$ being even):

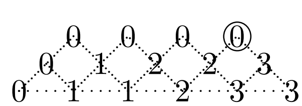


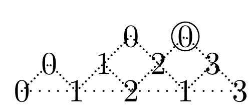
$\operatorname{char} M = (3,4)$



$\operatorname{char} M = (4,3)$

(3) *The numbers* $\operatorname{pd}\ \operatorname{rad}^i P$ *do not determine* $\operatorname{char} P$. Here are two projective modules P, P' with $\operatorname{char} P \ne \operatorname{char} P'$, but $\operatorname{pd}\operatorname{rad}^i P = \operatorname{pd}\operatorname{rad}^i P'$ for all i.





(4) We have seen in 1.6, Corollary 4: If A is a linear Nakayama algebra and M is an indecomposable module with $\operatorname{char} M = (z_1, \dots, z_m)$, then $\operatorname{pd} X \leq 1 + \max z_i$ for all subfactors X of M. Let us stress that the stronger inequality $\operatorname{pd} X \leq \max z_i$ holds true if M is odd, but is in general false. We may have $\operatorname{pd} X = 1 + \max z_i$, see for example the first two examples in (1).

(5) If A is a linear Nakayama algebra and M is an indecomposable module, and X a subfactor of M, then $\operatorname{pd} X$ is determined by $\operatorname{char} M$, see 1.6. In 4.8 we will provide an efficient algorithm for calculating $\operatorname{pd} X$ in terms of the coefficients of $\operatorname{char} M$, using piles.

2. The ascent algebra of a projective characteristic sequence.

Recall that a linear Nakayama algebra A is said to be ascending provided for indecomposable projective modules P, P' with $\operatorname{Hom}(P, P') \neq 0$, we have $|P| \leq |P'|$.

2.1. The left rotation λ. Assume that $m \geq 1$. We define the *left rotation* λ of projective characteristic sequences as follows: Let $\mathbf{z} = (z_0, \dots, z_m)$ be a projective characteristic sequence of length $m+1$. Let

$$\lambda(z_0, \dots, z_m) = \begin{cases} (0, z_0, \dots, z_{m-1}) & \text{if} \quad z_m = 0, \\ (z_0, \dots, z_{m-1}) & \text{if} \quad z_m = 1, \\ (z_m - 2, z_0, \dots, z_{m-1}) & \text{if} \quad z_m \geq 2. \end{cases}$$

2.2. Proposition. *Let A, A' be ascending Nakayama algebras with* $\operatorname{char} P(\omega_A) = \operatorname{char} P(\omega_{A'})$. *Then A and A' are isomorphic.*

Proof, by induction on the rank n of A. If the rank of A is $n = 1$, then $A = k$ and $\omega_A = P(\omega_A) = k$, thus $\operatorname{char} P(\omega_A) = (0)$. If also A' is a linear Nakayama algebra with $\operatorname{char} P(\omega_{A'}) = (0)$, then $P(\omega_{A'})$ has length 1. Thus $\omega_{A'}$ is projective. Since A' is connected, we must have $A' = k$, thus A and A' are isomorphic.

In general, in order to show that two linear Nakayama algebras A, A' of rank ≥ 2 are isomorphic, it is convenient to write $A = B[M],\ A' = B'[M']$ where B, B' are linear Nakayama algebras, such that M is an indecomposable B-module with $\operatorname{top} M = \omega_B$, and M' is an indecomposable B'-module with $\operatorname{top} M' = \omega_{B'}$. If we can show that B, B' are isomorphic, and that $|M| = |M'|$, then A, A' are isomorphic.

Thus, let us assume that A, A' are ascending Nakayama algebras such that A has rank $n \geq 2$ and $\operatorname{char} P(\omega_A) = \operatorname{char} P(\omega_{A'}) = \mathbf{z} = (z_0, \dots, z_m)$. In particular, the A-module $P(\omega_A)$ and the A'-module $P(\omega_{A'})$ both have length $m+1$, thus the A-module $M = \operatorname{rad} P(\omega_A)$ and the A'-module $M' = \operatorname{rad} P(\omega_{A'})$ both have length m. The Serre subcategory in $\operatorname{mod} A$ generated by the simple modules $\tau^i \omega_A$ with $i \geq 1$ is of the form $\operatorname{mod} B$, where B is an ascending Nakayama algebra of rank $n-1$. Of course, M is a B-module with $\operatorname{top} M = \omega_B$ and $A = B[M]$. Similarly, the Serre subcategory in $\operatorname{mod} A'$ generated by the simple modules $\tau^i \omega_{A'}$ with $i \geq 1$ is of the form $\operatorname{mod} B'$ for some ascending Nakayama algebra B'. The B'-module M' has $\operatorname{top} M' = \omega_{B'}$ and $A' = B'[M']$.

Thus, it remains to show that B and B' are isomorphic. We will show below that

$$(*) \qquad \operatorname{char} P(\omega_B) = \lambda\, \mathbf{z},$$

and similarly, char $P(\omega_{B'}) = \lambda\,\mathbf{z}$. Since the rank of B is $n-1$, and char $P(\omega_B) = \operatorname{char} P(\omega_{B'})$ we know by induction that B is isomorphic to B', as we want to show.

In order to verify $(*)$, three cases have to be distinguished.

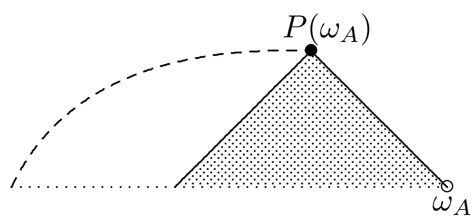
First case: $\operatorname{pd}\omega_A = 1$, thus $\mathbf{z} = (z_0, \dots, z_{m-1}, 1)$ and therefore $\lambda\,\mathbf{z} = (z_0, \dots, z_{m-1})$. In this case rad $P(\omega_A) = P(\omega_B)$ and char $P(\omega_B) = (z_0, \dots, z_{m-1})$, thus char $P(\omega_B) = \lambda\,\mathbf{z}$, as required.

If $\operatorname{pd}\omega_A \geq 2$, then rad $P(\omega_A)$ is a proper factor module of $P(\omega_B)$. Since A is ascending, we have $|P(\omega_B)| \leq |P(\omega_A)|$, and therefore rad $P(\omega_A) = P(\omega_B)/\operatorname{soc} P(\omega_B)$.

We consider now the second case: $\operatorname{pd}\omega_A \geq 2$, and rad $P(\omega_A)$ is odd. In this case soc $P(\omega_B)$ has to be even (since an indecomposable projective module has an even composition factor), thus char $P(\omega_B) = (0, z_0, \dots, z_{m-1})$ and by definition $\lambda\,\mathbf{z} = (0, z_0, \dots, z_{m-1})$. Thus, $(*)$ holds true also in this case.

Third case: $\operatorname{pd}\omega_A \geq 2$, and rad $P(\omega_A)$ is even. Since $\operatorname{pd}\omega_A \geq 2$, we see that rad $P(\omega_A)$ is not projective. Since rad $P(\omega_A)$ is even, ω_A is odd, thus $z_m = \operatorname{pd}\omega_A \geq 3$ and soc $P(\omega_B) = \Omega^2\omega_A$ has projective dimension $z_m - 2$. It follows that char $P(\omega_B) = (z_m - 2, z_0, \dots, z_{m-1})$, but this is just $\lambda\,\mathbf{z}$. □

2.3. Proposition. *For any projective characteristic sequence* $\mathbf{z} = (z_0, \dots, z_m)$*, there is a (necessarily unique) ascending algebra* $A = A(\mathbf{z})$ *with* char $P(\omega_A) = \mathbf{z}$. The algebra $A(\mathbf{z})$ will be called the *ascent algebra* of $\mathbf{z}$.



Proof of Proposition. If $\mathbf{z} = (z_0, \dots, z_m)$ is a projective characteristic sequence with $z_v = 0$, let $\epsilon\mathbf{z} = \sum_i z_i + v$, this is a non-negative integer. We use induction on $\epsilon\mathbf{z}$. The smallest possible value for $\epsilon\mathbf{z}$ is 0, it occurs for $\mathbf{z} = (0)$, and $A = k$ satisfies char $P(\omega_A) = (0)$. We assume now that $\epsilon\mathbf{z}$, thus $m \geq 1$.

Claim: *If* $m \geq 1$*, then* $\epsilon\lambda(z_0, \dots, z_m) < \epsilon(z_0, \dots, z_m)$.

Namely, if $z_m = 0$, then $\epsilon\lambda(z_0, \dots, z_m) = \sum_i z_i + 0 < \sum_i z_i + m = \epsilon(z_0, \dots, z_m)$. If $z_m = 1$, and z_v is even, then $\epsilon(z_0, \dots, z_m) = \sum_i z_i + v$, whereas $\epsilon(z_0, \dots, z_{m-1}) = \sum_{i=0}^{m-1} z_i + v = \sum_i z_i + v - 1$. Finally, if $z_m \geq 2$, and z_v is even, then the even entry of $\lambda(z_0, \dots, z_m)$ has index $v+1$, thus again $\epsilon\lambda(z_0, \dots, z_m) = \epsilon(z_m - 2, z_0, \dots, z_{m-1}) = \sum_i z_i - 2 + v + 1 = \sum_i z_i + v - 1$. This completes the proof of the claim.

By induction, there is the ascent algebra $B = A(\lambda(z_0, \dots, z_m))$ with last summit P' and $\omega' = \operatorname{top} P' = \omega_B$. We have char $P' = \lambda(z_0, \dots, z_m)$. We are going to construct A as a one-point extension of B, namely either $A = B[P']$ or $A = B[P'/\operatorname{soc} P']$. We denote the extension vertex by $\omega = \omega_A$, thus $P = P(\omega)$ has radical equal to P' or to $P'/\operatorname{soc} P'$, respectively. In both cases, with B also A is ascending (in case $A = B[P']$, we have $|P'| < |P|$; in case $A = B[P'/\operatorname{soc} P']$, we have $|P'| = |P|$). Note that for any B-module X, we have $\operatorname{pd}_B X = \operatorname{pd}_A X$ and the indecomposable A-modules which are not B-modules are factor modules of P.

First, let $z_m = 1$, thus $\lambda(z_0, \ldots, z_m) = (z_0, \ldots, z_{m-1})$. In this case, let $A = A'[P']$. Now $\operatorname{rad} P = P'$ shows that the composition factors of P going upwards are those of P' followed by $\omega = P/P'$. Since $\operatorname{pd} \omega = 1$, we see that $\operatorname{char} P = (z_0, \ldots, z_{m-1}, 1)$, as we want to show.

Next, let $z_m \neq 1$. Let $M = P'/\operatorname{soc} P'$ and $A = B[M]$. The composition factors of P going upwards are those of M followed by $\omega = P/M$. In order to calculate $\operatorname{char} P$, we have to distinguish two cases.

If $z_m = 0$, then the remaining entries z_i with $0 \le i \le m-1$ are odd and $\operatorname{char} P' = (0, z_0, \ldots, z_{m-1})$. Then $\operatorname{char} M = (z_0, \ldots, z_{m-1})$. The composition factors of P are those of M and in addition ω. Since the composition factors of M are odd, it follows that ω has to be even, therefore $\operatorname{char} P = (z_0, \ldots, z_{m-1}, 0) = (z_0, \ldots, z_m)$.

Now assume that $z_m \ge 2$ (and then z_m has to be odd, since we deal with a projective characteristic sequence). We have $\Omega^2\omega = \Omega M = \operatorname{soc} P'$, thus $\operatorname{pd} \omega = \operatorname{pd}(\operatorname{soc} P') + 2$. But $\operatorname{pd}(\operatorname{soc} P') = z_m - 2$, thus $\operatorname{pd} \omega = z_m$. Since $\operatorname{pd}(\operatorname{soc} P')$ is odd, M has an even composition factor. Thus $(z_0, \ldots, z_m)$ is obtained from the sequence of numbers $\operatorname{pd} F_i M$, replacing the even number by 0. In order to determine $\operatorname{char} P$, we have to take the sequence of numbers $\operatorname{pd} F_i P$, and replace the even number by 0. Since $\operatorname{pd} \omega$ is odd, we have to take the sequence of numbers $\operatorname{pd} F_i M$, replace the even number by 0 and add at the end $\operatorname{pd} \omega$. It follows that $\operatorname{char} P = (z_0, \ldots, z_m)$, as we want to show. □

Remark. The proof shows: *Let A be a linear Nakayama algebra. If S is a simple module, and $|P(\tau S)| \le |PS|$, then*

$$\lambda \operatorname{char} PS = \operatorname{char} P(\tau S).$$

□

2.4. Corollary. *If A is an ascending Nakayama algebra, $A = A(\operatorname{char}(P(\omega_A))$.*

Proof. For any projective characteristic sequence $\mathbf{z}$, we have $\operatorname{char} P(\omega_{A(\mathbf{z})}) = \mathbf{z}$. Thus, 2.2 asserts that A is isomorphic to $A(\operatorname{char} P(\omega_A))$. □

2.5. *For $h \ge 1$, the maps*

$$A \mapsto \operatorname{char} P(\omega_A).$$

and $\mathbf{z} \mapsto A(\mathbf{z})$ provide inverse bijections between
- *the ascending algebras A of height h, and*
- *the projective characteristic sequences $\mathbf{z}$ of length h.*

Proof. If we start with a characteristic sequence $\mathbf{z}$ and form $A = A(\mathbf{z})$, then, by construction $\operatorname{char} P(\omega_{A(\mathbf{z})}) = \mathbf{z}$. On the other hand, Corollary 2.3 asserts that $A = A(\operatorname{char} P(\omega_A))$. □

Let us describe the cases $h = 1$ and $h = 2$ in detail:

The only ascending algebra of height 1 is k. Similarly, there is just one projective characteristic sequence of length 1, namely (0).

The ascending algebras of height 2 are the radical-square-zero Nakayama algebras A of type $\mathbb{A}_n$ with $n \ge 2$. The module $P(\omega_A)$ has two composition factors, namely its socle with projective dimension $n-2$ and its top with projective dimension $n-1$, thus $\operatorname{char} P(\omega_A) = (0, n-1)$ in case n is even, and $\operatorname{char} P(\omega_A) = (n-2, 0)$ in case n is odd.

3. The partial d-closure of a linear Nakayama algebra.

3.1. Let A be a linear Nakayama algebra and d a positive natural number. A simple module S will be called *d-closed* provided S is torsionless or there is a module M with socle S and $\operatorname{pd} M \geq d$.

The algebra A will be called *d-closed* provided all simple modules are d-closed. Also, A will be called *almost d-closed* provided any simple module which is not d-closed is a composition factor of $P(\omega_A)$. And A will be called *partially d-closed* provided all composition factors of $P(\omega_A)$ are d-closed. Thus, *A is d-closed iff A is both almost d-closed and partially d-closed.*

3.2. Let A be a linear Nakayama algebra. The algebra B is said to be an *extension of A* provided $\operatorname{mod} A$ is a full subcategory of $\operatorname{mod} B$ which is closed under submodules, factor modules and projective covers. Note that in this case $\operatorname{mod} A$ is also closed under extensions, thus it is a Serre subcategory of $\operatorname{mod} B$; also, the simple B-modules which are not A-modules are of the form $\tau^{-t}\omega_A$ with $t \geq 1$.

An extension B of A is said to be *descending* provided

$$|P(\omega_A)| \geq |P(\tau_B^{-}\omega_A)| \geq |P(\tau_B^{-2}\omega_A)| \geq \cdots ,$$

thus provided $\operatorname{pd} \tau_B^{-i}\omega_A \neq 1$ for all $i \geq 1$.

3.3. Let A be a linear Nakayama algebra and d a positive natural number. The *d-cliff module* of A is the module $Y_A = P(\omega_A)/U$, where U is the maximal submodule of $P(\omega_A)$ whose composition factors are d-closed. Obviously, *the algebra A is partially d-closed iff the d-cliff module of A is zero.*

If the d-cliff module Y_A is non-zero, let $E_d(A) = A[Y_A]$ be the one-point extension using Y_A and call it the *d-cliff extension* of A. If A is partially d-closed, then we write $E_d(A) = A$. If we iterate this procedure, we get a sequence $A, E_d(A), E_d^2(A), \dots$, with all algebras being descending extensions of A.

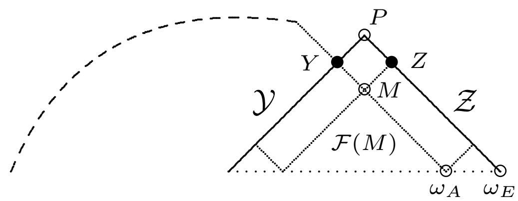
Proposition. *Let A be a linear Nakayama algebra and d a positive natural number. There is $t \geq 0$ such that $E_d^t(A)$ is partially d-closed.* If $E_d^t(A)$ is partially d-closed, we write $C_d(A) = E_d^t(A)$ and call it the *partial d-closure* of A.

Proof. Recall that for any indecomposable module M, $\mathcal{F}(M)$ denotes the set of non-zero subfactors. Let $||M||_d$ be the sum of the values $\operatorname{pd} N$, with $N \in \mathcal{F}(M)$ and $\operatorname{pd} N \leq d$.

(a) *Assume that the d-cliff module Y of A is non-zero. Let ω_E be the extension vertex of $E = E_d(A)$, then*

$$||P(\omega_E)/\operatorname{soc} P(\omega_E)||_d = ||Y||_d + |Y|.$$

Proof. Let $P = P(\omega_E)$, thus $\operatorname{rad} P = Y$. Let $M = Y/\operatorname{soc} Y$ and $Z = P/\operatorname{soc} P$. Let $\mathcal{Y}$ be the set of non-zero submodules of Y and $\mathcal{Z}$ the set of non-zero factor modules of Z. There is the bijection $\Omega\colon \mathcal{Z} \to \mathcal{Y}$ and we have $\operatorname{pd} \Omega N + 1 = \operatorname{pd} N$, for $N \in \mathcal{Z}$. By assumption, all $Y \in \mathcal{Y}$ have projective dimension at most $d-1$, thus all $N \in \mathcal{Z}$ have projective dimension at most d. We see that $\mathcal{F}(Y)$ is the disjoint union of the sets $\mathcal{Y}$ and $\mathcal{F}(M)$, and that $\mathcal{F}(Z)$ is the disjoint union of the sets $\mathcal{Z}$ and $\mathcal{F}(M)$.



Since $\operatorname{pd} N = 1 + \operatorname{pd} \Omega N$ for $N \in \mathcal{Z}$, we see that

$$\sum\nolimits_{N\in\mathcal{Z}} \operatorname{pd} N = |Y| + \sum\nolimits_{X\in\mathcal{Y}} \operatorname{pd} X,$$

therefore $||Z||_d = |Y| + ||Y||_d$. □

(b) *If Y is the d-cliff module, then $||Y||_d \le d|Y|(|Y|+1)$.*

Proof. The set $\mathcal{F}(Y)$ has cardinality $\frac{1}{2}|Y|(|Y|+1)$, and any summand of $||Y||_d = \sum_N \operatorname{pd} N$ is bounded by d. □

Consider now a sequence of d-cliff extensions A, $E_d(A)$, ..., $E_d^t(A)$, say $E_d^i(A) = E_d^{i-1}(A)[Y_{i-1}]$ for $1 \le i \le t$, where $Y_{i-1} \neq 0$ is the d-cliff module of $E_d^{i-1}(A)$.

Let $\omega_0 = \omega_A$ and let $\omega_i = \tau_B^{-i}\omega_0$ be the extension vertex for the extension $E_d^{i-1}(A) \subset E_d^i(A)$. Then we have $|P(\omega_i)| = |Y_{i-1}|+1$ and $|P(\omega_0)| \ge |P(\omega_1)| \ge |P(\omega_2)| \ge \cdots$. Assume that we have equalities

$$|P(\omega_0)| = |P(\omega_1)| = \cdots = |P(\omega_s)|,$$

thus $|Y_i| = |Y_0|$ for $1 \le i < s$. Applying (a) several times, we have

$$||Y_{s-1}||_d = ||Y_0||_d + (s-1)|Y_0|.$$

According to (b), we see that s is bounded. It follows that there is $s > 0$ such that $|Y_0| > |Y_s|$. By induction, it follows that t has to be bounded. Thus there is $t > 0$ such that the d-cliff module of $E_d^t(A)$ is zero. □

3.4. Proposition. *Let A be a non-zero linear Nakayama algebra of global dimension $1 \le d(A) \le d$. Let B be a descending extension of A. If B is d-closed and has global dimension at most d, then $B = C_d(A)$ and the global dimension of B is equal to d.*

Proof, by induction on the number m of simple B-modules which are not A-modules. If $m = 0$, then $A = B$ is d-closed, thus, by definition, $C_d(A) = A$. Since $d(A) \ge 1$, the module ω_A is not projective. Since ω_A is d-closed, we have $\operatorname{pd} \omega_A = d$, thus the global dimension of $A = B$ is d.

Assume now that $m \ge 1$. Let $T = \tau_B^- \omega_A$ and $M = \operatorname{rad} P_B(T)$. We claim that M is the d-cliff module of A. First of all, M is a factor module of $P(\omega_A)$, say $M = P(\omega_A)/U$, where U is a submodule of $P(\omega_A)$. Since B is a descending extension of A, we have $|P(\omega_A)| \ge |P_B(T)|$, therefore M is a proper factor module of $P(\omega_A)$. In particular, no composition factor of M is torsionless (as an A-module).

We show that the composition factors S of U are d-closed as A-modules. First of all, there is the socle $S = \operatorname{soc} U = \operatorname{soc} P(\omega_A)$. Of course, this module is torsionless as an A-module, thus a d-closed simple A-module. Let S be different from $\operatorname{soc} P(\omega_A)$. Then S is not torsionless as a B-module. Since S is d-closed as a B-module, there is a B-module M' with $\operatorname{soc} M' = S$ and $\operatorname{pd}_B M' = d$. Clearly, M' has to be an A-module, and $\operatorname{pd}_A M' = \operatorname{pd}_B M' = d$. This shows that S is d-closed as an A-module.

On the other hand, $S = \operatorname{soc} M$ is not d-closed as an A-module. Namely, otherwise there would exist an A-module N with socle S and $\operatorname{pd} N = d$. But then N is a submodule of $P_B(T)$ and therefore $\operatorname{pd}_B P_B(T)/N = 1 + \operatorname{pd} N = d + 1$. But this contradicts the assumption that the global dimension of B is at most d.

Altogether, we see that U is the maximal submodule of $P(\omega_A)$ whose composition factors are d-closed. This shows that $M = P(\omega_A)/U$ is the d-cliff module of A.

Let $\operatorname{mod} A'$ be the Serre subcategory of $\operatorname{mod} B$ generated by $\operatorname{mod} A$ and the simple module T. Then $A' = A[M] = E_d(A)$ is a linear Nakayama algebra and B is a descending extension of A. The number of simple B-modules which are not A'-modules is $m - 1$. Thus, by induction $B = C_d(A') = C_d(E_d(A)) = C_d(A)$. $\square$

Corollary. *Let B be a concave Nakayama algebra which is d-closed and has global dimension at most d. Let P be a summit of B. Then $B = C_dA(\operatorname{char} P)$.*

Proof. Let $T = \operatorname{top} P$ and $\operatorname{mod} A$ the Serre subcategory of $\operatorname{mod} B$ generated by the simple modules $\tau^i T$ with $i \geq 0$. Since P is a summit of B, the algebra A is ascending and the algebra B is a descending extension of A. By construction, $P = P(\omega_A)$. According to 2.2, $A = A(\operatorname{char} P)$. Thus $B = C_d(A) = C_dA(\operatorname{char} P))$. $\square$

3.5. Let A be a linear Nakayama algebra and d a positive natural number. Here are some properties of $C_d(A)$.

(1) *If X is an indecomposable $C_d(A)$-module, but not an A-module, then* $\operatorname{pd} X \leq d$.

(2) *If $C = C_d(A)$ is a proper extension of A, then* $\operatorname{pd} \omega_C = d$.

(3) *If A is a non-zero algebra, then the global dimension of $C_d(A)$ is the maximum of d and of the global dimension of A.*

(4) *If A is concave, also $C_d(A)$ is concave.*

Proof. (1), (2) and (4) follow immediately from the definitions involved. (3) is a direct consequence of (1) and (2). $\square$

3.6. Proposition. *Let A be a linear Nakayama algebra and d a positive natural number. If A is almost d-closed, then $E_d(A)$ is almost d-closed.*

Proof. Let A be almost d-closed and Y the d-cliff module of A. Let $E = E_d(A) = A[Y]$ with extension vertex ω_E. Let $P = P(\omega_A)$ with socle S and let $P' = P(\omega_E)$ with socle S'. We have to show that all simple modules T which are predecessors of S' are d-closed in E. The simple modules which are predecessors of S are d-closed in A, thus in E, since A is almost d-closed. If $T = S'$, then T is torsionless in E, since it is the socle of P'. Thus, it remains to assume that T is a proper predecessor of S' and a proper successor of S. The definition of Y implies that IT has a submodule of projective dimension at least d, thus T is d-closed. □

Corollary. *Let A be a linear Nakayama algebra and d a positive natural number. If A is almost d-closed, then $C_d(A)$ is d-closed.* □

3.7. Finally, let us provide another characterization of the ascending algebras which shows that ascending algebras are almost d-closed.

Lemma. *A linear Nakayama algebra A is ascending iff any simple module is torsionless or a composition factor of $P(\omega_A)$.*

Proof. First, assume that A is ascending. Let S be a simple module which is not torsionless. Then IS is not projective, thus $|PIS| > |IS|$. Assume that S is also not a composition factor of $P(\omega_A)$. Then $\operatorname{top} IS \neq \omega_A$, thus there is a simple module T with $\tau T = \operatorname{top} IS$. Since IS is injective, we must have $|IS| \geq |PT|$. Therefore $|P(\tau T)| = |PIS| > |IS| \geq |PT|$, but this contradicts our assumption that A is ascending.

Conversely, let A be a linear Nakayama algebra and assume that A is not ascending, thus there is a simple module T with $|P(\tau T)| > |PT|$. We show that there is a simple module S which is not torsionless and not a composition factor of $P(\omega_A)$. Let X be the factor module of $P(\tau T)$ with $|X| = |PT|$. Since $|P(\tau T)| > |PT|$, we see that X is a proper non-zero factor module of an indecomposable projective module, thus not projective. On the other hand, it follows from $|X| = |PT|$, that X is injective. Let $S = \operatorname{soc} X$. Then $X = IS$. Since IS is not projective, S is not torsionless. Since $\operatorname{top} IS = \tau T$ for some simple module T, we see that $\operatorname{top} IS \neq \omega_A$, and therefore S is not a composition factor of $P(\omega_A)$. □

Corollary. *Let A be an ascending Nakayama algebra and d a positive natural number. Then A is almost d-closed, thus $C_d(A)$ is d-closed.*

Proof. Let A be an ascending Nakayama algebra. Then all the simple modules which are not composition factors of $P(\omega_A)$ are torsionless, thus d-closed. Therefore A is almost d-closed. Corollary 3.6 shows that then $C_d(A)$ is d-closed. □

4. The right rotation ρ (and memory piles).

In this section $m \geq 1$ will be a fixed positive integer.

4.1. The rotation ρ. Let $\mathcal{Z}_m$ be the set of m-tuples of integers with at most one even entry; in particular, $\mathcal{Z}_1 = \mathbb{Z}$. Note that the elements of $\mathcal{Z}_m$ with non-negative entries are just the characteristic sequences.

We define $\rho = \rho_m \colon \mathcal{Z}_m \to \mathcal{Z}_m$ (and call it the *right rotation* for $\mathcal{Z}_m$) as follows: for $\mathbf{z} = (z_1, \dots, z_m) \in \mathcal{Z}_m$, let

$$\rho\,\mathbf{z} = \begin{cases} (z_2, \dots, z_m, z_1{+}1), & \text{if } z_2, \dots, z_m \text{ are odd,} \\ (z_2, \dots, z_{v-1}, z_1{+}1, z_{v+1}, \dots, z_m, z_v{+}1), & \text{if } v > 1 \text{ and } z_v \text{ is even.} \end{cases}$$

(For $m = 1$, $\rho(z) = z + 1$ is just the addition by 1.) Note that if z_1 is even, then $\rho\,\mathbf{z}$ has only odd entries; otherwise, $\rho\,\mathbf{z}$ has an even entry (of course, just one).

Clearly, ρ is bijective (thus invertible), and it is easy to write down the corresponding formula for ρ^{-1}. Let $\mathbf{z} = (z_1, \dots, z_m) \in \mathcal{Z}_m$.

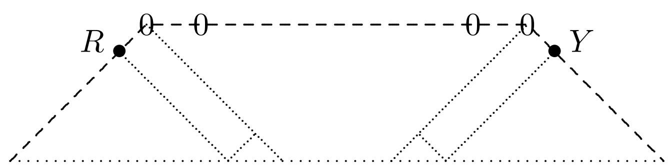
$$\rho^{-1}\mathbf{z} = \begin{cases} (z_m - 1, z_1, \dots, z_{m-1}), & \text{if } z_1, \dots, z_{m-1} \text{ are odd,} \\ (z_v - 1, z_1, \dots, z_{v-1}, z_m - 1, z_{v+1}, \dots, z_{m-1}), & \text{if } v < m \text{ and } z_v \text{ is even.} \end{cases}$$

Lemma. *Let $\mathbf{z} \in \mathcal{Z}_m$ be a characteristic sequence. Then $\rho^{-1}(\mathbf{z})$ has a negative entry iff $\mathbf{z}$ is a projective characteristic sequence.*

Proof. If $\mathbf{z}$ is projective, then $z_i = 0$ for some i, thus the first entry of $\rho^{-1}\mathbf{z}$ is -1 (and all other entries are non-negative). If $\mathbf{z}$ is not projective, then all entries of $\mathbf{z}$ are positive, thus all entries of $\rho^{-1}\mathbf{z}$ are non-negative. □

4.2. Memory piles. A *pile* Γ of height $h \geq 2$ is the Auslander-Reiten quiver of a linear Nakayama algebra A with Kupisch series of the form $(1, 2, \dots, h{-}1, h, \dots, h)$. A pile has a unique projective vertex R of length $h-1$, it is called its *radical*. Similarly, it has a unique injective vertex Y of length $h-1$, it is called its *cliff*. If Γ has t summits, we may label the simple modules of A by $S_1, \dots, S_{t+h-1}$, with $\tau S_i = S_{i-1}$ for $i \geq 2$; then IS_{t+1} is the cliff module and the A-modules with socle S_{t+1} form the *cliff ray*.

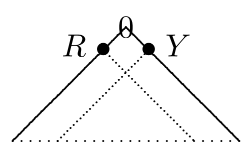
A *memory pile* is a pile with a memory function μ such that $\mu(x) = 0$ for all summits x (actually, we only need to assume that $\mu(x) = 0$ for the first summit x).



Serre subcategories which are piles play an important role in the sequel, in particular the summit pile and the descent piles mentioned in 6.5.

Dealing with a Serre subcategory of a module category may lead to confusion. Whenever it seems necessary, we add a corresponding subscript. If Γ is a pile, and x is a non-projective vertex in Γ, we may write $\Omega_\Gamma x$ for the first syzygy of x in Γ, and $\pi_\Gamma x$ for the Auslander-Reiten translate of x in Γ. Note that for a memory pile (Γ, μ), and x a non-projective vertex of Γ, we have $\mu(\Omega_\Gamma M) = -1 + \mu(M)$. Also, if (Γ, μ) is a memory quiver, and x is a vertex of Γ, we may write $\operatorname{char}_\mu x$ for the characteristic sequence of x with respect to μ.

4.3. Lemma (The module theoretic interpretation of ρ). *If (Γ, μ) is a memory pile with a single summit, with radical R and cliff Y, then $\rho(\operatorname{char} R) = \operatorname{char} Y$.*



Proof. Let $\operatorname{char} R = (z_1, \dots, z_m)$.

First, let $z_2, \dots, z_m$ be odd. Then $\operatorname{char} Y = (z_2, \dots, z_m, x)$ for some x. If z_1 is odd, then $z_1 = \operatorname{pd} \operatorname{soc} R$. Now $Y = IR/\operatorname{soc} R$, thus $\operatorname{pd} Y = 1 + \operatorname{pd} \operatorname{soc} R = 1 + z_1$. This shows that Y is even, therefore $\operatorname{top} Y$ has to be even, and $x = \operatorname{pd} Y = z_1 + 1$. If z_1 is even, then $z_1 = \operatorname{pd} R$. Now $R = \Omega \operatorname{top} Y$, thus $\operatorname{pd} \operatorname{top} Y = 1 + \operatorname{pd} \operatorname{top} R$ is odd, therefore $x = \operatorname{pd} \operatorname{top} Y = 1 + z_1$.

Second, assume that z_v is even, where $v > 1$. Then $z_v = \operatorname{pd} R$. Since $\Omega \operatorname{top} Y = R$, we see that the last entry of $\operatorname{char} Y$ is $z_v + 1$. We have $F_i(Y) = F_{i+1}(R)$ for $1 \leq i < m$. In particular, $F_{v-1}(Y) = F_v(R)$ is even, therefore the entry of $\rho(\operatorname{char} R)$ with index $v - 1$ is $\operatorname{pd} Y = \operatorname{pd} F_1(R) + 1 = z_1 + 1$. The remaining entries are obtained by the obvious index shift from entries of $\mathbf{z}$. □

By induction on t, we obtain:

Corollary. *Let (Γ, μ) be the memory pile with radical R and t summits. Let Y be its cliff. Then* $\operatorname{char} Y = \rho^t \operatorname{char} R$. □

4.4. Shift-Lemma. For all characteristic sequences $\mathbf{z} \in \mathcal{Z}_m$, we have

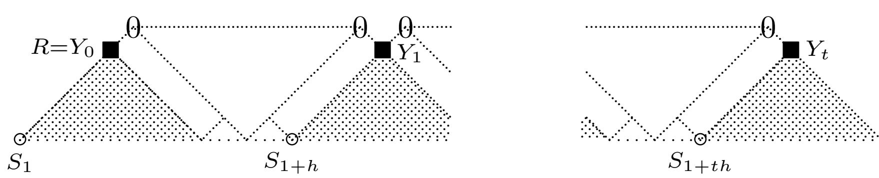
$$\rho^{m+1}(\mathbf{z}) = \mathbf{z} + (2, \dots, 2).$$

Proof. Let Γ be the pile of height $m + 1$ with $m + 1$ summits. Let R be the radical of Γ. We assume that (Γ, μ) is a memory pile with $\operatorname{char}_\mu R = \mathbf{z} = (z_1, \dots, z_m)$.

Let Y be the cliff of Γ. Let $\operatorname{char}_\mu Y = (z'_1, \dots, z'_m)$. If z_i is odd, then $z_i = \operatorname{pd} F_i R$ and $\operatorname{pd} F_i Y = 2 + \operatorname{pd} F_i R$ is also odd, thus $z'_i = \operatorname{pd} F_i Y = 2 + z_i$. If z_i is even, then $\operatorname{pd} F_i R$ is even and $z_i = \operatorname{pd} R$. In this case, also $\operatorname{pd} F_i Y = 2 + \operatorname{pd} F_i R$ is even and $z'_i = \operatorname{pd} Y = 2 + \operatorname{pd} R = 2 + z_i$. Altogether, we see that $z'_i = 2 + z_i$ for all i, thus $\operatorname{char} Y = \operatorname{char} R + (2, 2, \dots, 2)$. □

Corollary. *Let $\mathbf{z} = (z_1, \dots, z_{h-1})$ and $t \geq 1$. Then $\rho^{th}(\mathbf{z}) = \mathbf{z} + t(2, \dots, 2)$.* □

Here is the pile Γ of height h with th summits and with radical R. The black squares mark the vertices $Y_i = \tau_\Gamma^{-ih} R$, the small circles are the vertices $S_{1+ih} = \tau_\Gamma^{-ih}(\operatorname{soc} R)$.



4.5. The piles used in section 6. We will consider piles of height $m+1$ with radical R, cliff Y, and s summits. In (1) and (2), we start with $\operatorname{char} R$ and calculate $\operatorname{char} Y = \rho^s \operatorname{char} R$. In (1′) and (2′), we start with $\operatorname{char} Y$ and calculate $\operatorname{char} R = \rho^{-s} \operatorname{char} Y$.

Let Γ be a pile with radical R, cliff Y and s summits.

(1) *If* $\operatorname{char} R = (0, c_2, \dots, c_m)$, $t \geq 0$ *and* $s = (m + 1)t + 1$, *then, of course, the numbers* c_i *are odd. We have* $\operatorname{char} Y = (c_2 + 2t, \dots, c_m + 2t, 1 + 2t)$.

(1′) *Let* $\operatorname{char} Y = (c'_1, c'_2, \dots, c'_m)$, *with all numbers* c'_i *odd, let* $0 \leq t < \frac{1}{2} c'_i$ *for all* c'_i *and let* $s = (m + 1)t + 1$. *Then* $\operatorname{char} R = (c'_m - 2t - 1, c'_1 - 2t, \dots, c'_{m-1} - 2t)$.

(2) *If* char $R = (c_1, \dots, c_m)$, $t \geq 0$ *and* $s = (m+1)t$, *then* char $Y = (c_1 + 2t, \dots, c_m + 2t)$.

(2′) *Let* char $Y = (c'_1, \dots, c'_m)$, *with all numbers* c'_i *being odd. Let* $0 \leq t < \frac{1}{2}c'_i$ *for all* i *and let* $s = (m+1)t$. *Then* char $R = (c'_1 - 2t, \dots, c'_m - 2t)$.

Proof: Assertion (2) is just the shift lemma 4.4. In order to establish (1), we first use the definition of ρ which asserts that $\rho(0, c_2, \dots, c_m) = (c_2, \dots, c_m, 1)$. Then, according to (2), we have $\rho^{(m+1)t}(c_2, \dots, c_m, 1) = (c_2 + 2t, \dots, c_m + 2t, 1 + 2t)$. Altogether, we get $\rho^{(m+1)t+1}(0, c_2, \dots, c_m) = (c_2 + 2t, \dots, c_m + 2t, 1 + 2t)$.

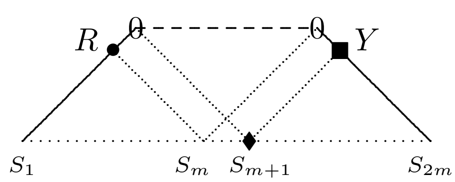
(2′) is again the shift lemma 4.4, but now formulated for ρ^{-1}. It remains to prove (1′). According to (2′), $\rho^{-(m+1)t}(c'_1, \dots, c'_m) = (c'_1 - 2t, \dots, c'_m - 2t)$. We have to apply ρ^{-1} once more. We have $\rho^{-1}(c'_1 - 2t, \dots, c'_m - 2t) = (c'_m - 2t - 1, c'_1 - 2t, \dots, c'_{m-1} - 2t)$. Altogether we see that char $R = (c'_m - 2t - 1, c'_1 - 2t, \dots, c'_{m-1} - 2t)$. □

We will use these assertions in section 6 for a given concave algebra. The assertions (1) and (1′) concern the "summit pile", whereas the assertions (2) and (2′) concern the "descent piles", see 6.5.

4.6. Lemma. *Let* $c_1, \dots, c_m$ *be odd natural numbers. Then*

$$\rho^m(c_1, \dots, c_m) = (c_m + 1, c_1 + 2, \dots, c_{m-1} + 2).$$

Proof. Let R be the radical, Y the cliff of Γ. Let $S_i = \tau_\Gamma^{-i+1} \operatorname{soc} R$ for $1 \leq i \leq 2m$. Let $\rho^m(c_1, \dots, c_m) = (c'_1, \dots, c'_m)$.



We have $\Omega S_{m+1} = R$, and $\operatorname{pd} R = c = \max\{c_1, \dots, c_m\}$. Thus S_{m+1} is even. It follows that $c'_1 = \operatorname{pd} Y$. Since $\Omega Y = S_m$, we have $\operatorname{pd} Y = 1 + c_m$. According to 4.4, we have $\rho^{m+1}(c_1, \dots, c_m) = (c_1 + 2, \dots, c_m + 2)$, therefore $c'_i = c_{i-1} + 2$ for $2 \leq i \leq m$. □

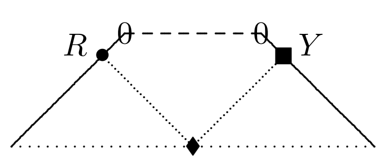
4.7. Lemma. *Let* $m \geq 2$. *Let* $(z_1, \dots, z_m)$ *be a characteristic sequence.*

(a) *Let* z_m *be even. Then*

$$\rho^{m-1}(z_1, \dots, z_m) = (z_{m-1} + 1, z_m + 1, z_1 + 2, \dots, z_{m-2} + 2).$$

(b) *Let* $(z_1, \dots, z_m)$ *be non-projective with* z_1 *even. Then*

$$\rho^{-m+1}(z_1, \dots, z_m) = (z_3 - 2, \dots, z_m - 2, z_1 - 1, z_2 - 1).$$



Proof. We consider the pile (Γ) with $m-1$ summits and radical R, where $\operatorname{char} R = (z_1, \dots, z_m)$. Let Y be its cliff. We denote the simple objects by $S_1, \dots, S_{2m-1}$ with $\tau S_i = S_{i-1}$ for $2 \le i \le 2m-1$. We have $\Omega S_{m+1} = R$, $\Omega Y = S_{m-1}$ and $\Omega^2 S_{i+m-1} = S_{i-2}$ for $3 \le i \le m$.

Since z_m is even, we have $\operatorname{char} R = (z_1, \dots, z_m) = (\operatorname{pd} S_1, \dots, \operatorname{pd} S_{m-1}, \operatorname{pd} R)$ and $\operatorname{char} Y = (\operatorname{pd} Y, \operatorname{pd} S_{m+1}, \dots, \operatorname{pd} S_{2m-1}) = (z'_1, \dots, z'_m)$.

Since $\Omega S_{m+1} = R$, we have $z'_2 = \operatorname{pd} S_{m+1} = \operatorname{pd} R + 1 = z_m + 1$. Since $\Omega Y = S_{m-1}$, we have $z'_1 = \operatorname{pd} Y = \operatorname{pd} S_{m-1} + 1 = z_{m-1} + 1$. Finally, for $3 \le i \le m$, we have $z'_i = \operatorname{pd} S_{i+m-1} = z_{i-2} + 2$, since $\Omega^2 S_{i+m-1} = S_{i-2}$. This yields (a).

Since ρ is invertible, $\rho^{-m+1}(z'_1, \dots, z'_m) = (z_1, \dots, z_m)$, and we have $z_i = z'_{i+2} - 2$, for $1 \le i \le m-2$ and $z_{m-1} = z'_1 - 1$, $z_m = z'_2 - 1$. This yields the assertion (b). $\square$

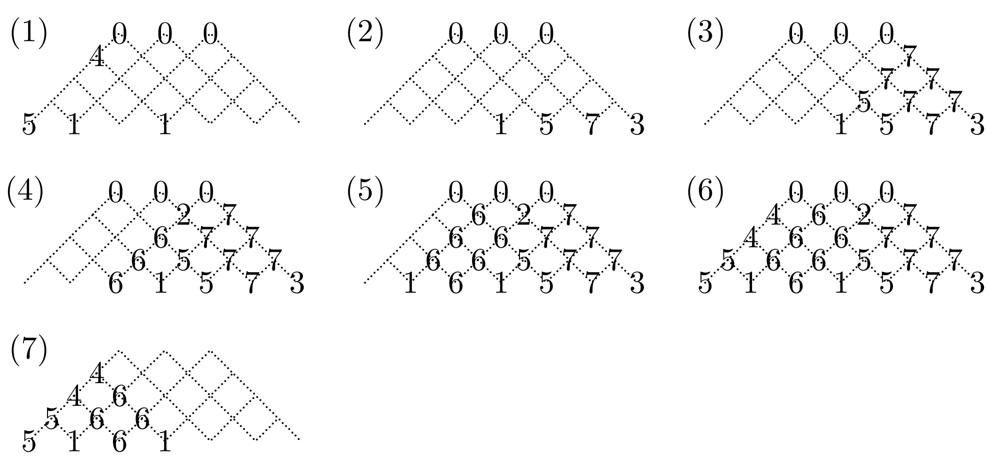
4.8. An algorithm for determining the projective dimension of subfactors of an indecomposable module. Let A be a linear Nakayama algebra and M an indecomposable module with $\operatorname{char} M = (z_1, \dots, z_m)$. We have seen in 1.6 that $\operatorname{char} M$ determines the projective dimension of any subfactor of M, but we have refrained from writing down an explicit formula.

Here we outline an effective algorithm in order to obtain the projective dimension of all subfactors of M. There is no problem in case M is odd, since one just uses the maximum principle. Thus, we can assume that M is **even**. So, let $\mathbf{z} = (z_1, \dots, z_m)$ and assume that some z_v is even. Let Γ be the pile of height $m+1$ with v summits. We want to determine a memory function μ on Γ, with μ being obtained from $\mathbf{z}$ by applying $\rho, \rho^2, \dots, \rho^v$. Let Y be the cliff of Γ, thus

$$\operatorname{char} Y = \rho^v \mathbf{z} = (z_{v+1}, \dots, z_m, z_v{+}1, z_1{+}2, \dots, z_{v-1}{+}2).$$

In particular, *all coefficients of $\rho^v\mathbf{z}$ are odd,* thus the maximum principle yields immediately the values $\mu(x)$ for all subfactors x of Y. We now use downward induction, in order to calculate $\mu(x)$ for the vertices x in the ray with index i, where $1 \le i \le v$.

Example. Let $\mathbf{z} = (5, 1, 4, 1)$. Here $v = 3$ (since $z_3 = 4$ is even). Thus, we have to deal with the pile of height 5 with 3 summits.



We have labeled the pictures. Picture (1) shows the given data (namely, the entries of $\mathbf{z}$ are the projective dimension of M itself and of 3 of the 4 composition factors of M, where

M is an indecomposable module with $\operatorname{char} M = \mathbf{z}$), whereas the last picture (7) shows the projective dimension of all non-zero subfactors of M. The formula for $\rho^3\mathbf{z}$ yields the numbers exhibited in (2) (then we may forget the remaining numbers mentioned in (1); they will systematically be recovered). We use the maximum principle in order to obtain (3). Next, we calculate $\mu(x)$ for the vertices in the third ray, this yields (4). Then we deal with the second ray, this yields (5), and finally with the first ray, this yields (6). The information which we were aiming at is presented in picture (7).

How to get from picture (1) to picture (2)? We have mentioned that one may use the formula for ρ^v. However, the following procedure seems to be more efficient: Note that we present in (1) the characteristic of an **even** module M of length m. Let us label the vertices at the lower boundary by $S_1, S_2, \dots$ going from left to right. In (1) there is given $\operatorname{pd} M$ as well as $\operatorname{pd} S_i$ for $m-1$ composition factors $S_i = F_i(M)$ of M, namely $\operatorname{pd} S_i$ with $1 \le i < v$ as well as $v < i \le m$. Since $\Omega S_{m+1} = M$, we obtain $\operatorname{pd} S_{m+1} = 1 + \operatorname{pd} M$ (in our example, this yields the entry $5 = 1 + \operatorname{pd} M = \operatorname{pd} S_5$ in (2)). For $1 \le i < v$, we obtain $\operatorname{pd} S_{m+i}$ by the formula $\operatorname{pd} S_{m+1+i} = 2 + \operatorname{pd} S_i$, since $\Omega^2 S_{m+1+i} = S_i$ (in our example, this yields the numbers $7 = 2 + \operatorname{pd} S_1 = \operatorname{pd} S_6$, and $3 = 2 + \operatorname{pd} S_2 = \operatorname{pd} S_7$). In this way, we know $\operatorname{pd} S_i$ for all the indices i with $v < i < m+1+v$, thus for m consecutive vertices at the lower boundary. All these numbers $\operatorname{pd} S_i$ are odd, thus we can use the maximum principle in order to get picture (3).

Warning. On a first sight, this procedure can be interpreted as being purely formal! On the other hand, the procedure also has a module theoretic interpretation. But we should stress the following: In case we start with an A-module M and consider "a pile Γ with radical M", say with $s \ge 1$ summits, this pile Γ usually cannot be realized by A-modules, but we have to use B-modules for some related linear Nakayama algebra B. Here is the recipe for obtaining such an algebra B. Let T be the top of M. Let $\operatorname{mod} A_0$ be the Serre subcategory of $\operatorname{mod} A$ generated by the simple modules $\tau^t T$, with $t \ge 0$; this yields a linear Nakayama algebra A_0 and we may consider $M = M_0$ as an A_0-module (note that M, considered as an A_0-module, is injective). Second, we construct inductively s one-point extensions as follows: we start with $A_1 = A_0[M_0]$, then we form $A_2 = A_1[M_1]$, where $M_1 = \tau^-_{A_1} M_0$, and so on, finally let $B = A_s$. Note that the operator Ω mentioned in the previous paragraph is the syzygy functor $\Omega = \Omega_B$ in the category $\operatorname{mod} B$.

4.9. We have seen in Corollary 4 of 1.6: *Let A be a linear Nakayama algebra and M an indecomposable module with* $\operatorname{char} M = (z_1, \dots, z_m)$. *If X is a subfactor of M, then* $\operatorname{pd} X \le 1 + \max z_i$.

The algorithm 4.8 provides a new proof: If M is odd, then we even have $\operatorname{pd} X \le \max z_i$. Thus, let us assume that M is even. Say let $F_v(M)$ be even for some $1 \le v \le m$. Then we have

$$\rho^v(z_1, \dots, z_m) = (z_{v+1}, \dots, z_m, z_v + 1, z_1 + 2, \dots, z_{v-1} + 2),$$

and all these numbers are odd, thus $\operatorname{pd} Y \le 2 + \max z_i$, where Y is the cliff of the pile Γ with radical M and v summits. On the other hand, $F_v(M) = \Omega_\Gamma Y$, therefore $\operatorname{pd} F_v(M) = -1 + \operatorname{pd} Y \le 1 + \max z_i$.

If X is a subfactor of M, then the set of composition factors of X is a subset of the set of composition factors of M, and therefore $\operatorname{pd} X \le \max_i F_i(M) \le 1 + \max z_i$. □

5. Higher Auslander algebras.

5.1. Let A be an artin algebra. By definition, the algebra A is a *higher Auslander algebra* provided $d(A)$ is finite and bounded by the dominant dimension of A. *An artin algebra which is a higher Auslander algebra is either semisimple (and then $d(A) = 0$, and the dominant dimension is infinite), or else the global dimension and the dominant dimension are non-zero, finite and equal.*

Proof. Let A be an artin algebra which is a higher Nakayama algebra. If A is semisimple, then $d(A) = 0$ and the dominant dimension t of A is infinite.

We assume now that $d = d(A) > 0$ and want to show that $d = t$. By assumption, $d \le t$. Assume that $d < t$. Let

$$0 \to {}_AA \xrightarrow{\delta_0} I_0 \xrightarrow{\delta_1} I_1 \to \cdots \xrightarrow{\delta_d} I_d \to \cdots$$

be a minimal injective coresolution of ${}_AA$. Since $d < t$, the modules I_i with $0 \le i \le d$ are projective. In particular,

$$0 \to \operatorname{Cok}(\delta_0) \xrightarrow{\delta_1} I_1 \to \cdots \xrightarrow{\delta_d} I_d \to \operatorname{Cok}(\delta_d) \to 0$$

is a projective resolution of $\operatorname{Cok}(\delta_d)$. Since $d = d(A)$, we see that $\operatorname{Cok}(\delta_0)$ is projective. Thus, δ_0 splits. Since ${}_AA$ is a direct summand of I_0, it is injective. This shows that A is self-injective. A self-injective artin algebra of finite global dimension is semi-simple, thus $d(A) = 0$, a contradiction. □

We are going to show that for a Nakayama algebra which is a higher Auslander algebra A of global dimension d, any indecomposable module M is torsionless or satisfies $\operatorname{pd} M = d$, see Proposition 5.5.

In general, there is a converse implication, see Proposition 5.7: If A is an artin algebra of finite global dimension d and of dominant dimension at least 1, and any indecomposable module M is torsionless or satisfies $\operatorname{pd} M = d$, then A is a higher Auslander algebra.

For the proof of Proposition 5.5, we need three preliminnary results, namely 5.2, 5.3, and 5.4.

5.2. Lemma. *Let A be an artin algebra with dominant dimension $t \ge 2$. Let S be a simple module. Then S is torsionless or $\operatorname{Ext}^i(S, A) = 0$ for $1 \le i < t$.*

Proof. We calculate $\operatorname{Ext}^i(S, A)$ using a minimal injective coresolution of ${}_AA$, say

$$0 \to {}_AA \to I_0 \to \cdots \to I_{t-1} \to I_t \to \cdots .$$

Since the dominant dimension of A is at least t, the modules $I_0, \dots, I_{t-1}$ are projective. Now let $1 \le i < t$. The group $\operatorname{Ext}^i(S, A) \ne 0$ is the homology of the complex $\operatorname{Hom}(S, I_\bullet)$ at the position i:

$$\operatorname{Hom}(S, I_{i-1}) \to \operatorname{Hom}(S, I_i) \to \operatorname{Hom}(S, I_{i+1}).$$

Assume that $\operatorname{Ext}^i(S, A) \ne 0$. Then, we must have $\operatorname{Hom}(S, I_i) \ne 0$. Since $1 \le i < t$, the module I_i is projective, thus S is torsionless. □

5.3. Lemma. *Let A be an artin algebra which is a higher Auslander algebra of global dimension d. If P is indecomposable projective,* $\operatorname{id} P \in \{0, d\}$. *If I is indecomposable injective,* $\operatorname{pd} I \in \{0, d\}$.

Proof. We show the first assertion, the second follows by duality. Let P be indecomposable projective. Assume that $1 \le e = \operatorname{id} P < d$. We take a minimal injective coresolution of P, say

$$0 \to P \to I_0 \to \cdots \to I_{e-1} \to I_e \to 0.$$

Since the dominant dimension of P is d, all the modules $I_0, \dots, I_e$ are projective. In particular, I_e is projective, thus the map $I_{e-1} \to I_e$ is split epi. But this is impossible. □

5.4. Lemma (Madsen). *Let A be a connected Nakayama algebra. Let S be a simple module with* $\operatorname{pd} S = \operatorname{pd} IS$. *Then A has finite global dimension and* $\operatorname{pd} M = \operatorname{pd} S$ *for all modules M with* $\operatorname{soc} M = S$.

Proof. Let M be a module with $\operatorname{soc} M = S$, thus $IM = IS$. According to [6], Theorem 3.6, $\operatorname{pd} S$ is finite. Thus A has finite global dimension (see for example [6], 3.7 (6)). In particular, $\operatorname{pd} M$ is finite, thus even or odd. Now we use [6] 3.2 (a) and (b). If $\operatorname{pd} M$ is odd, then $\operatorname{pd} S \le \operatorname{pd} M$ and $\operatorname{pd} S$ is also odd. Thus $\operatorname{pd} IS$ is odd, and therefore $M \subseteq IS$ shows that $\operatorname{pd} M \le \operatorname{pd} IS$. Similarly, if $\operatorname{pd} M$ is even, then $\operatorname{pd} M \ge \operatorname{pd} IS$ and $\operatorname{pd} IS$ is even. Thus $\operatorname{pd} S$ is even, and therefore $S \subseteq M$ implies that $\operatorname{pd} S \ge \operatorname{pd} M$. □

5.5. Proposition. *Let A be a Nakayama algebra. If A is a higher Auslander algebra of global dimension d, and M is an indecomposable module, then M is torsionless or* $\operatorname{pd} M = d$.

Proof. We assume that A is an artin algebra which is a higher Auslander algebra of global dimension d. If $d = 0$, then A is semi-simple, thus all modules are torsionless.

Thus, we can assume that $d \ge 1$. Let M be an indecomposable module which is not torsionless. We have to show that $\operatorname{pd} M = d$. Since M is not torsionless, it cannot be projective, thus $1 \le \operatorname{pd} M \le d$. If $d = 1$, then $\operatorname{pd} M = d$. Thus, we can assume that $d \ge 2$.

Let $S = \operatorname{soc} M$ and $I = IM = IS$. Since M is not torsionless, also S is not torsionless. Let $e = \operatorname{pd} S$. Since S is not torsionless, it cannot be projective, thus $e \ge 1$. Since the global dimension of A is d, we have $e \le d$. Now $\Omega^e S$ is projective and a minimal projective resolution of S yields a non-zero element in $\operatorname{Ext}^e(S, \Omega^e S)$. This shows that $\operatorname{Ext}^e(S, {}_AA) \ne 0$. Lemma 5.2 implies that $e = d$. According to 5.3, $\operatorname{pd} I = d$. Thus 5.4 asserts that $\operatorname{pd} M = d$. □

Corollary. *Let A be a Nakayama algebra. If A is a higher Auslander algebra of global dimension d, then A is d-closed.* □

5.6. We now want to look at the converse implication: We look at Nakayama algebras such that any indecomposable module M is torsionless or satisfies $\operatorname{pd} M = d$ for some fixed number d, or, equivalently, that any indecomposable module M with $\operatorname{pd} M \ne d$ is torsionless.

Lemma. *Let A be an artin algebra of dominant dimension at least* 1*. Let $t \geq 1$. Assume that any module of projective dimension smaller than t is torsionless. Then the dominant dimension of A is at least t.*

Proof. If $t = 1$, nothing has to be shown. Thus, we may assume that $t \geq 2$. Since the dominant dimension of A is at least 1, the injective envelope of any projective module P is projective, thus $\Sigma^1 P$ has projective dimension at most 1, thus $\Sigma^1 P$ is torsionless and $\Sigma^2 P$ has projective dimension at most 2. By induction, $\Sigma^{t-1} P$ has projective dimension at most $t-1$, thus is torsionless, thus its injective envelope is projective (a torsionless module N is a submodule of a projective module Q, thus the injective envelope of N is a submodule, and therefore a direct summand, of the injective envelope of Q). In this way, we see that the first t terms of a minimal injective coresolution of P are projective. □

5.7. Proposition. *Let A be a artin algebra of finite global dimension and of dominant dimension at least* 1*. Assume that any indecomposable module M is torsionless or satisfies* $\operatorname{pd} M = d$*. Then either $d(A) = 0$, thus A is semi-simple (and the global dimension is* 0*, the dominant dimension is ∞), or else $d(A) > 0$ and both the global dimension $d(A)$ and the dominant dimension of A are equal to d.*

In particular, A is a higher Auslander algebra.

Proof. We assume that every indecomposable module M is torsionless or satisfies $\operatorname{pd} M = d$. Let us show that the global dimension of A cannot be greater than d. Namely, otherwise there exist an indecomposable module M_0 with $\operatorname{pd} M_0 = e > d$. Then M_0 has to be torsionless, say a submodule of the projective module P. Then $M_1 = P/M_0$ satisfies $\operatorname{pd} M_1 = \operatorname{pd} M_0 + 1 = e + 1$. Inductively, we obtain modules M_i with $\operatorname{pd} M_i = e + i$ for all $i \geq 0$. Since by assumption A has finite global dimension, we obtain a contradiction. Thus, $d(A) \leq d$.

First, let us assume that all modules are torsionless, then the injective modules are torsionless, thus A is selfinjective. A selfinjective algebra of finite global dimension is semi-simple.

Second, we assume that there exists a module M which is not torsionless, thus, by assumption $\operatorname{pd} M = d$. Since M is not projective, $d > 0$. Since we know already that $d(A) \leq d$, we see that $d(A) = d$. It remains to be seen that the dominant dimension of A is at least d. By assumption, the dominant dimension of A is at least 1.

According to Lemma 5.6, the dominant dimension of A is at least d. Let us assume that the dominant dimension of A is greater than or equal to $d+1$. Then there is an exact sequence

$$0 \to {}_AA \to I_0 \to I_1 \to \cdots \to I_d \to Z \to 0$$

with all modules I_i injective and projective. Since we know already that the global dimension of A is d, we see that the image of the map $I_0 \to I_1$ is projective. But then the embedding ${}_AA \to I_0$ splits and A is self-injective. But then there is no module with projective dimension $d > 0$. □

5.8. Remark. *Let A be an artin algebra of global dimension $d \geq 1$. Then any torsionless indecomposable module M satisfies* $\operatorname{pd} M < d$*.*

Proof. Let M be indecomposable and torsionless. If M is projective, $\operatorname{pd} M = 0$. Thus we can assume that M is not projective. Since M is torsionless, there is an embedding

$u\colon M \to P$ where P is a projective module. Since M is indecomposable and not projective, we can assume that the image of u is contained in the radical of P. Thus $P \to \operatorname{Cok}(u)$ is a projective cover, therefore $M = \Omega \operatorname{Cok}(u)$ and $\operatorname{pd}\operatorname{Cok}(u) = 1 + \operatorname{pd} M$. Since the global dimension of A is d, we have $\operatorname{pd}\operatorname{Cok}(u) \le d$, thus $\operatorname{pd} M < d$. □

5.9. Remark: The ray property. The modules which are not torsionless form rays: *If the Nakayama algebra A is a higher Auslander algebra of global dimension d, then the modules M in a fixed ray are either all torsionless with* $\operatorname{pd} M < d$ *or else have* $\operatorname{pd} M = d$.

There is also the converse: *If A is a Nakayama algebra such that the modules M in a fixed ray are either all torsionless with* $\operatorname{pd} M < d$ *or else have* $\operatorname{pd} M = d$, *then A is a higher Auslander algebra of global dimension d.*

Proof. The first assertion follows directly from 5.5 and 5.8. Since the dominant dimension of a Nakayama algebra is at least 1, the second assertion follows from 5.7. □

6. d-bound algebras and d-piles.

6.1. Lemma. *Let A be a linear Nakayama algebra with global dimension at most d. The following conditions are equivalent.*
(i) *A is d-closed.*
(ii) $\operatorname{pd} I \in \{0, d\}$ *for all indecomposable injective modules I.*

If A is a linear Nakayama algebra with global dimension at most d and the (equivalent) conditions of Lemma are satisfied, then A is said to be *d-bound.* According to Corollary 5.5, a higher Auslander algebra of global dimension d is d-bound. But there are other d-bound algebras, see the last algebras exhibited in Remark 1.8 (1) and (3).

Proof of Lemma. (i) implies (ii). We assume that A is d-closed and that I is indecomposable injective. If I is projective, then $\operatorname{pd} I = 0$. Thus, let I be non-projective and $S = \operatorname{soc} I$. Since S is not torsionless, and A is d-closed, there is a module M with $\operatorname{soc} M = S$ and $\operatorname{pd} M \ge d$. Since the global dimension of A is bounded by d, we have $\operatorname{pd} Z \le d = \operatorname{pd} M$ for all subfactors Z of M. According to Lemma 1.7, I is pd-controlled and $\operatorname{pd} I = d$.

(ii) implies (i). Let S be simple, and $I = IS$. According to (ii), we have $\operatorname{pd} I = 0$, thus S is torsionless, or else $\operatorname{pd} I = d$. This shows that A is d-closed. □

6.2. Lemma. *Let (Γ, μ) be a memory pile with cliff Y. Let e be the maximum of $\mu(Y')$, where Y' is a subfactor of Y. If X is not a subfactor of Y, then $\mu(X) < e$.*

Proof. Assume that X is not a subfactor of Y. If X is a summit, then $\mu(X) = 0$. Otherwise $X = \Omega X'$ for some vertex X' of the pile, which again is not a summit. By induction, we see that $X = \Omega^i Y'$ for a subfactor Y' of Y and $i \ge 1$. Therefore $\mu(X) = \mu(Y') - i < \mu(Y') \le e$. □

Definition of a d-pile. A memory pile (Γ, μ) is said to be a *d-pile* provided the cliff module Y is pd-controlled and $\operatorname{pd} Y = d$ (thus, provided $d = \mu(Y) \ge \mu(Z)$ for all subfactors Z of Y, or, equivalently, provided the maximum of the values $\mu(U)$, where U is a submodule of the cliff Y, is d, and second, $\mu(Z) \le d$ for all non-zero subfactors Z of Y).

Corollary. *Let (Γ, μ) be a d-pile, with cliff module Y. If X is a vertex of Γ and not a subfactor of Y, then $\mu(X) < d$.* $\square$

6.3. The radical of a d-pile. We are going to characterize the radicals of the d-piles.

Lemma. *Let A be a linear Nakayama algebra and R an indecomposable module.*

(a) *The module R is the radical of a d-pile iff $\operatorname{pd} U < d$ for all submodules U of R and $\operatorname{pd} X \le d$ for all subfactors X of R.*

(b) *If R is the radical of a d-pile (Γ, μ), then (Γ, μ) is uniquely determined by* $\operatorname{char} R$.

Proof. First, let (Γ, μ) be a d-pile and R its radical. Since $\mu(x) \le d$ for all vertices of Γ, we have $\operatorname{pd} X \le d$ for all subfactors X of R. Also, if U is a submodule of R, then there is $z \in \Gamma$ such that $U = \Omega_\Gamma z$ and therefore $\operatorname{pd} U = -1 + \mu(z) < d$.

Conversely, let us assume that R is an indecomposable module of length m such that $\operatorname{pd} U < d$ for all submodules U of R and that $\operatorname{pd} X \le d$ for all subfactors X of R. We want to show that R is the radical of a d-pile. We may assume that $\operatorname{top} R = \omega_A$. We consider the one-point extension $A[R]$, say with extension vertex S. Then $\operatorname{rad} PS = R$. Since all submodules of R have projective dimension at most $d-1$, the factor modules of PS have projective dimension at most d. It follows that all subfactors of PS have projective dimension at most d.

Let $\mathbf{z} = \operatorname{char} PS$ and $C = C_d(A(\mathbf{z}))$. We have $|PS| = m+1$. Let s be minimal such that $|P(\tau^{-s}S)| \le m$, and let $Q = P(\tau^{-s+1}S)$. Note that all the projective modules $P(\tau^{-i}S)$ with $0 \le i < s$ have length $m+1$. The Serre subcategory generated by these modules $P(\tau^{-i}S)$ with $0 \le i < s$ is a pile Γ with s summits.

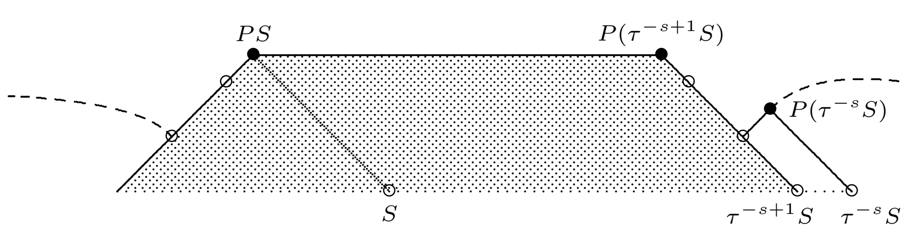
We claim that Γ is a d-pile. Since all subfactors of PS have projective dimension at most d, the definition of E_d shows that the same is true for all subfactors of $P(\tau^{-i}S)$ with $i \ge 0$. Also, let Y be the cliff of Γ, thus $Y = Q/\operatorname{soc} Q$. Then there has to be a submodule U of Y of projective dimension d, since C is d-closed and Y is not torsionless.

Finally, we have to show that Γ is the unique d-pile with radical R. We show this by induction on the number of summits s of Γ. If $s = 1$, then Γ is the pile with radical R and one summit. Now assume that $s \ge 2$. Then R determines uniquely the module $R' = \pi_\Gamma^- R$, and we have $\operatorname{char} R' = \rho(\operatorname{char} R)$. Since $s \ge 2$, we see that R' is the radical of a d-pile Γ' with $s-1$ summits, and by induction Γ' is uniquely determined by R'. $\square$

6.4. Proposition. *Let A be a d-bound linear Nakayama algebra. Let S be a simple module and $s \ge 1$ such that*

$$|PS| = |P(\tau^- S)| = \cdots = |P(\tau^{-s+1}S)| > |P(\tau^{-s}S|.$$

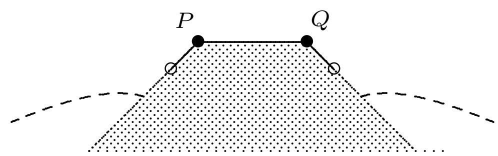
Let $\mathcal{C}$ be the Serre subcategory of $\operatorname{mod} A$ *generated by the modules $P(\tau^{-i}S)$ with $0 \le i < s$. Let Γ be the Auslander-Reiten quiver of $\mathcal{C}$. Then $(\Gamma, \operatorname{pd}_A)$ is a d-pile.*



Proof: Of course, Γ is a pile. Since A has global dimension at most d, we have $\mu(x) \le d$ for all vertices x of Γ. Let Y be the cliff of Γ, thus $Y = P(\tau^{-s+1}S)/\operatorname{soc} P(\tau^{-s+1}S)$. Then Y is not torsionless. Since A is d-closed, there is a submodule U of Y with $\operatorname{pd} U = d$. $\square$

6.5. Dealing with a Nakayama algebra A, Serre subcategories which are piles play an important role. In the present paper, we are mainly interested in concave algebras. Such an algebra has a summit pile and usually several descent piles, defined as follows.

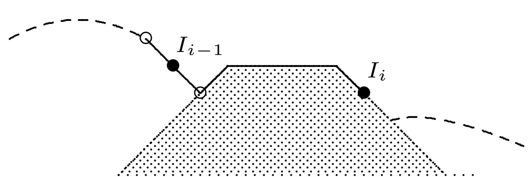
The summit pile. Let A be concave with first summit P and last summit Q. The non-zero subfactors of the indecomposable injective modules which are successors of P and predecessors of Q form a pile, the *summit pile* of A.



Clearly, *the cliff module of the summit pile is the principal cliff module.*

The descent piles. Let A be concave. Let h be the height of A. Let $0 \le i \le h-2$. It is easy to see that there is a unique indecomposable injective module I_i of length $h-i-1$ which is not projective. The module I_0 is the principal cliff module, and we have $I_{h-2} = \omega_A$.

Assume that $i \ge 1$ and that I_i is not a factor module of I_{i-1}. Then the non-zero subfactors of the indecomposable injective modules which are successors of $I_{i-1}/\operatorname{soc} I_{i-1}$ and predecessors of I_i form a pile. These piles are called the *descent piles* of A.



Corollary. *Let A be a d-bound concave Nakayama algebra. Then the summit pile and the descent piles are d-piles.* $\square$

6.6. Lemma. *Let A be a d-bound concave Nakayama algebra and R an indecomposable module. Let Γ be a d-pile with radical R.*

(a) *If $R = \operatorname{rad} P$, where P is the first summit, then Γ is the summit pile.*

(b) *If R is not injective, and $R = I/\operatorname{soc} I$, where I is indecomposable injective and not projective, then Γ is a descent pile.*

Proof. (a) Let $R = \operatorname{rad} P$ with P the first summit. The summit pile is a d-pile with radical R. According to 6.3 (b), there is at most one d-pile with a given radical.

(b) Assume that R is not injective, and $R = I/\operatorname{soc} I$, where I is indecomposable injective and not projective. The corresponding descent pile with radical R is a d-pile. According to 6.3 (a), there is at most one d-pile with a given radical. $\square$

6.7. Remark. We have seen in 6.2: Given a pile (Γ, μ), the maximum of μ will be obtained only on subfactors of the cliff. On the other hand, one should be aware that *any indecomposable non-projective module M occurs as the cliff of a uniquely determined pile, namely of the summit pile of $A(\operatorname{char} M)$.*

7. The case d odd.

In this section, d is always an odd natural number.

First, we present a general property of odd modules over Nakayama algebras which are higher Auslander algebras of odd global dimension.

Let A be a Nakayama algebra. An indecomposable module N will be said to be *decreasing* provided N is odd and $\operatorname{char} N$ is a decreasing sequence of odd numbers. An indecomposable module M will be said to be *plus-decreasing* provided $\rho \operatorname{char} M$ is a decreasing sequence of odd numbers.

7.1. Proposition. *Let A be a Nakayama algebra which is a higher Auslander algebra with odd global dimension. Any odd indecomposable module is decreasing.*

Proof. We assume that A is a Nakayama algebra which is a higher Auslander algebra with odd global dimension d. Let us first show the following property:

(a) *If M' is a non-zero submodule of an odd indecomposable module M, then* $\operatorname{pd} M' = \operatorname{pd} M$. We use downward induction on $e = \operatorname{pd} M$. Since A has global dimension d, we have $e \le d$. Since M is odd, we have $\operatorname{pd} M' = e' \le e$. If $e = d$, then M cannot be torsionless, since otherwise IM is projective and $\operatorname{pd} IM/M = \operatorname{pd} M + 1 > d$. Thus, also M' is not torsionless. According to 5.5, $\operatorname{pd} M' = d$.

Now assume that $e < d$. Then, according to 5.5, M is torsionless, thus IM is projective. There are non-split short exact sequences $0 \to M \to IM \to \Sigma M \to 0$ and $0 \to M' \to IM \to \Sigma M' \to 0$. Since IM is projective, $\operatorname{pd} \Sigma M = e + 1$ and $\operatorname{pd} \Sigma M' = e' + 1$. Since $e' + 1 \le e + 1 < d$, the modules ΣM and $\Sigma M'$ both are again torsionless (and non-projective), thus there are non-split short exact sequences $0 \to \Sigma M \to I\Sigma M \to \Sigma^2 M \to 0$ and $0 \to \Sigma M' \to I\Sigma M' \to \Sigma^2 M' \to 0$. Since $I\Sigma M$ and $I\Sigma M'$ both are projective, $\operatorname{pd} \Sigma^2 M = e + 2$ and $\operatorname{pd} \Sigma^2 M' = e' + 2$. Note that $\Sigma^2 M'$ is a (non-zero) submodule of $\Sigma^2 M$. By induction, $e' + 2 = \operatorname{pd} \Sigma^2 M' = \operatorname{pd} \Sigma^2 M = e + 2$ and therefore $e' = e$. This completes the proof of (a).

There is the following consequence: (b) *If S is a simple module and both S and τS are odd, then* $\operatorname{pd} \tau S \ge \operatorname{pd} S$.

Namely, if S is simple and not projective, then there exists an indecomposable module M of length 2 with $\operatorname{top} M = S$. Now $\operatorname{soc} M = \tau S$ and if both S and τS are odd, then the maximum principle shows that also M is odd, with $\operatorname{pd} M = \max\{\operatorname{pd} S, \operatorname{pd} \tau S\}$. According to (a), $\operatorname{pd} M = \operatorname{pd} \tau S$, thus $\operatorname{pd} \tau S = \operatorname{pd} M \ge \operatorname{pd} S$.

The assertion mentioned in the Proposition is an immediate consequence: Let M be an odd indecomposable module, say with $S = \operatorname{top} M$. Let $\operatorname{char} M = (c_1, \dots, c_m)$. By the definition of a characteristic sequence, $c_i = \operatorname{pd} \tau^{m-i} S$, for $1 \le i \le m$. Since M is odd, the maximum principle asserts that all the numbers c_i are odd. According to (b), we have $\operatorname{pd} \tau^{m-i} S \ge \tau^{m-i-1} S$ for $1 \le i < m$. But this just means that $c_i \ge c_{i+1}$ for $1 \le i < m$. $\square$

Let us stress that the property described in Proposition 7.1 has some interesting consequences (and actually is equivalent to these properties, see our proof of 7.1), namely: (a) *If M' is a non-zero submodule of an odd indecomposable module M, then* $\operatorname{pd} M' = \operatorname{pd} M$. And: (b) *If S is a simple module and both S and τS are odd, then* $\operatorname{pd} \tau S \geq \operatorname{pd} S$.

Corollary. *Let d be odd. Let A be a Nakayama algebra which is a higher Auslander algebra of global dimension d. Let I be indecomposable injective, not projective. Then* $\operatorname{pd} I = d$ *and I is decreasing.* □

7.2. Proposition. *Let $d \geq c_1 \geq c_2 \geq \cdots \geq c_u$ be odd numbers. Then the algebra $C_d A(0, c_1, \dots, c_u, 1)$ is a concave higher Auslander algebra of global dimension d.*

Proof. Let $A = A(0, c_1, \dots, c_u, 1)$ and $H = C_d A$. According to 2.3, the algebra A is ascending. By definition, H is a descending extension of A, thus H is concave. We show that the cliff modules of the summit pile and of the descent piles have projective dimension d and decreasing characteristic sequences.

We start with the summit pile. Its radical R has $\operatorname{char} R = (0, c_1, \dots, c_u)$ with decreasing odd numbers $c_1 \geq c_2 \geq \cdots \geq c_u$. Thus, let $t = \frac{1}{2}(d - c_1)$ and $s = (u+2)t + 1$ and let Γ be the pile with radical R and s summits. Then, according to 4.5 (1), the cliff module Y of Γ has $\operatorname{char} Y = (c_1 + 2t, \dots, c_u + 2t, 1 + 2t)$, thus $\operatorname{char} Y$ is decreasing. Now, all the entries are odd, $c_1 + 2t = d$, and the remaining entries of $\operatorname{char} Y$ are bounded by d. Thus, we see that Y is odd, thus pd-controlled. Also, $\operatorname{pd} Y = d$. It follows that Γ is a d-pile. According to 6.6, Γ is the summit pile of H. In this way, we have shown that the cliff module of the summit pile has projective dimension d and a decreasing characteristic sequence.

Since A is concave and of height h, there is a unique indecomposable injective non-projective module I_i of length $h - i$ for $1 \leq i \leq h - 1$. We use induction on i in order to show that I_i has projective dimension d and a decreasing characteristic sequence. We have already seen that I_1 has projective dimension d and that $\operatorname{char} I_1$ is a decreasing sequence. Now assume we know that $\operatorname{pd} I_i = d$ and that $\operatorname{char} I_i$ is a decreasing characteristic sequence, for some $1 \leq i < h - 1$. Let $\operatorname{char} I_i = (c_1, \dots, c_{h-i})$. Of course, $c_1 = d$. There are two possibilities.

First case: I_{i+1} is a factor module of I_i, thus $I_{i+1} = I_i / \operatorname{soc} I_i$. In this case $\operatorname{char} I_{i+1} = (c_2, \dots, c_{h-1})$ is a decreasing sequence and $\operatorname{pd} I_{i+1} = c_2$. Since c_2 is odd, $\operatorname{pd} U \leq c_2$ for all submodules U of I_{i+1}. Since A is d-closed and I_{i+1} is not torsionless, we must have $c_2 = d$.

Second, I_{i+1} is not a factor module of I_i. Let $R_i = I_i / \operatorname{soc} I_i$. Then there is the pile Γ with radical R_i and with s summits, where $t = \frac{1}{2}(d - c_2)$ and $s = (h - i)t$. According to 4.5 (2), the cliff module of Γ is Y_i with $\operatorname{char} Y_i = (c_2 + 2t, \dots, c_{h-i} + 2t)$. Again, we use 6.6. It asserts that Γ is the descent pile of A with radical R_i, thus $Y_i = I_{i+1}$. We see that $\operatorname{char} I_{i+1} = (c_2 + 2t, \dots, c_{h-i} + 2t)$ is a decreasing sequence. Also, $\operatorname{pd} I_{i+1} = c_2 + 2t = d$.

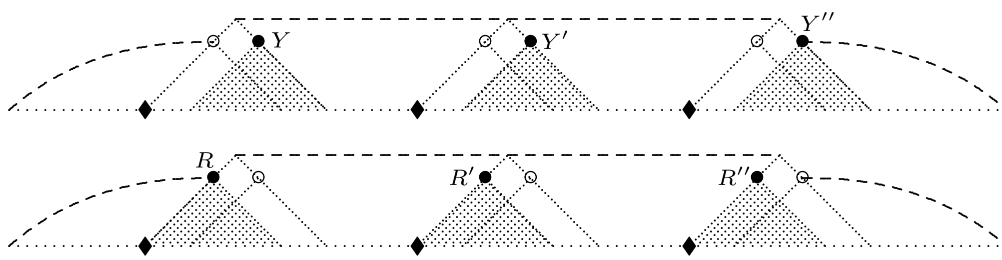
Let I be an indecomposable module which is injective and not projective. As we have seen, $\operatorname{pd} I = d$ and $\operatorname{char} I$ is decreasing. Now, if M is an odd module with decreasing characteristic sequence, then $\operatorname{pd} U = \operatorname{pd} M$ for any non-zero submodule U of M. This shows that any non-zero submodule of I has projective dimension d. According to 5.7, H is a higher Auslander algebra of global dimension d. □

7.3. Lemma. *Let A be a Nakayama algebra. Let M be an indecomposable module* *which is plus-decreasing. Then* $\operatorname{soc} M$ *is even. If* U *is a submodule of* M*, then* $\operatorname{pd} U \leq \operatorname{pd} \operatorname{soc} M$.

Proof. Let $\rho \operatorname{char} M = (c_1, \ldots, c_m)$, then $c_1 \geq \cdots \geq c_m$, and all these numbers are odd. Let (Γ, μ) be the pile with a unique summit and radical M. The maximum principle asserts that $\mu(\tau_\Gamma^- M) = c_1$ and that $\mu(y) \leq c_1$ for all vertices y which do not lie on the first ray, thus $\mu(x) \leq c_1 - 1$ for all vertices x on the first ray. Also, $\mu(\tau_\Gamma^- M) = c_1$ implies that $\operatorname{pd} \operatorname{soc} M = c_1 - 1$. Altogether, we see: If U is a non-zero submodule of M, then $\operatorname{pd} U \leq c_1 - 1 = \operatorname{pd} \operatorname{soc} M$. □

7.4. Proposition. *Let* d *be odd and* $m \geq 1$*. Let* A *be a concave Nakayama algebra of height* $m+1$ *which is* d*-bound. The following assertions are equivalent:*
(1) *A is a higher Auslander algebra of global dimension d.*
(2) *A has a decreasing module Y of length m with* $\operatorname{pd} \operatorname{top} Y = 1$.
(2′) *A has a decreasing module Y' of length m.*
(2″) *A has an injective decreasing module Y'' of length m.*
(3) *A has a projective plus-decreasing module R of length m.*
(3′) *A has a plus-decreasing module R' of length m.*
(3″) *A has a plus-decreasing module R'' of length m with* $\operatorname{pd} \operatorname{soc} R'' = d - 1$.
(4) $A = C_d A(0, c_1, \ldots, c_{m-1}, 1)$ *with odd numbers* c_i *such that* $d \geq c_1 \geq c_2 \geq \cdots \geq c_{m-1}$.

For A a concave Nakayama algebra of height $m+1$ which is a higher Auslander algebra of global dimension d, the assertions (2) to (3″) concern the indecomposable modules N of length m such that N or $\tau^- N$ is odd. Here are sketches which show the possible positions of such modules $N = Y'$ or $N = R'$:



Proof of Proposition. We assume that A is a concave Nakayama algebra of height $m+1$, that A is d-closed and has global dimension at most d.

(1) implies (2″). Here we assume in addition that A is a higher Auslander algebra of global dimension d. Let Y'' be the principal cliff module. Then Y'' is indecomposable injective and not projective. Thus Y'' is decreasing by Corollary 7.1.

(2″) implies (2′). Trivial.

(2′) implies (2): Proof. Let Y' be a decreasing module of length m. Let $(c'_1, \ldots, c'_m)$ be the characteristic of Y'. Then Y' is odd and $c'_1 \geq c'_2 \geq \cdots \geq c'_m$. Let $t = \frac{1}{2}(c'_m - 1)$, thus t is an integer with $0 \leq t < \frac{1}{2} c'_i$ for all c'_i. Let $s = (m+1)t$. We consider the pile Γ with s summits and cliff Y'. Let Y be its radical. According to 4.5 (2′), $\operatorname{char} Y = (c'_1 - 2t, \ldots, c'_m - 2t)$ and $c'_m - 2t = 1$, by definition of t. This shows that Y is decreasing with $\operatorname{pd} \operatorname{top} Y = 1$.

(2) implies (3): Since Y is odd, Y is not projective. Let $R = \tau Y$. Then R is indecomposable, not injective, and $Y = \tau^- R$ is decreasing, thus R is plus-decreasing. Also, $R = \operatorname{rad} PY$. Since $\operatorname{pd}\operatorname{top} Y = \operatorname{pd}\operatorname{top} PY = 1$, we see that R is projective.

(3) implies (3′). Trivial.

(3′) implies (3″): Here we use again the shift lemma. Let R' be plus-decreasing of length m. Let $e = \operatorname{pd}\operatorname{soc} R'$. According to Lemma 7.3, all submodules U of R' have $\operatorname{pd} U \le e$. Let $t = \frac{1}{2}(d-1-e)$ and $s = (m+1)t$. Let (Γ, μ) be the pile with $s = 2t$ summits and radical R'. Now A is d-closed and has global dimension at most d. Since the height of A is $m+1$ and the length of R' is m, we see that Γ is part of the Auslander-Reiten quiver of A. It follows that $R'' = \tau^{-s} R'$ is indecomposable with $\operatorname{char} R'' = \rho^s R'$. The shift lemma asserts that R'' is plus-decreasing and that $\operatorname{pd}\operatorname{soc} R'' = e + 2t = d - 1$.

(3″) implies (2″): Let R'' be a plus-decreasing module of length m with $\operatorname{pd}\operatorname{soc} R'' = d - 1$. Let $\rho \operatorname{char} R'' = (c_1, \dots, c_m)$, thus $c_1 \ge \cdots \ge c_m$ are odd numbers. Let (Γ, μ) be the pile with a unique summit and radical R''. The maximum principle asserts that $\mu(\tau_\mu^- R'') = c_1$, thus $c_1 = 1 + \operatorname{pd}\operatorname{soc} R'' = d$. Since A is d-closed, R'' is not injective. Since A has height $m+1$ and R'' has length m, the injective envelope of R'' has length $m+1$. As a consequence, $Y'' = \tau^- R''$ is the cliff of Γ and $\operatorname{char} Y'' = \rho^- \operatorname{char} R'' = (c_1, \dots, c_m)$ is decreasing and $\operatorname{pd} Y'' = d$. Since the global dimension of A is at most d, the module Y'' has to be injective.

(2) implies (4). As in the proof that (2) implies (3), we consider the projective cover PY of Y. This is a summit, and since $R = \operatorname{rad} PY$ is projective, PY is the first summit. Let $\operatorname{mod} B$ be the predecessors of $\operatorname{top} P$. This is an ascending algebra, namely $B = A(P)$. Since C is a concave higher Auslander algebra of global dimension d, it follows that $C = C_d(B)$. On the other hand, let $\operatorname{char} Y = (c_1, \dots, c_m)$. Since Y is odd, we must have $\operatorname{char} P = (0, c_1, \dots, c_m)$. As we know, $d \ge c_1 \ge c_2 \ge \cdots \ge c_m = 1$.

(4) implies (1). This is 7.2. □

7.5. Proof of Theorems 1, 2, and 3. First, let $d \ge c_1 \ge \cdots \ge c_u$ be odd numbers and $H = H_d(c_1, \dots, c_u) = C_d A(0, c_1, \dots, c_u, 1)$. Now $A(0, c_1, \dots, c_u, 1)$ is ascending, thus $C_d A(0, c_1, \dots, c_u, 1)$ is concave. The equivalence of (4) and (1) in 7.4 yields Theorems 1 and 2.

Let P be the first summit of H and $R = \operatorname{rad} P$. By construction, we have $\operatorname{char} P = (0, c_1, \dots, c_u, 1)$ and $\operatorname{char} R = (0, c_1, \dots, c_u)$. If $u = 0$, then the Auslander-Reiten quiver of H is just the pile of height 2 with d summits, and $\operatorname{char} Q = (0, d)$. Thus, we assume that $u \ge 1$. Let $t = \frac{1}{2}(d - c_1)$ and $s = (u+2)t$. The shift lemma asserts that $\tau^{-s} R$ has characteristic $(2t, c_1 + 2t, \dots, c_u + 2t)$. It follows that $Y = \tau^{-s-1} R$ has $\operatorname{char} Y = \rho \operatorname{char} \tau^s Y = (c_1 + 2t, \dots, c_u + 2t, 2t + 1)$. Of course, Y is the principal cliff module and its projective cover Q (the last summit) has $\operatorname{char} Q = (0, c_1 + 2t, \dots, c_u + 2t, 2t + 1)$. □

7.6. The characterizations presented in 7.4 refer to the existence of an indecomposable module of length $h - 1$, where h is the height of the algebra. Actually, it is sufficient to assume the existence of a corresponding sequence of simple modules:

Proposition. *Let A be a concave Nakayama algebra of height h and let d be an odd integer.*

First, let A be a higher Auslander algebra with global dimension d. Then A is d-closed. If S is the top of the principal cliff module, then $d \geq \operatorname{pd}\tau^{h-2}S \geq \cdots \geq \operatorname{pd}\tau S \geq \operatorname{pd}S$ are odd numbers.

Conversely, assume that A is d-closed and that A has global dinension d. Let S be a simple module S such that $d \geq \operatorname{pd}\tau^{h-2}S \geq \cdots \geq \operatorname{pd}\tau S \geq \operatorname{pd}S$ are odd numbers. Then A is a higher Auslander algebra with global dimension d.

Proof. First, assume that A is a higher Auslander algebras of odd global dimension d and height h. According to Proposition 7.4, A is d-closed and there is an indecomposable module M with $\operatorname{char} M = (c_1, \dots, c_{h-1})$, where $d \geq c_1 \geq \cdots \geq c_{h-1}$ is a sequence of odd numbers. Let $S = \operatorname{top} M$. Then $\operatorname{pd}\tau^i S = c_{h-1-i}$ for $0 \leq i \leq h-2$. These numbers are odd and bounded by d.

Conversely, assume that A is d-closed, has global dimension d and height h, with a simple module S such that

$$d \geq \operatorname{pd}\tau^{h-2}S \geq \cdots \geq \operatorname{pd}\tau S \geq \operatorname{pd}S$$

are odd numbers. According to 1.1 (4), we have $|PS| \geq h$. Since h is the height of A, we have $|PS| = h$. Let $Y = PS/\operatorname{soc}PS$. Then $\operatorname{char} Y = (c_1, \dots, c_{h-1})$ with $d \geq c_1 \geq \cdots \geq c_{h-1}$, thus 7.4 asserts that A is a higher Auslander algebra of global dimension d. □

8. The case d even.

In this section, d always will be even.

8.1. Proposition. *Let A be a Nakayama algebras which is a higher Auslander algebra with even global dimension.*

(1) *There is no chain $P_1 \subset P_2 \subset P_3$ of indecomposable projective modules.*
(2) *There is no composition $I_1 \to I_2 \to I_3$ of proper epimorphisms between indecomposable injective modules.*

Proof. We show the second assertion (2); the first one follows by duality. We assume that A is a higher Auslander algebra with global dimension d being even and that there is a composition $I_1 \to I_2 \to I_3$ of proper epimorphisms between indecomposable injective modules. Then I_2 and I_3 are not projective. We may assume that the kernel S of $I_2 \to I_3$ is simple. Then $\tau^- S$ is the socle of I_3. Both modules S and $\tau^- S$ are not torsionless, thus $\operatorname{pd}S = d = \operatorname{pd}\tau^- S$, according to 5.5. In particular, both module $S, \tau^- S$ are even. This is a contradiction to 1.2 (1). □

8.2. The indecomposable module Y is called *plus-strictly-increasing* provided $\operatorname{char} Y = (e, c_2, \dots, c_m)$ with e even and $c_2 < c_3 < \cdots < c_m < e$, or, equivalently, provided $\rho \operatorname{char} Y$ has strictly increasing odd entries. The indecomposable module R is called *minus-strict-increasing* provided $\operatorname{char} R = (c_1, \dots, c_{m-1}, e)$ with e even and $e - 1 < c_1 < \cdots < c_{m-1}$, or, equivalently, provided $\rho^{-1} \operatorname{char} R$ has odd strictly increasing entries (since we allow that $e = 0$, the first entry of $\rho^{-1} \operatorname{char} R$ can be negative). Note that a simple module is plus-strictly-increasing iff it is even iff it is minus-strictly-increasing.

8.3. Proposition. *Let d be even. Let A be a concave Nakayama algebra which is a higher Auslander algebra of global dimension d. Let I be indecomposable injective, not projective and of length at least 2. Then I is plus-strictly-increasing with* $\operatorname{pd} I = d$.

Proof. Note that $\operatorname{soc} I$ is not torsionless, therefore 5.5 asserts that $\operatorname{pd}\operatorname{soc} I = d$. Since $\operatorname{soc} I$ is even, $N = I/\operatorname{soc} I$ has to be odd.

We use induction on $|I|$ in order to show that $\operatorname{char} N$ is strictly increasing. If $|I| = 2$, nothing has to be shown. Thus, let $m = |I| \geq 3$. Let $\operatorname{char} N = (c_2, \dots, c_m)$. Thus $\operatorname{pd} N = c = \max c_i$.

Let Γ be the descent pile with radical N. Let s be the number of summits of Γ. Let us denote the simple modules which belong to Γ by $S_1, S_2, \dots, S_{s+m-1}$, going from left to right. According to 5.5 and 5.8, we have $\operatorname{pd} S_i < d$, for $1 \leq i \leq s$, and $\operatorname{pd} S_{s+1} = d$. The characteristic of N asserts that $\operatorname{pd} S_i = c_{i+1}$ for $1 \leq i < m$ and we have $\operatorname{pd} S_m = c+1$, since $\Omega S_m = N$. For $i > m$, we have $\Omega^2 S_i = S_{i-m}$, thus $\operatorname{pd} S_i = 2 + \operatorname{pd} S_{i-m}$. It follows that $\operatorname{pd} S_i$ is even iff m divides i. Since $\operatorname{pd} S_{s+1} = d$ is even, we must have $s+1 = (t+1)m$ for some $t \geq 0$. Let Y be the cliff module of Γ, thus $\operatorname{char} Y = \rho^s(c_2, \dots, c_m)$ and $s = tm+m-1$. According to the shift lemma, $\rho^{tm} \operatorname{char} N = \operatorname{char} N + t(2, \dots, 2) = (c'_2, \dots, c'_m)$. We use Lemma 4.6 and get

$$\begin{aligned}\operatorname{char} Y &= \rho^s(c_2, \dots, c_m) = \rho^{m-1}(c'_2, \dots, c'_m) \\ &= (c'_m + 1, c'_2 + 2, \dots, c'_{m-1} + 2) \\ &= (c_m + 2t + 1, c_2 + 2t + 2, \dots, c_{m-1} + 2t + 2).\end{aligned}$$

The first coefficient $c_m + 2t + 1$ is even, thus equal to $\operatorname{pd} Y = d$. Since the global dimension of A is d, all the coefficients are bounded by d, thus for $2 \leq i \leq m$, we have $c_i + 2t + 2 \leq d = c_m + 2t + 1 < c_m + 2t + 2$, and therefore $c_i < c_m$.

Since Γ is a descent pile of A, the module Y is injective and not projective. Since $|Y| = |N| = |I| - 1$, we know by induction that $Y/\operatorname{soc} Y$ is odd with strictly increasing characteristic sequence. Now $\operatorname{char}(Y/\operatorname{soc} Y) = (c_2 + 2t + 2, \dots, c_{m-1} + 2t + 2)$, thus $c_2 < \cdots < c_{m-1}$. Since we know already that $c_{m-1} < c_m$, we see that $\operatorname{char} N$ is strictly increasing.

Altogether, we see that $\operatorname{char} I = (d, c_2, \dots, c_m)$ with $c_2 < \cdots < c_m < d$. □

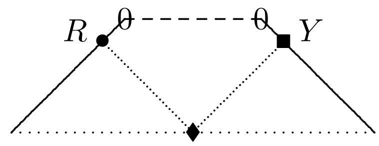
8.4. Lemma. *Let A be a Nakayama algebra of height $m+1$.*

(a) *If Y is an indecomposable module of length m which is plus-strictly-increasing, then $\tau^{m-1} Y$ is minus-strictly-increasing.*

(b) *If R is an indecomposable module of length m which is minus-strictly-increasing, then $\tau^{-m+1} R$ is plus-strictly-increasing or zero.*

(c) *Let Y be an indecomposable module which is plus-strictly-increasing. If U is a non-zero submodule of Y, then* $\operatorname{pd} U = \operatorname{pd} Y$. *If Z is a subfactor of Y, then* $\operatorname{pd} Z \leq \operatorname{pd} Y$. *In particular, Y is* pd*-controlled.*

The relationship between the modules R and Y considered in (a) and (b) is seen in the following picture (it is important that $\operatorname{top} R = \operatorname{soc} Y$ is even):



Proof of Lemma. For the proof of (a) and (b), we use Lemma 4.7.

(a) We assume that Y is an indecomposable module of length m which is plus-strictly-increasing. Thus, $\operatorname{char} Y = (e, c_2, \dots, c_m)$ with $e > 0$ even and $c_2 < c_3 < \cdots < c_m < e$. Then 4.7 (b) asserts that $\rho^{-m+1} \operatorname{char} Y = (c_3 - 2, \dots, c_m - 2, e - 1, c_2 - 1)$. Since c_2 is odd, $c_2 - 1$ is even, and we have $c_2 - 2 < c_3 - 2 < \cdots < c_m - 2 < e - 1$. It remains to be seen that $\operatorname{char} \tau^{m-1} Y = \rho^{-m+1} \operatorname{char} Y$. But, by assumption, $\operatorname{pd} Y = e > 0$, thus Y itself is not projective, and for $1 \le i \le m-2$ we have $\operatorname{pd} \tau^i Y = c_{m-i+1} - 1 > c_1 > 0$, therefore also $\tau^i Y$ is not projective.

(b) Assume that R is an indecomposable module of length m which is minus-strictly-increasing with $\operatorname{char} R = (c_1, \dots, c_{m-1}, e)$. According to 4.7 (a), $\rho^{m-1} R = (c_{m-1} + 1, e + 1, c_1 + 2, \dots, c_{m-2} + 2)$. Since c_{m-1} is odd, $c_{m-1} + 1$ is even and we have $e + 1 < c_1 + 2 < \cdots < c_{m-2} + 2 < c_{m-1} + 1$. If $\tau^{-m+1} R$ is non-zero, then $\operatorname{char} \tau^{-m+1} R = \rho^{m-1} \operatorname{char} R$.

(c) Again, let Y be plus-strictly-increasing. Let $\operatorname{char} Y = (e, c_2, \dots, c_t)$, thus $\rho \operatorname{char} Y = (c_2, \dots, c_t, e + 1)$. Let (Γ, μ) be the memory pile with radical Y and a unique summit, let Y' be the cliff of Γ. Since $e + 1 \ge c_i$ for $2 \le i \le t$, and these are odd numbers, the maximum principle asserts that $\mu(z) = e + 1$ for all all non-zero factor modules z of Y', thus $\mu(U) = e$ for all non-zero submodules U of Y. Also, if Z is a subfactor of Y, and not a submodule of Y, then Z is a subfactor of $Y/\operatorname{soc} Y$. Since $Y/\operatorname{soc} Y$ is odd with projective dimension c_t, it follows that $\operatorname{pd} Z \le c_t < e$. □

8.5. Lemma. *Let d be even and $u \ge 1$. Let $c_1 < c_2 < \cdots < c_u$ be odd numbers bounded by d. Let $A = C_d A(c_1, \dots, c_u, 0, 1)$.*

(a) *Let $t = \frac{1}{2}(d - c_u - 1)$. Then $t \ge 0$ and A has $(u+2)t + u$ summits. If Y is the principal cliff of A, then* $\operatorname{char} Y = (d, 1 + 2t, c_1 + 2t + 2, \dots, c_{u-1} + 2t + 2)$.

(b) *Let I be an indecomposable module which is injective and not projective. Then I is plus-strictly-increasing with* $\operatorname{pd} I = d$.

(c) *A is a concave higher Auslander algebra of global dimension d.*

Proof. Let $A = C_d A(c_1, \dots, c_u, 0, 1)$.

(a) Let P be the first summit, and $R = \operatorname{rad} P$. Thus $\operatorname{char} R = (c_1, \dots, c_u, 0)$. Let $t = \frac{1}{2}(d - c_u - 1)$ and $s = (u+2)t + u$. Let (Γ, μ) be the pile with radical R and s summits. Note that R is minus-strictly-increasing, thus we can use 4.7 (see also 8.4). According to 4.7 (a), we have $x = \rho^u \operatorname{char} R = (c_u + 1, 1, c_1 + 2, \dots, c_{u-1} + 2)$, thus x is plus-strictly-increasing with $\mu(x) = c_u + 1$. The shift lemma 4.4 shows that $\rho^s \operatorname{char} R = \rho^{(u+2)t} x = (c_u + 1 + 2t, 1 + 2t, c_1 + 2 + 2t, \dots, c_{u-1} + 2 + 2t)$. By definition of t, we have $c_u + 1 + 2t = d$. Thus, $y = \rho^s(\operatorname{char} R)$ is plus-strictly-increasing and $\mu(y) = d$. This shows that Γ is a d-pile. According to 6.6, we have constructed the summit pile of A. Note that y is the principal cliff of A.

(b) Let I_i be the indecomposable injective non-projective module which has length $u + 1 - i$, where $0 \le i \le u$ (these are all the indecomposable injective modules which are not projective). We show by induction on i that I_i is plus-strictly-increasing and $\operatorname{pd} I_i = d$.

The module I_0 is just the principal cliff-module, thus (a) shows that I_0 is plus-strictly-increasing and that $\operatorname{pd} I_0 = d$.

Now, let $1 \le i \le u$. Let $R_i = I_{i-1}/\operatorname{soc} I_{i-1}$. We want to construct the descent pile with radical R_i and cliff module I_i. Since I_{i-1} is plus-strictly-increasing, we have

$\operatorname{char} R_i = (z_1, \ldots, z_m)$ with odd numbers $z_1 < \cdots < z_m$ (here, $m = |R_i| = u+1-i$, but this is not relevant). Of course, all $z_i < d$; let $t = \frac{1}{2}(d-1-z_m)$. We construct the pile (Γ, μ) with radical R_i and $s = (m+1)t+m$ summits. According to 4.6, we have $x = \rho^m(z_1, \ldots, z_m) = (z_m + 1, z_1 + 2, \ldots, z_{m-1} + 2)$. Note that x is plus-strictly-increasing and $\mu(x) = z_m + 1$. According to the shift lemma 4.4, $y = \rho^s(\operatorname{char} R) = \rho^{(m+1)t+m}(\operatorname{char} R) = \rho^{(m+1)t}x$ is also plus-strictly-increasing, and $\mu(y) = z_m + 1 + 2t = d$. This shows that Γ is a d-pile. According to 6.6, we have constructed the descent pile of A with radical R_i, thus with cliff module $I_i = y$. As we have shown, I_i is plus-strictly-increasing and $\operatorname{pd} I_i = d$.

(c) If M is indecomposable and not torsionless, then IM is indecomposable, injective and non-projective, thus equal to I_i for some $0 \le i \le u$. According to (b), I_i is plus-strictly-increasing with $\operatorname{pd} I_i = d$. According to Lemma 8.4 (c), we have $\operatorname{pd} M = d$, since M is a non-zero submodule of IM. According to 5.7, A is a higher Auslander algebra of global dimension d. □

8.6. Proposition. *Let d be even and $m \ge 2$. Let A be a concave Nakayama algebra of height $m+1$ which is d-bound.*

The following assertions are equivalent:

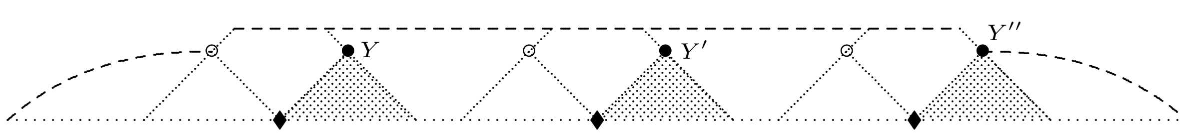
(1) *A is a higher Auslander algebra of global dimension d.*
(2) *A has a plus-strictly-increasing module Y of length m with* $\operatorname{pd}\operatorname{soc}(Y/\operatorname{soc} Y) = 1$.
(2′) *A has a plus-strictly-increasing module Y' of length m.*
(2″) *A has a plus-strictly-increasing module Y'' of length m which is injective.*
(3) *A has a minus-strictly-increasing module R of length m which is projective.*
(3′) *A has a minus-strictly-increasing module R' of length m.*
(3″) *A has a minus-strictly-increasing module R'' of length m with* $\operatorname{pd}\operatorname{top} R'' = d$.
(4) $A = C_dA(c_1, \ldots, c_{m-1}, 0, 1)$ *with odd numbers c_i such that $c_1 < c_2 < \cdots < c_{m-1} < d$.*

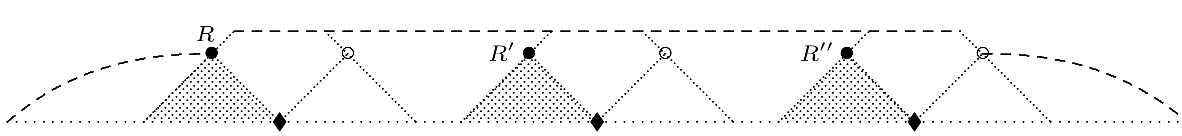
There is also the following condition:

(5) *A has an odd indecomposable module N of length m with* $\operatorname{char} N$ *strictly increasing.*

It implies the equivalent conditions (1) *to* (4)*, but the converse is not true.*

For A a concave Nakayama algebra of height $m+1$ which is a higher Auslander algebra of global dimension d, the assertions (2) to (3″) concern the indecomposable modules N of length m which are plus-strictly-increasing or minus-strictly-increasing. Here are sketches which show the possible positions of such modules $N = Y'$ or $N = R'$:





Proof of Proposition. We assume that A is a concave Nakayama algebra of height $m+1$ which is d-closed and has global dimension at most d.

(1) implies (2″). Here, we assume in addition that A is a higher Auslander algebra of global dimension d.

Let Y'' be the principal cliff module. It has length m and is indecomposable, injective and not projective. Thus Y'' is plus-strictly-increasing, see 8.3.

(2″) implies (2′). Trivial.

(2′) implies (2). Let Y' be plus-strictly-increasing of length m with $\operatorname{char} Y' = (z'_1, \dots, z'_m)$. Let $t = \frac{1}{2}(z'_2 - 1)$ and $s = (m+1)t$. We consider the pile with cliff Y' and s summits. Let Y be its radical. The shift lemma asserts that $\operatorname{char} Y = \operatorname{char} Y' - 2t(1, \dots, 1)$, thus Y is plus-strictly-increasing with $\operatorname{char} Y = (z'_1 - 2t, \dots, z'_m - 2t)$, where $z_2 = z'_2 - 2t = 1$.

The equivalences of (2) and (3), of (2′) and (3′) as well as of (2″) and (3″) are given by Lemma 8.4.

(4) implies (3). Trivial.

(3) implies (4). We assume that (3) holds. Let $\operatorname{char} R = (c_1, \dots, c_{m-1}, 0)$. Clearly, A is a descending extension of $A(c_1, \dots, c_{m-1}, 0, 1)$ and $c_1 < c_2 < \dots < c_{m-1} < d$. According to Lemma 8.5 (c), $A(c_1, \dots, c_{m-1}, 0, 1)$ has global dimension at most d. By assumption, A is d-closed. Also, A is a descending extension of $A(c_1, \dots, c_{m-1}, 0, 1)$. It follows from Proposition 3.4 that $A = C_d A(c_1, \dots, c_{m-1}, 0, 1)$.

(4) implies (1): see Lemma 8.5.

Finally, let us deal with condition (5). If we assume that there exists an odd indecomposable module of length m with $\operatorname{char} N$ strictly increasing, then N is not projective and τN is plus-strictly-increasing, thus the condition (2′) is satisfied.

However there are many concave higher Auslander algebras with even global dimension and height $m+1$, which have no odd indecomposable module of length m, for example $H_4(3)$ or $H_4(1, 3)$. □

8.7. Proof of Theorems 1′, 2′, and 3′. Let $c_1 < \dots < c_u$ be odd numbers bounded by d and let $H = C_d A(c_1, \dots, c_u, 0, 1)$. According to 8.5, H is a concave higher Auslander algebra of global dimension d and according to 8.6 any such algebra is obtained in this way.

Let P be the first summit of H, and $R = \operatorname{rad} P$. By construction, we have $\operatorname{char} P = (c_1, \dots, c_u, 0, 1)$ and $\operatorname{char} R = (c_1, \dots, c_u, 0)$. Let Q be the last summit and $Y = Q/\operatorname{soc} Q$. We have calculated $\operatorname{char} Y$ in 8.5 (a). It follows that $\operatorname{pd} \operatorname{soc} Q = d - 1$, since $\operatorname{soc} Q = \Omega Y$. In this way, we obtain the explicit form of $\operatorname{char} Q$ as mentioned. □

8.8. The characterizations presented in 8.6 refer to the existence of indecomposable modules of length $h - 1$, where h is the height of the algebra. Actually, it is sufficient to assume the existence of corresponding sequences of simple modules.

Proposition. *Let d be even and $m \ge 2$. Let A be a concave Nakayama algebra of height $m+1$ and global dimension d.*

Let A be a higher Auslander algebra. Then A is d-closed. If S is the top of the principal cliff module and $z_i = \operatorname{pd} \tau^i S$ with $0 \le i \le m-1$, then the numbers $z_0, \dots, z_{m-2}$ are odd, z_{m-1} is even, and $z_{m-2} < z_{m-3} < \dots < z_0 < z_{m-1}$.

Conversely, assume that A is d-closed and that there exists a simple module S such that the numbers $z_i = \operatorname{pd} \tau^i S$ with $0 \le i \le m-1$ satisfy the following conditions: $z_0, \dots, z_{m-2}$

are odd, z_{m-1} is even, and $z_{m-2} < z_{m-3} < \cdots < z_0 < z_{m-1}$. Then A is a higher Auslander algebra.

Proof. First, we assume that A is a higher Auslander algebra. According to Corollary 5.5, A is d-closed. Let Y be the principal cliff module and S its top. According to 8.5 (b), Y is plus-strictly-increasing and $\operatorname{pd} Y = d$.

Let $\operatorname{char} Y = (z_1, \ldots, z_m)$. Since $\operatorname{pd} Y = d$, Lemma 1.4 (6) asserts that $z_i = \operatorname{pd} F_{m-i}(M)$ for all i. But $F_{m-i}(M) = \tau^i S$, thus $z_i = \operatorname{pd} \tau^i S$. Since Y is plus-strictly-increasing, we have $z_{m-2} < \cdots < z_0 < z_{m-1}$. Since $\operatorname{pd} Y = d$ is even, one of the numbers z_i is equal to $\operatorname{pd} Y$, thus equal to d. Now for all j, we have $z_j \leq d$, therefore $z_{m-1} = d$. Since only one of the numbers z_j is even, all other are odd, thus the numbers $z_0, \ldots, z_{m-2}$ are odd. This completes the proof of the first part.

For the converse, we assume that A is d-closed and that there exists a simple module S such that the numbers $z_i = \operatorname{pd} \tau^i S$ with $0 \leq i \leq m-1$ satisfy the following conditions: $z_0, \ldots, z_{m-2}$ are odd, z_{m-1} is even, and $z_{m-2} < z_{m-3} < \cdots < z_0 < z_{m-1}$.

According to 1.1 (4), we have $|PS| \geq m$. If we would have $|PS| = m$, then $\operatorname{soc} PS = \tau^{m-1} S$. Thus $\operatorname{pd} \operatorname{soc} PS = z_{m-1}$ and therefore $\operatorname{pd}(PS/\operatorname{soc} PS) = z_{m-1} + 1$, whereas the maximum principle asserts that $\operatorname{pd}(PS/\operatorname{soc} PS) = z_0$, a contradiction. This shows that $|PS| = m + 1$. Let $Y = PS/\operatorname{soc} PS$. Then Y is plus-strictly-increasing (with $\operatorname{char} Y = (z_{m-1}, \ldots, z_0)$), thus condition $(2')$ of 8.6 is satisfied. □

9. Final remark: Parity.

The paper shows that the higher Auslander algebras A with global dimension $d(A)$ have a quite extreme homological behavior, and that the extreme conditions satisfied by higher Auslander algebras A depend on the parity of $d(A)$: they are of completely different nature, depending on whether $d(A)$ is even or odd.

Let A be a higher Auslander algebra. Let I be indecomposable injective. Then $\operatorname{pd} I \in \{0, d(A)\}$. For d even, there are many Nakayama algebras with $\operatorname{pd} I \in \{0, d\}$ for all indecomposable injective modules (for example, this happens always, if the Kupisch series is of the form $(2m, 2m+1)$, but for $m \geq 2$, these algebras have infinite global dimension), whereas for d odd, this condition implies that A has finite global dimension.

Similarly, if A is a higher Auslander algebra and S is simple, then S is torsionless or else $\operatorname{pd} S = d(A)$. For d odd, there are many Nakayama algebras such that any simple module is torsionless or has projective dimension d (for example, if the Kupisch series is of the form $(2m-1, 2m)$, then A has two simple modules, one of them is torsionless, the other has projective dimension 1, and for $m \geq 2$, the algebra A has infinite global dimension). On the other hand, if A is a Nakayama algebra such that any simple module is torsionless or has even projective dimension, then A has to be self-injective.

Acknowledgment. The author thanks the referee for a very careful reading of two preliminary versions, pointing out a huge number of inaccuracies and misprints. In fact, on the basis of the first comments, the paper was completely rewritten.

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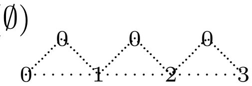
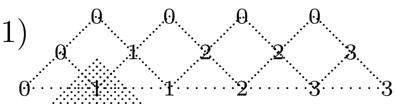
Some examples: the concave higher Auslander algebras $H = H_d(\mathbf{c})$ of height $2, 3, 4$ with $d = 3$ and $d = 4$. The set of subfactors of $Z(H)$ (for d odd) or $Z'(H)$ (for d even) has been shaded. First, we deal with $d = 3$.

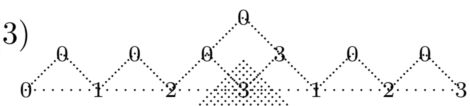
$H_3(\emptyset)$



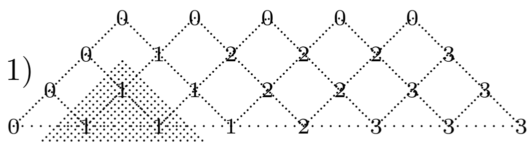
$H_3(1)$



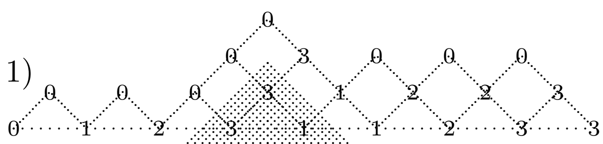
$H_3(3)$



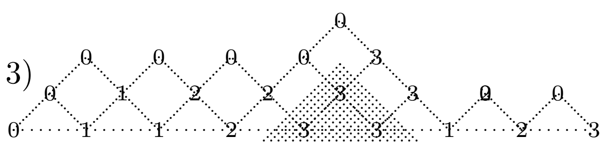
$H_3(1, 1)$



$H_3(3, 1)$

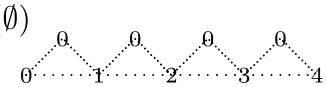


$H_3(3, 3)$

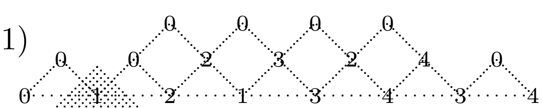


Second, $d = 4$

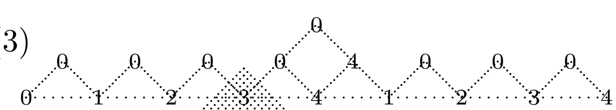
$H_4(\emptyset)$



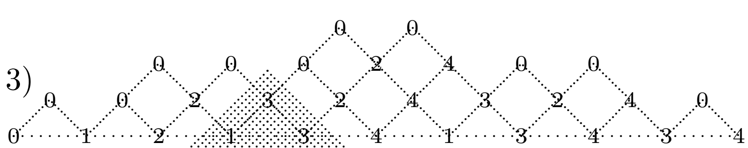
$H_4(1)$



$H_4(3)$



$H_4(1, 3)$



In general, for arbitrary height h, there are $h - 1$ algebras $H_3(\mathbf{c})$ of height h. But there are no algebras $H_4(\mathbf{c})$ of height greater than 4.