

The Gorenstein projective modules for the Nakayama algebras. I.

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Abstract: The aim of this paper is to outline the structure of the category of the Gorenstein projective Λ -modules, where Λ is a Nakayama algebra. In addition, we are going to introduce the resolution quiver of Λ . It provides a fast algorithm in order to obtain the Gorenstein projective Λ -modules and to decide whether Λ is a Gorenstein algebra or not, and whether it is CM-free or not.

Throughout the paper, Λ will be a Nakayama algebra without simple projective modules, and the modules will be left Λ -modules of finite length. Let $\text{mod } \Lambda$ be the category of all such modules. The subcategories of $\text{mod } \Lambda$ which we will deal with will be assumed to be full and closed under direct sums and direct summands. If \mathcal{M} is a class of modules, we denote by $\text{add } \mathcal{M}$ the smallest subcategory containing \mathcal{M} and $(\text{add } \mathcal{M})$ -approximations will be just called \mathcal{M} -approximations.

We denote by $\text{gp } \Lambda$ the full subcategory of $\text{mod } \Lambda$ given by the Gorenstein projective modules, and $\text{gp}_0 \Lambda$ denotes the subcategory of all Gorenstein projective modules without any indecomposable projective direct summand. Recall that Λ is said to be CM-free [C] provided all Gorenstein projective modules are projective, thus provided $\text{gp}_0 \Lambda$ is the zero category. Our aim is to describe the subcategory $\mathcal{C} = \mathcal{C}(\Lambda)$ whose indecomposable objects are the indecomposable non-projective Gorenstein projective modules as well as their projective covers. We call \mathcal{C} the *Gorenstein core*, clearly

$$\text{gp}_0 \Lambda \subseteq \mathcal{C} \subseteq \text{gp } \Lambda.$$

The first assertions concern the structure of $\mathcal{C}(\Lambda)$. Here, $\underline{\text{gp}} \Lambda$ denotes the factor category of $\text{gp } \Lambda$ obtained by factoring out the ideal of all maps which factor through a projective module. Similarly, $\underline{\text{mod}} \Lambda'$ is obtained from $\text{mod } \Lambda'$ by factoring out the ideal of all maps which factor through a projective Λ' -module.

Proposition 1. *Let Λ be a Nakayama algebra. The Gorenstein core $\mathcal{C} = \mathcal{C}(\Lambda)$ is a full exact abelian subcategory of $\text{mod } \Lambda$ which is closed under extensions, projective covers and minimal left Λ -approximations; \mathcal{C} is equivalent to $\text{mod } \Lambda'$, where Λ' is a self-injective Nakayama algebra and the inclusion functor $\mathcal{C} \rightarrow \text{gp } \Lambda$ induces an equivalence $\underline{\text{mod}} \Lambda' \rightarrow \underline{\text{gp}} \Lambda$. If \mathcal{C} is not zero and Λ is connected, then also Λ' is connected.*

Given a class \mathcal{M} of modules, we denote by $\mathcal{F}(\mathcal{M})$ the class of all modules with a filtration with factors in \mathcal{M} .

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Proposition 2. *Let Λ be a Nakayama algebra. Let $\mathcal{E} = \mathcal{E}(\Lambda)$ be the class of non-zero modules in $\text{gp}_0 \Lambda$ such that no proper non-zero factor module is Gorenstein projective.*

(a) *We have $\mathcal{C}(\Lambda) = \mathcal{F}(\mathcal{E})$.*

(b) *If Λ is connected and $\mathcal{C}(\Lambda)$ is not zero, let E_1, \dots, E_g be representatives of the isomorphism classes in \mathcal{E} . Then any simple Λ -module occurs with multiplicity 1 in $\bigoplus_{i=1}^g E_i$. In particular, E_1, \dots, E_g are pairwise orthogonal bricks.*

(c) *Let \mathcal{E}' be the class of non-zero modules in $\text{gp}_0 \Lambda$ such that no proper non-zero submodule is Gorenstein projective. Then $\mathcal{E} = \mathcal{E}'$.*

Since we deal with a set of orthogonal bricks, the elements of $\mathcal{E}(\Lambda)$ are just the simple objects of $\mathcal{C}(\Lambda)$ (see for example [R1]), we call them the *elementary* Gorenstein projective modules. Note that assertion (a) implies that $\mathcal{C}(\Lambda)$ is the extension closure of gp_0 (since $\mathcal{E} \subseteq \text{gp}_0 \subseteq \mathcal{C}(\Lambda)$).

Given a module M , we denote by $P(M)$ the projective cover, by $\Omega(M)$ the first syzygy module and by $\tau M = D \text{Tr } M$, $\tau^- M = \text{Tr } DM$ the Auslander-Reiten translates of M . If \mathcal{M} is a class of modules, we write $\tau \mathcal{M}$ for the class of modules τM with $M \in \mathcal{M}$, and similarly, $\tau^- \mathcal{M}$ is the class of modules $\tau^- M$ with $M \in \mathcal{M}$.

Proposition 3. *Let Λ be a Nakayama algebra. Let $\mathcal{X} = \mathcal{X}(\Lambda)$ be the class of simple modules S with $P(S)$ belonging to $\mathcal{C}(\Lambda)$. The following conditions are equivalent for a module M :*

- (i) *M belongs to $\mathcal{C}(\Lambda)$.*
- (ii) *$\text{top } M$ belongs to $\text{add } \mathcal{X}$, and $\text{soc } M$ belongs to $\text{add } \tau^- \mathcal{X}$.*
- (iii) *$\text{top } M$ and $\text{top } \Omega M$ both belong to $\text{add } \mathcal{X}$.*

We see that $\mathcal{C}(\Lambda)$ may be obtained by deleting rays and corays from the Auslander-Reiten quiver of Λ . Namely, we have to delete the rays consisting of the indecomposable modules with top not in \mathcal{X} , as well as the corays consisting of the indecomposable modules with socle not in $\tau^- \mathcal{X}$.

The basic observation which we use is the following characterization of the indecomposable non-projective Gorenstein projective modules. Here, an indecomposable projective module P is said to be *minimal projective* provided no proper non-zero submodule of P is projective, or, equivalently, provided the projective dimension of $\text{top } P$ is at least 2.

Proposition 4. *Let Λ be a Nakayama algebra. Let M be an indecomposable non-projective module. The following assertions are equivalent:*

- (i) *The module M is Gorenstein projective.*
- (ii) *All the projective modules occurring in a minimal projective resolution of M are minimal projective.*
- (iii) *There is an exact sequence*

$$(*) \quad 0 \rightarrow M \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

such that all the modules P_i are minimal projective.

Of course, since any Nakayama algebra is representation-finite, any Gorenstein projective module M has a periodic projective resolution, thus there is an exact sequence of the

form $(*)$ with projective modules P_0, \dots, P_{n-1} . But usually not all modules with a periodic projective resolution are Gorenstein projective. The interesting feature here is the fact that the isomorphism classes of the modules P_i in a periodic projective resolution of M determine whether M is Gorenstein projective or not.

Throughout the paper, we will denote the number of simple modules by $s = s(\Lambda)$, the minimal length of an indecomposable projective module by $p = p(\Lambda)$. Since we assume that there are no simple projective modules, we have $p \geq 2$.

Note that $p > s$ if and only if no projective module is a brick. Examples which we will exhibit in section 7 show that algebras with $p \leq s$ may have some irregularities, thus some of our results require the condition $p > s$.

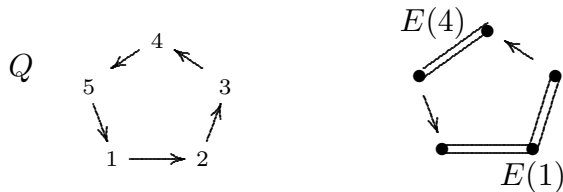
We will work with the *resolution quiver* $R = R(\Lambda)$ of Λ : The vertices of R are the simple modules and for every vertex S , there is an arrow from S to $\tau \text{soc} P(S)$. Since any vertex in R is the start of a unique arrow, any connected component of R contains precisely one cycle. A vertex S of R is said to be *black* provided the projective dimension of S is at least 2 (thus if and only if $P(S)$ is a minimal projective module), otherwise it is said to be *red*. As we will see the modules in $\mathcal{X}(\Lambda)$ are precisely the simple modules S which belong to a cycle of black vertices provided $p > s$, see Corollary 3 to Lemma 5.

Proposition 5. *Let Λ be a Nakayama algebra and assume that $p > s$.*

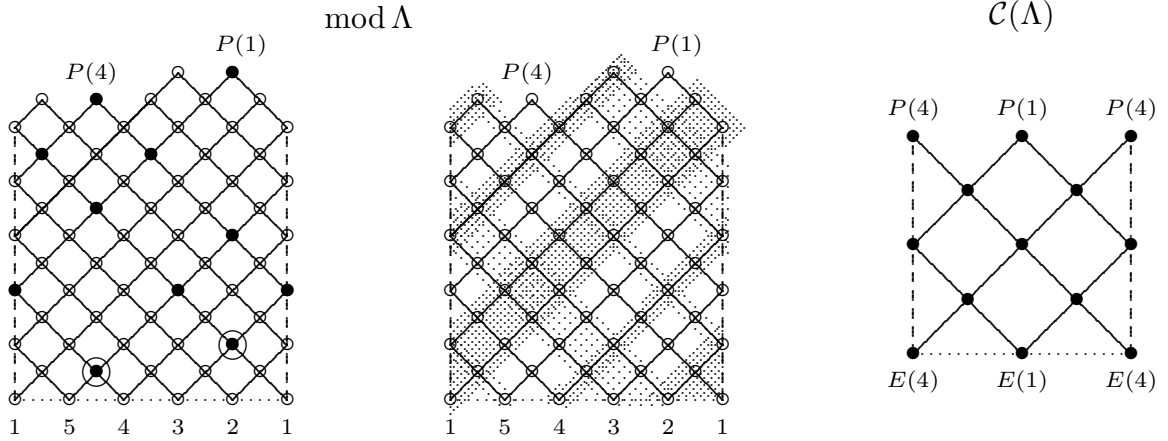
- (a) *The algebra Λ is a Gorenstein algebra if and only if any cycle in $R(\Lambda)$ contains only black vertices.*
- (b) *The algebra Λ is CM-free if and only if any cycle in $R(\Lambda)$ contains at least one red vertex.*

In a second part [R2] we will describe some further properties of the resolution quiver. In particular, we will show that for a connected Nakayama algebra Λ there is either no loop in $R(\Lambda)$, or else all cycles in $R(\Lambda)$ are loops. This result has also been obtained (independently and with a different proof) by Dawei Shen [S].

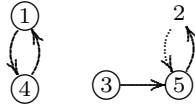
An example. Let Q be the quiver of type $\tilde{\mathbb{A}}_4$ with cyclic orientation, say with vertices $1, 2, 3, 4, 5$ and arrows $i \rightarrow i+1$ (modulo 5). Since a Nakayama algebra is defined by zero relations (monomials), it is sufficient to mention the length of the indecomposable projectives, instead of writing down the relations. Here we consider the case where $|P(i)| = 13$ for $i = 1, 2$ and $|P(i)| = 12$ for $i = 3, 4, 5$, thus we deal with the algebra Λ with Kupisch series $(13, 13, 12, 12, 12)$. There are two elementary Gorenstein projective modules, namely $E(1)$ with composition factors $1, 2, 3$ and $E(4)$ with composition factors $4, 5$. The right picture below shows the support of these modules $E(1)$ and $E(4)$, thus the Ext-quiver for $\mathcal{E} = \{E(1), E(4)\}$ will be the quiver of type $\tilde{\mathbb{A}}_1$ with cyclic orientation.



Next, let us draw twice the Auslander-Reiten quiver of Λ (always, the left dashed boundary has to be identified with the right dashed boundary) and label the simple module $S(i)$ just by i . On the left, we use bullets to mark the indecomposable objects in $\mathcal{C}(\Lambda)$; the encircled ones are the elementary Gorenstein projective modules $E(1)$ and $E(4)$). Then in the middle, we shade the rays and the corays which have to be deleted: note that $\mathcal{X}(\Lambda) = \{S(1), S(4)\}$, thus $\tau^- \mathcal{X}(\Lambda) = \{S(5), S(3)\}$, this means that we have to delete the corays ending in $S(2), S(3), S(5)$, and the rays starting in $S(1), S(2), S(4)$. On the right, we present the Auslander-Reiten quiver of $\mathcal{C}(\Lambda)$.



The resolution quiver $R(\Lambda)$ has the following form:



The black vertices 1, 3, 4, 5 have been encircled, the arrow $2 \rightarrow 5$ has been dotted in order to stress that it starts at a red vertex. We see that there are two cycles, one containing only black vertices, the other one containing one black and one red vertex. The modules in $\mathcal{X}(\Lambda)$ are precisely the simple modules S which belong to a cycle of black vertices, thus we see that $\mathcal{X}(\Lambda) = \{1, 4\}$ (as we have mentioned already).

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1. Notation.

We denote by $Q = Q(\Lambda)$ the quiver of Λ ; its vertices are the (isomorphism classes of the) simple Λ -modules and there is an arrow $S \rightarrow S'$ provided there is a length 2 module with top S and socle S' . Since we assume that no simple module is projective, the quiver

Q is just a cycle. Let τ be the Auslander-Reiten translation. Since Λ is a Nakayama algebra without simple projective modules, an arrow $S \rightarrow S'$ in Q corresponds to the assertion $\tau(S) = S'$. Often, we will use the vertices x of the quiver Q in order to index the corresponding modules: thus, we write $S(x)$ for the simple module itself, $P(x)$ for the projective cover of $S(x)$.

Given a module M , recall that $P(M)$ denotes the projective cover, and we denote by $\Omega(M)$ the first syzygy module. Inductively, let $P_0(M) = P(M)$, $\Omega_0(M) = M$, and for $n \geq 1$, let $P_n(M) = P(\Omega_n(M))$ and $\Omega_n(M) = \Omega(\Omega_{n-1}(M))$. Thus, there are exact sequences

$$0 \rightarrow \Omega_{n+1}(M) \rightarrow P_n(M) \rightarrow \Omega_n(M) \rightarrow 0$$

for all $n \geq 0$. A minimal projective resolution of M has the form

$$\cdots \rightarrow P_n(M) \rightarrow P_{n-1}(M) \rightarrow \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0,$$

with $\Omega_n(M)$ being the image of $P_n(M) \rightarrow P_{n-1}(M)$.

Recall that a complex $P_\bullet = (P_i, \delta_i)$ with maps $\delta : P_i \rightarrow P_{i-1}$ is called a *complete projective resolution* provided it is an exact complex of projective modules P_i such that also the complex $\text{Hom}_\Lambda(P_\bullet, \Lambda)$ is exact. The latter condition is equivalent to the requirement that the inclusion map $\text{Im}(\delta_i) \subseteq P_{i-1}$ is a left Λ -approximation, for each i . A module M is said to be *Gorenstein projective* provided there is a complete projective resolution (P_i, δ_i) such that $M = \text{Im}(\delta_0)$.

An artin algebra Λ is said to be a *Gorenstein algebra* provided the injective dimension of ${}_\Lambda\Lambda$ as well as of Λ_Λ is finite. If this is the case, these dimensions are equal and called the *Gorenstein dimension* (or also the *virtual dimension*) of Λ ; we will denote it by $v(\Lambda)$. Note that Λ is a Gorenstein algebra with Gorenstein dimension at most v if and only if $\Omega_v(M)$ is a Gorenstein projective module, for every module M .

2. Projective resolutions of indecomposable Λ -modules.

The aim of this section is to point out the relevance of the minimal projective modules. In particular, we will provide the proof of Proposition 4. Recall that an indecomposable projective module P is said to be *minimal projective* provided its radical is non-projective (thus provided $P = P(S)$ for some simple module S of projective dimension at least 2). Note that a proper non-zero submodule of a minimal projective module is not projective (this explains the name). Recall that a module M is said to be *torsionless* provided it is a submodule of a projective module.

Lemma 1. *Let M be an indecomposable non-projective module. If M is torsionless, then M can be embedded into a minimal projective module and any such embedding is a minimal left Λ -approximation.*

Proof. If there exists an embedding $M \rightarrow P$, with P projective, then there is such an embedding $M \rightarrow P_0$ with P_0 indecomposable (since the socle of M is simple). Since we assume that M is not projective, there is such an embedding $M \rightarrow P_0$, where P_0 is

in addition minimal projective. This shows that any torsionless module can be embedded into a minimal projective module.

Let us fix an embedding $\iota: M \rightarrow P_0$ with P_0 minimal projective. In order to see that ι is a left Λ -approximation, we have to show that any non-zero map $f: M \rightarrow P_1$ with P_1 indecomposable projective factors through ι . Here, we can assume that also P_1 is minimal projective (namely, since M is not projective, it will map into $\text{rad } P_1$, thus, if $\text{rad } P_1$ is projective, we may replace P_1 by $\text{rad } P_1$, and so on).

We want to show that f factors through ι . One possibility is to look at the Auslander-Reiten quiver of Λ and consider various right almost split maps. Here is a proof which uses the fact that any automorphism of a submodule of an indecomposable module X can be extended to an automorphism of X .

First, consider the case where f is a monomorphism. Thus P_0 is a submodule of P_1 and the image of f coincides with the image of ι . It follows that f factors through ι .

Thus, we can assume that f is not a monomorphism, thus $\text{Ker}(f) \neq 0$. Let S be the socle of the image of f and take the composition series

$$\text{Ker}(f) = K_0 \supset K_1 \supset \cdots \supset K_t = 0$$

of $\text{Ker}(f)$. Note that $t \geq 1$ and $K_{i-1}/K_i = \tau^i S$ for $1 \leq i \leq t$.

Now f maps into $N = \text{rad } P_1$, and by assumption N is not projective. Let $q: P = P(N) \rightarrow N$ be the projective cover of N . Note that $P(N)$ is both projective and injective. Let U be the kernel of q and

$$U = U_0 \supset U_1 \supset \cdots \supset U_u = 0$$

the composition series of U . Then also $U_{i-1}/U_i = \tau^i S$ for $1 \leq i \leq u$. For $1 \leq i \leq u-1$, the module P/U_i is injective and not projective, thus the modules $\tau^i S = U_{i-1}/U_i$ for $1 \leq i \leq u-1$ are not torsionless.

Now assume that $t < u$. Then $1 \leq t \leq u-1$ and therefore $K_{t-1}/K_t = \tau^t S$ is not torsionless. But this is the socle of the module M and by assumption there is the embedding $\iota: M \rightarrow P_0$. This contradiction shows that we must have $u \leq t$.

Let M' be the image of f and write $f = f_3 f_2 f_1$, where $f_1: M \rightarrow M'$, whereas $f_2: M' \rightarrow N$ and $f_3: N \rightarrow P_1$ are the inclusion maps. Let $M'' = q^{-1}(M')$ with inclusion map $f'_2: M'' \rightarrow P(N)$ and let $q': M'' \rightarrow M'$ be the restriction of q , thus $q f'_2 = f_2 q'$. The length of M'' is

$$|M''| = |M'| + |\text{Ker}(q)| = |M'| + u \leq |M'| + t = |M|,$$

and therefore we can lift the map $f_1: M \rightarrow M'$ to M'' , thus there is $f'_1: M \rightarrow M''$ with $q' f'_1 = f_1$ and therefore $f_2 f_1 = f_2 q' f'_1 = q f'_2 f'_1$.

Since $P(N)$ is injective, there is a map $g: P_0 \rightarrow P(N)$ such that $g \iota = f'_2 f'_1$. Thus $f = f_3 f_2 f_1 = f_3 q f'_2 f'_1 = f_3 q g \iota$ shows that f factors through ι .

Since minimal left approximations are uniquely determined up to isomorphisms, it follows that *an embedding $M \rightarrow P$ with M indecomposable and P projective is a minimal left Λ -approximation if and only if P is a minimal projective module.*

Proof of Proposition 4. (i) \implies (ii). This follows from Lemma 1.

(ii) \implies (iii). We assume that we deal with the projective resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Now all the images $\Omega_n(M)$ are indecomposable, thus there are natural numbers $0 \leq n' < n$ such that $\Omega_n(M) = \Omega_{n'}(M)$. Choose n minimal and assume that $n' \geq 1$. Now $\Omega_n(M)$ is a submodule of P_{n-1} , and $\Omega_{n'}(M)$ is a submodule of $P_{n'-1}$. Since $\Omega_n(M) = \Omega_{n'}(M)$, the minimality of P_{n-1} and $P_{n'-1}$ implies that $P_{n-1} = P_{n'-1}$ and therefore $\Omega_{n-1}(M) = \Omega_{n'-1}(M)$. But this contradicts the minimality of n . Thus $n' = 0$ and $\Omega_n(M) = \Omega_0(M) = M$.

(iii) \implies (i). We assume that there is given an exact sequence

$$0 \rightarrow M \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

such that all the modules P_i are minimal projective modules. Concatenation of countably many such sequences yields a complete resolution with M as one of the images. This shows that M is Gorenstein projective.

3. The resolution quiver of Λ .

Let $\gamma S = \tau \text{soc } P(S)$. The importance of the map γ stems from two observations, see Lemmas 2 and 3. As Xiao-Wu Chen has pointed out, the map γ has been considered already by W. H. Gustafson [G].

Lemma 2. *Let M be an indecomposable module. Then either the projective dimension of M is at most 1 and $\Omega_2(M) = 0$, or else $\text{top } \Omega_2(M) = \gamma \text{ top } M$.*

Proof: Write $M = P(M)/U$ for some submodule U of $P(M)$. We can assume that U is a proper submodule of $P(M)$. There are exact sequences of the following form

$$\begin{aligned} 0 \rightarrow \Omega_2(M) \rightarrow P_1(M) \rightarrow \Omega_1(M) \rightarrow 0 \\ 0 \rightarrow \Omega_1(M) \rightarrow P(M) \rightarrow M \rightarrow 0. \end{aligned}$$

Clearly, $\text{soc } \Omega_1(M) = \text{soc } P(M)$ and the first exact sequence shows that either $\Omega_2(M) = 0$ (thus the projective dimension of M is at most 1) or else $\text{top } \Omega_2(M) = \tau \text{soc } \Omega_1(M)$. In the latter case, $\text{top } \Omega_2(M) = \gamma \text{ top } M$.

Inductively, we see:

Corollary. *Let M be an indecomposable module and $m \in \mathbb{N}$. Then either $\Omega_{2m}(M) = 0$ or else $\text{top } \Omega_{2m}(M) = \gamma^m \text{ top } M$.*

If $|P(S)| \geq s$, we define $H(S)$ to be the factor module of $P(S)$ of length s , where S is any simple module. We call this module $H(S)$ a *primitive* module. In case $p > s$, these modules $H(S)$ do exist for all simple modules S and are non-projective.

Lemma 3. *Let S be a simple module with $|P(S)| > s$. Then $|P(\gamma S)| \geq s$ and*

$$\Omega_2 H(S) = H(\gamma S).$$

Proof. Since $H(S)$ has length s , and $|P(S)| > s$, the minimal projective presentation of $H(S)$ is

$$P(S) \rightarrow P(S) \rightarrow H(S) \rightarrow 0,$$

and therefore $\Omega_2 H(S)$ has length s . On the other hand, according to Lemma 2, we know that $\Omega_2 H(S)$ is a factor module of $P(\gamma S)$. This yields both assertions.

Corollary. *Let $p > s$. Let S be a simple module and m a natural number. Then*

$$\Omega_{2m} H(S) = H(\gamma^m S).$$

Note that this corollary implies that the projective dimension of $H(S)$ is infinite. Thus, any Nakayama algebra with $p > s$ has infinite global dimension. This implies the following result of Gustafsen [G]: if the Loewy length of Λ is greater than or equal to $2s$, then Λ has infinite global dimension (since in this case $p > s$).

The map γ is used in order to obtain the *resolution quiver* $R = R(\Lambda)$ (as introduced in the introduction): its vertices are the (isomorphism classes of the) simple Λ -modules, and for any simple module S , there is an arrow $S \rightarrow \gamma S$.

Note that *any connected component of the resolution quiver has a unique cycle*. This follows from the fact that at any vertex x precisely one arrow starts; thus given any connected component, the number of arrows in the component is equal to the number of vertices in the component.

We say that a vertex x or the corresponding simple or projective modules $S(x)$ and $P(x)$ are *black* provided $P(x)$ is minimal projective, otherwise x (and $S(x)$ and $P(x)$) will be said to be *red*. Note that x is red if and only if the projective dimension of $S(x)$ is equal to 1, and black if and only if the projective dimension of $S(x)$ is greater than or equal to 2. A cycle in R will be said to be *black* provided all the vertices occurring in the cycle are black. A vertex x is said to be *cyclically black* provided x belongs to a black cycle.

The resolution quiver has the following property: *any vertex y is end point of at most one arrow $x \rightarrow y$ with x black*. Namely, if S, S' are black simple modules with $\gamma S = \gamma S'$, then $\tau \text{soc } P(S) = \tau \text{soc } P(S')$, thus $\text{soc } P(S) = \text{soc } P(S')$ and therefore $P(S) \subseteq P(S')$ or $P(S') \subseteq P(S)$. However, since both $P(S)$ and $P(S')$ are minimal projective modules, it follows that $P(S) = P(S')$ and thus $S = S'$.

As a consequence we obtain:

Lemma 4 (Red Entrance Lemma). *If $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_a \rightarrow y$ is a path such that y is cyclically black, whereas x_a is not cyclically black, then x_a is red.*

Proof. Since y is cyclically black, there is an arrow $x' \rightarrow y$ such that also x' is cyclically black. Since x_a is not cyclically black, we have $x_a \neq x'$, and it follows that x_a cannot be black, thus it is red.

There is the following consequence: *Let x be a vertex which is not cyclically black. Then there exists $m \geq 0$ such that $\gamma^m x$ is red.*

Lemma 5. *Let M be indecomposable and not projective. Then the following conditions are equivalent:*

- (i) *M is Gorenstein projective.*
- (ii) *Both $\text{top } M$ and $\text{top } \Omega(M)$ are cyclically black.*
- (iii) *Both $\text{top } M$ and $\tau \text{soc } M$ are cyclically black.*

Proof: The assertions (ii) and (iii) are equivalent, since $\tau \text{soc } M = \text{top } \Omega(M)$.

(ii) \implies (i). If $\text{top } M$ is cyclically black, then M has infinite projective dimension and all the modules $P_{2m}(M)$ are black, for $m \geq 0$. If $\text{top } \Omega(M)$ is cyclically black, then all the modules $P_{2m+1}(M)$ are black, for $m \geq 0$. Thus, if both conditions are satisfied, then all the projective modules occurring in the minimal resolution of M are minimal, thus M is Gorenstein projective, according to Lemma 2.

(i) \implies (ii). Assume that $\text{top } M$ is not cyclically black. Then there is some $m \geq 0$ such that $\gamma^m \text{top } M$ is red, but $P_{2m}(M) = P(\gamma^m \text{top } M)$. Similarly, if $\text{top } \Omega M$ is not cyclically black, then there is some $m \geq 0$ such that $\gamma^m \text{top } \Omega M$ is red, and $P_{2m+1}(M) = P(\gamma^m \text{top } \Omega M)$. Thus, M cannot be Gorenstein projective, according to Lemma 2.

Several consequences are of interest. First, we look at part (b) of Proposition 5. We see that one direction works without the assumption $p > s$.

Corollary 1. *Let Λ be a Nakayama algebra and assume that no cycle of $R(\Lambda)$ is black. Then Λ is CM-free.*

Corollary 2. *If S is a cyclically black simple module and $|P(S)| > s$, then $H(S)$ is Gorenstein projective and not projective.*

Proof: Let S be cyclically black and $|P(S)| > s$. Then $H(S)$ exists and is non-projective. Both $\text{top } H(S)$ and $\text{top } \Omega H(S)$ are equal to S , thus cyclically black. According to Lemma 5, we see that $H(S)$ is Gorenstein projective.

Corollary 3. *The simple modules in $\mathcal{X}(\Lambda)$ are cyclically black. Conversely, if S is a cyclically black simple module and $|P(S)| > s$, then S belongs to $\mathcal{X}(\Lambda)$.*

Proof. Let S be in $\mathcal{X}(\Lambda)$. Then $P(S)$ has a factor module M which is Gorenstein projective, but not projective. Lemma 5 asserts that $S = \text{top } M$ is cyclically black.

On the other hand, let S be cyclically black and $|P(S)| > s$. Then $H(S)$ exists and is non-projective. According to Corollary 2, $H(S)$ is Gorenstein projective. Since $P(S)$ has a non-projective factor module which is Gorenstein projective, we see that S belongs to $\mathcal{X}(\Lambda)$.

Proof of Proposition 5 (b). One direction is covered by Corollary 1. For the converse, let us assume that $p > s$ and that there is a black cycle, say containing the vertex x . According to Corollary 2, the module $H(x)$ is Gorenstein projective and not projective, thus Λ is not CM-free.

Proof of Proposition 3. By definition, $\mathcal{X} = \mathcal{X}(\Lambda)$ is the class of simple modules S such that $P(S)$ has a factor module which is non-projective and Gorenstein projective, thus S belongs to \mathcal{X} if and only if there is a non-projective Gorenstein projective module N with $S = \text{top } N$.

To show the equivalence of (i), (ii), (iii), it is sufficient to consider the case of M being indecomposable.

First, assume that M is projective, say $M = P(S)$ for some simple module S . If $P(S) \in \mathcal{C}$, then $P(S)$ is the projective cover of a non-projective Gorenstein projective module M' , thus $S \in \mathcal{X}$, this shows that (i) implies (iii). In order to show that (i) implies (ii), we have to show in addition that $\tau \text{soc } M$ belongs to \mathcal{X} . Since M' is non-projective and Gorenstein projective, also $\Omega^2 M'$ is non-projective and Gorenstein projective, thus $\tau \text{soc } M = \tau \text{soc } \Omega M' = \text{top } \Omega^2 M'$ belongs to \mathcal{X} . Conversely, if (ii) or (iii) is satisfied, then $\text{top } M$ belongs to \mathcal{X} , thus $M = P(\text{top } M)$ belongs to \mathcal{C} .

Next, assume that M is non-projective. The assertions (ii) and (iii) are equivalent since for M indecomposable and non-projective, $\tau \text{soc } M = \text{top } \Omega M$. If M is Gorenstein projective, then also ΩM is Gorenstein projective (and non-projective), thus $\text{top } M$ and $\text{top } \Omega M$ both belong to \mathcal{X} . This shows that (i) implies (iii). Conversely, assume that $\text{top } M$ and $\text{top } \Omega M$ are in \mathcal{X} . Then, according to the Corollary 3, $\text{top } M$ and $\text{top } \Omega M$ are cyclically black and therefore Lemma 5 asserts that M is Gorenstein projective. This shows that (iii) implies (i) and completes the proof of Proposition 3.

Remark 1. It seems to be of interest to identify the sources of the resolution quiver R : *The simple module S is a source of R if and only if $\tau^- S$ cannot be embedded into Λ (thus if and only if $I(\tau^- S)$ is not projective).* Namely, if S is in the image of γ , then there is a simple module S' with $S = \gamma S' = \tau \text{soc } P(S')$, thus $\tau^- S = \text{soc } P(S') \subseteq \text{soc } \Lambda$. And conversely, if $\tau^- S$ can be embedded into Λ , then it can be embedded into some indecomposable projective module $P(S')$, and then $\gamma S' = \tau \text{soc } P(S') = \tau \tau^- S = S$, thus S is in the image of γ .

As a consequence, *the number of sources of R is equal to $s - t$* , where t is the number of indecomposable modules which are both projective and injective (and this is also the number of minimal projective modules).

The same argument shows that in general *the number of arrows ending in S is equal to the number of projective modules with socle $\tau^- S$.*

Remark 2. The referee has pointed out that a simple module S is cyclically black if and only if it is perfect in the sense of [CY].

4. The elementary Gorenstein projective modules.

We are going to use Lemma 5 (or else Proposition 3) in order to show some important closure properties of gp_0 . We need them in order to prove Proposition 1. After Proposition 1 is established, they are direct consequences.

Lemma 6. *Let X, Y be indecomposable modules which are Gorenstein projective and not projective. If $f: X \rightarrow Y$ is a homomorphism, then kernel, image and cokernel of f are Gorenstein projective modules.*

Proof: We can assume that f is non-zero. Then the image Z of f is indecomposable and not projective. According to Lemma 5, $\text{top } X, \text{top } Y, \tau \text{soc } X, \tau \text{soc } Y$ all are cyclically black. But $\text{top } Z = \text{top } X$ and $\text{soc } Z = \text{soc } Y$, thus $\text{top } Z$ and $\tau \text{soc } Z$ are cyclically black. Using again Lemma 5, we see that Z is Gorenstein projective. If the kernel K of f is non-zero, then K is indecomposable and non-projective. Also, $\text{soc } K = \text{soc } X$, and $\text{top } K = \tau \text{soc } Z = \tau \text{soc } Y$, thus Lemma 5 implies again that K is Gorenstein projective. Finally, if the cokernel Q of f is non-zero, it is indecomposable and non-projective, and $\text{top } Q = \text{top } Y$, and $\tau \text{soc } Q = \text{top } Z = \text{top } X$ are cyclically black, so that also Q is Gorenstein projective.

Lemma 7. *Let X be an indecomposable module which is Gorenstein projective and not projective. If $X' \subseteq X''$ are submodules of X such that X''/X' is Gorenstein projective and non-zero, then also X' and X/X'' are Gorenstein projective.*

Proof: Again, we use Lemma 5. Of course, also X''/X' is non-projective, thus $\text{top } X, \tau \text{soc } X, \text{top}(X''/X'), \tau \text{soc}(X''/X')$ are cyclically black. If $X' \neq 0$, then this is a non-projective indecomposable module with $\text{top } X' = \tau \text{soc}(X''/X')$ and $\tau \text{soc } X' = \tau \text{soc } X$ cyclically black. If $X/X'' \neq 0$, then this is a non-projective indecomposable module with $\text{top}(X/X'') = \text{top } X$ and $\tau \text{soc}(X/X'') = \text{top}(X''/X')$ cyclically black.

Let S be a simple module. If $P(S)$ has a non-projective factor module which is Gorenstein projective, then we denote the smallest module of this kind by $E(S)$. If M is any module, then M is of the form $E(S)$ provided M is non-projective, but Gorenstein projective, and the only proper factor module of M which is Gorenstein projective, is the zero module (and $S = \text{top } M$). Thus, the modules $E(S)$ are the elementary Gorenstein projective modules as defined in the introduction.

Let g be the number of isomorphism classes of elementary Gorenstein projective modules. Note that g equals the number of simple modules in $\mathcal{X}(\Lambda)$.

Proof of Proposition 2. Let E, E' be elementary modules and $f: E \rightarrow E'$ a non-zero homomorphism. The image of f is a non-projective Gorenstein projective factor module of E , thus f has to be injective, according to Lemma 6. If f is not surjective, then the cokernel of f is a non-projective Gorenstein projective proper factor module of E' , impossible (again we use Lemma 6). This shows that f is an isomorphism. Thus, if E_1, \dots, E_g are representatives of the isomorphism classes of \mathcal{E} , then this is a set of pairwise orthogonal bricks.

If U is a proper submodule of E which is Gorenstein projective, then E/U is a non-zero factor module of E which is Gorenstein projective, according to Lemma 7. But this implies that $U = 0$. This shows that $\mathcal{E} \subseteq \mathcal{E}'$.

Conversely, assume that X belongs to \mathcal{E}' , thus X is an indecomposable module which is Gorenstein projective and non-projective, such that no proper non-zero submodule is Gorenstein projective. Assume that there is a proper factor module X/U of X which is Gorenstein projective. Then Lemma 6 asserts that U is Gorenstein projective and therefore $U = X$. This shows that $\mathcal{E}' \subseteq \mathcal{E}$. Thus we have established (c).

Next, let us show that $\mathcal{C} = \mathcal{F}(\mathcal{E})$. Since we deal with a set of orthogonal bricks, $\mathcal{F}(\mathcal{E})$ is an abelian subcategory. Since all the modules in \mathcal{E} are Gorenstein projective, the modules in $\mathcal{F}(\mathcal{E})$ are Gorenstein projective. Assume that P is indecomposable projective and in

$\mathcal{F}(\mathcal{E})$. Then P has a non-zero submodule U such that P/U belongs to \mathcal{E} . But this shows that P is the projective cover of a Gorenstein projective module which is not projective. This shows that $\mathcal{F}(\mathcal{E}) \subseteq \mathcal{C}$.

Conversely, consider an indecomposable module X in \mathcal{C} . For the non-projective modules we use induction on the length. Thus, let X be non-projective, let $P(X) = P(S)$ be its projective cover, with S simple. Since X is Gorenstein projective, the module $E(S)$ exists and is a factor module of X , say $X/U = E(S)$. Now U itself is Gorenstein projective, according to Lemma 6, thus by induction it belongs to $\mathcal{F}(\mathcal{E})$. Then also X belongs to $\mathcal{F}(\mathcal{E})$.

If $X = P(S)$ is projective, then X is the projective cover of a Gorenstein projective module which is not projective, thus the factor module $E(S)$ exists, say $E(S) = X/X'$. But X' is Gorenstein projective and not projective, thus we know already that X' belongs to $\mathcal{F}(\mathcal{E})$. This completes the proof of (a).

It remains to establish (b). We assume now that Λ is connected and \mathcal{C} is non-zero, and we denote by E_1, \dots, E_g representatives of the isomorphism classes in \mathcal{E} . On the one hand, these modules are pairwise orthogonal bricks. On the other hand, according to Lemma 7, no module E_i is a proper subquotient of some E_j , thus the modules E_i must have pairwise different support. Thus, any simple module occurs with multiplicity at most 1 in $\bigoplus_{i=1}^g E_i$.

It remains to be seen that any simple module occurs in the support of some E_i . Since \mathcal{C} is not zero, we have $g \geq 1$, thus there is a simple module S which belongs to the support of $\bigoplus_{i=1}^g E_i$. We show: if S is a composition factor of an elementary module E , then τS is a composition factor of an elementary module E' . Thus assume that S is a composition factor of E . If S is not the socle of E , then τS is also composition factor of E . Thus, we have to consider the case that $S = \text{soc } E$. The following lemma shows that there is an elementary module E' such that τS is a composition factor of E' . Since we assume that Λ is connected, the set of simple Λ -modules is a single τ -orbit, therefore any simple module occurs as a composition factor of some elementary module. This completes the proof of (b).

Lemma 8. *If E is an elementary module, then there is an elementary module E' with $\text{top } E' = \tau \text{soc } E$ and $\text{Ext}^1(E, E') \neq 0$.*

Proof. Let $P(E)$ be the projective cover of E . This is a module in $\mathcal{C} = \mathcal{F}(\mathcal{E})$, thus $P(E)$ has a filtration using elementary Gorenstein projective modules, say

$$P(E) = M_0 \supset M_1 \supset \dots \supset M_t = 0,$$

such that M_{i-1}/M_i is elementary, for all $1 \leq i \leq t$. Note that $M_0/M_1 = E$. Since E is not projective, we have $t \geq 2$, thus let $E' = M_1/M_2$. Then

$$\text{top } E' = \text{top}(M_1/M_2) = \tau \text{soc}(M_0/M_1) = \tau \text{soc } E.$$

Since the module M_0/M_2 is indecomposable, the exact sequence

$$0 \rightarrow M_1/M_2 \rightarrow M_0/M_2 \rightarrow M_0/M_1 \rightarrow 0$$

does not split, therefore $\text{Ext}^1(E, E') \neq 0$.

Proof of Proposition 1. Since $\mathcal{C} = \mathcal{F}(\mathcal{E})$, and \mathcal{E} is a set of pairwise orthogonal bricks, \mathcal{C} is a full exact subcategory of $\text{mod } \Lambda$ which is closed under extensions. By definition, the category \mathcal{C} is closed under projective covers. In order to see that \mathcal{C} is closed under minimal left Λ -approximations, consider an indecomposable object M in \mathcal{C} . We may assume that M is not projective. According to Proposition 4, there is an exact sequence

$$0 \rightarrow M \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

such that all the modules P_i are minimal projective modules. With M also $\Omega_{n-1}(M)$ is Gorenstein projective and not projective, and P_{n-1} is its projective cover. By the construction of \mathcal{C} , the module P_{n-1} belongs to \mathcal{C} . Lemma 1 asserts that the embedding $M \rightarrow P_{n-1}$ is a minimal left Λ -approximation.

As an abelian length category with only finitely many indecomposable objects, the category \mathcal{C} is equivalent to the module category of an algebra Λ' , and Λ' has precisely g simple modules. The indecomposable objects in \mathcal{C} are indecomposable Λ -modules: since they have a unique composition series as Λ -modules, they also have a unique composition series inside the category \mathcal{C} . This shows that Λ' is again a Nakayama algebra. The indecomposable projective objects in \mathcal{C} are the projective Λ -modules which belong to \mathcal{C} : these are some of the minimal projective Λ -modules. But a minimal projective Λ -module cannot be properly embedded into any other minimal projective Λ -module. This shows that the indecomposable projective objects in \mathcal{C} are injective in \mathcal{C} , thus Λ' is self-injective.

If Λ is connected and $\mathcal{C} \neq 0$, then we want to see that Λ' is connected. But this follows immediately from Proposition 2 and Lemma 8.

It remains to show that the embedding $\text{mod } \Lambda' \rightarrow \text{mod } \Lambda$ induces an embedding $\underline{\text{mod}} \Lambda' \rightarrow \underline{\text{mod}} \Lambda$ with image just $\underline{\text{gp}}$. Let X, Y be indecomposable in gp_0 , and $f: X \rightarrow Y$ a morphism which factors through a projective Λ -module, say P . Then the morphism $X \rightarrow P$ factors through P' , where $X \rightarrow P'$ is the minimal left Λ -approximation of X . But P' belongs to \mathcal{C} and is projective in \mathcal{C} , thus f is zero in $\underline{\text{gp}}$.

More information about Λ' . We know that the number of simple Λ' -modules is g . Let us insert a formula for the length p' of the indecomposable projective Λ' -modules. We assume that Λ is a connected Nakayama algebra and that \mathcal{C} is non-zero. Let E_1, \dots, E_g be a complete set of elementary Gorenstein projective modules and let $p_i = |P(E_i)|$. Then *the length p' of the indecomposable projective Λ' -modules is given by the formula*

$$p' = \frac{1}{s} \sum_{i=1}^g p_i$$

Proof: Let $E_i = E_j$ provided $i \equiv j \pmod{g}$. We can assume that $\text{Ext}^1(E_i, E_{i+1}) \neq 0$ for all i , thus $P(E_i)$ has in \mathcal{C} a composition series with factors $E_i, E_{i+1}, \dots, E_{i+p'-1}$ going down. Then

$$\begin{aligned} \sum_{i=1}^g p_i &= \sum_{i=1}^g |P(E_i)| = \sum_{i=1}^g \sum_{j=0}^{p'-1} |E_{i+j}| = \sum_{j=0}^{p'-1} \sum_{i=1}^g |E_{i+j}| \\ &= \sum_{i=0}^{p'-1} s = p' s. \end{aligned}$$

Here, we have used that for any j the sequence E_{j+1}, \dots, E_{j+g} is obtained from the sequence E_1, \dots, E_g just by permutation and that $\sum_{i=1}^g |E_i| = s$, see Proposition 2.

5. Gorenstein algebras.

We want to present a proof of part (a) of Proposition 5. We assume in this section that $p > s$.

Proposition 6. *Let $p > s$. The Nakayama algebra Λ is a Gorenstein algebra if and only if all cycles in the resolution quiver of Λ are black. In this case, the Gorenstein dimension of Λ is equal to $2d$, where d is the maximal distance between vertices and the black cycles.*

We need the following lemmas.

Lemma 9. *Let $x = x_0 \rightarrow \dots \rightarrow x_d$ be a path in the resolution quiver such that x_d is cyclically black, whereas x_{d-1} is not. Then $G\text{-dim } H(x) = 2d$.*

Proof. We show that $\Omega_{2d-1}H(x)$ is not Gorenstein projective, whereas $\Omega_{2d}H(x)$ is Gorenstein projective. According to Lemma 4 we know that $\Omega_{2d}H(x) = H(\gamma^d x) = H(x_d)$ and this is a Gorenstein projective module. On the other hand, there is the following exact sequence

$$0 \rightarrow H(x_d) \rightarrow P(x_{d-1}) \xrightarrow{f} P(x_{d-1}) \rightarrow H(x_{d-1}) \rightarrow 0$$

and $\Omega_{2d-1}H(x)$ is the image of f . In particular, we see that the top of $\Omega_{2d-1}H(x)$ is $S(x_{d-1})$ and this is a red vertex, according to the red entrance lemma. Since the top of $\Omega_{2d-1}H(x)$ is not even black, $\Omega_{2d-1}H(x)$ cannot be Gorenstein projective.

Lemma 10. *Assume that for any path $x_0 \rightarrow \dots \rightarrow x_d$ in the resolution quiver, the vertex x_d is cyclically black. Then the G-dimension of any Λ -module is at most $2d$.*

Proof. We show that $\Omega_{2d}(M)$ is Gorenstein projective, for any indecomposable module M . According to Lemma 5, we have to show that the modules $\text{top } \Omega_{2d}(M)$ and $\text{top } \Omega_{2d+1}(M)$ are zero or cyclically black. But $\text{top } \Omega_{2d}(M)$ is zero or equal to $\gamma^d \text{top } M$, and $\text{top } \Omega_{2d+1}(M)$ is zero or equal to $\gamma^d \text{top } \Omega M$. The assumption of the lemma can be rephrased by saying that $\gamma^d S$ is cyclically black for all simple modules S . This completes the proof.

Proof of Proposition 6. First, assume that there is a cycle in the resolution quiver which involves a red vertex, say the red vertex x . Then a minimal projective resolution of $H(x)$ involves infinitely many copies of $P(x)$ and $P(x)$ is not minimal projective, thus $H(x)$ has infinite G -dimension. This shows that Λ is not a Gorenstein algebra.

On the other hand, if all the cycles in the resolution quiver are black, then all indecomposable modules have finite G -dimension, according to Lemma 8. Thus Λ is a Gorenstein algebra.

Proposition 7. *If Λ is a Gorenstein algebra with $p > s$, then $v(\Lambda) \leq 2s - 2$.*

Proof. Any path $x = x_0 \rightarrow \cdots \rightarrow x_d$ in R such that x_{d-1} does not belong to a cycle involves pairwise different vertices, thus $d \leq s - 1$. Thus, Lemma 8 asserts that the G -dimension of any Λ -module is at most $2s - 2$.

As Xiao-Wu Chen has pointed out, this result (and its proof) corresponds to Gustafson's bound for the finitistic dimension of a Nakayama algebra.

6. Calculation of resolution quivers.

Recall that (p_1, \dots, p_s) is said to be a *Kupisch series* for Λ , provided we have labeled the indecomposable projective modules P_1, \dots, P_s such that $\text{rad } P_i$ is a factor module of P_{i+1} (thus provided there is an arrow $S_i \rightarrow S_{i+1}$) and $p_i = |P_i|$. The Kupisch series for Λ are obtained from any one by cyclic permutation.

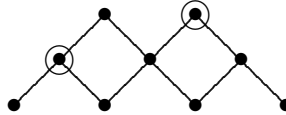
It is an easy exercise to draw the resolution quiver of Λ if a Kupisch series is known. Namely, *if (p_1, \dots, p_s) is a Kupisch series for Λ , then*

$$\gamma(i) \equiv i + p_i \pmod{s}.$$

The *roof* of the Nakayama algebra Λ is the factor algebra $r(\Lambda) = \Lambda / \text{soc}_{p-2} \Lambda$. (Here, for any left module M , the socle sequence $0 = \text{soc}_0 M \subseteq \text{soc}_1 M \subseteq \text{soc}_2 M \subseteq \cdots \subseteq M$ is defined inductively by $\text{soc}_i M / \text{soc}_{i-1} M = \text{soc}(M / \text{soc}_{i-1} M)$, for all $i \geq 1$; note that for $M = {}_\Lambda \Lambda$, all the submodules $\text{soc}_i M$ are two-sided ideals.) It is easy to check that two Nakayama algebras Λ, Λ' have the same roof provided they have Kupisch series (p_1, \dots, p_s) and (p'_1, \dots, p'_s) such that $p_i - p'_i = c$, for $1 \leq i \leq s$, where $c = c(\Lambda, \Lambda')$ is a constant. Let us also stress: *If $c(\Lambda, \Lambda')$ is a multiple of s , then Λ and Λ' have the same resolution quiver.*

In order to visualize a roof $r(\Lambda)$, we will draw its Auslander-Reiten quiver. Actually, instead of drawing the Auslander-Reiten quiver of $r(\Lambda)$, we draw the Auslander-Reiten quiver Γ of a Nakayama algebra with linearly directed quiver of type \mathbb{A}_{s+1} such that the Auslander-Reiten quiver of $r(\Lambda)$ is obtained from Γ by identifying the simple projective with the simple injective module.

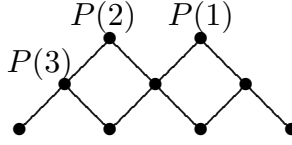
For example, here is the way we present the roof of the Nakayama algebra Λ with Kupisch series $(m+3, m+3, m+2)$, note that the roof $r(\Lambda)$ is the Nakayama algebra with Kupisch series $(3, 3, 2)$:



This is the Auslander-Reiten quiver of a Nakayama algebra with linearly directed quiver of type \mathbb{A}_4 , in order to obtain the Auslander-Reiten quiver of $r(\Lambda)$, the vertex far right has to be identified with the vertex far left (so that the Auslander-Reiten of $r(\Lambda)$ quiver lives on a cylinder).

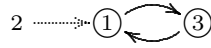
Let us stress the conventions which we use: When we draw an Auslander-Reiten quiver, always the direction of the arrows will be deleted: all arrows are supposed to be directed from left to right. The projective vertices of $r(\Lambda)$ (and thus of Λ) should be labeled **going**

from right to left as $P(1), P(2), P(3)$, see



In the previous picture, two of the projective vertices, namely $P(1)$ and $P(3)$, have been encircled, since they correspond to the minimal projective modules of $r(\Lambda)$ and of Λ .

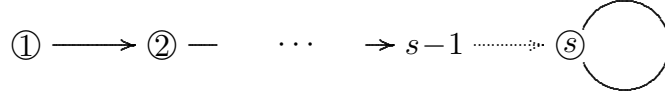
In order to draw the resolution quiver, we use as vertices the numbers $1, 2, \dots, s$ and we draw an arrow $i \rightarrow j$ provided $j \equiv i + p_i \pmod s$. For the convenience of the reader, the black vertices will be encircled and the arrows which start at a red vertex will be dotted (this is quite redundant, but maybe helpful). Thus, the resolution quiver for the algebra with Kupisch series $(5, 5, 4)$ (or, more generally, with Kupisch series $(3m+2, 3m+2, 3m+1)$ for some $m \geq 0$) looks as follows:



The bound $2s - 2$ in Proposition 7 is optimal as the algebras with Kupisch series

$$(ms + 1, ms + 1, \dots, ms + 1, ms)$$

and $m \in \mathbb{N}_1$ show. Namely, the resolution quiver looks as follows

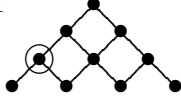
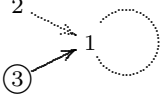
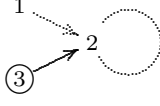
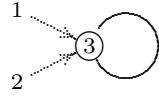
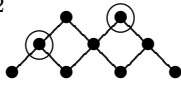
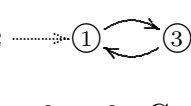

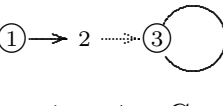
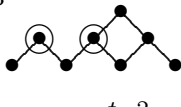
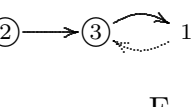
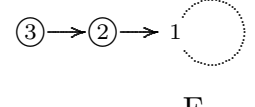
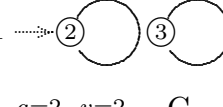
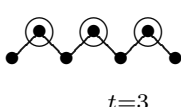
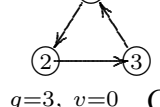
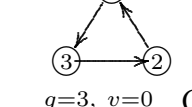
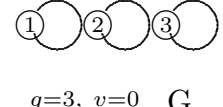


(since $\gamma s = s + p_s = s + ms \equiv s \pmod s$, whereas $\gamma i = i + p_i = i + ms + 1 \equiv i + 1 \pmod s$ for $1 \leq i \leq s - 1$). The vertex $s - 1$ is red, all others are black. In particular, the loop at the vertex s is black. Thus, according to Lemma 9, the path $1 \rightarrow 2 \rightarrow \dots \rightarrow s - 1 \rightarrow s$ shows that the G -dimension of $H(s)$ is equal to $2s - 2$.

Examples. We present the different types of the connected Nakayama algebras with $s = 3, 4, 5$ (and $p > s$). Note that for $s = 3$, this classification can be found already in the paper [CY] by Chen and Ye.

In the tables, we show on the left the different roofs (or at least the upper boundary of the roofs). The remaining columns are indexed by the numbers a with $1 \leq a \leq s$ and show for $s = 3, 4$ the resolution quiver $R(\Lambda)$ of the Nakayama algebras Λ with given roof and $p \equiv a \pmod s$, for $s = 5$ only the cycles of $R(\Lambda)$ are exhibited.

The cases $s = 3$.

	$p \equiv 1 \pmod 3$	$p \equiv 2 \pmod 3$	$p \equiv 0 \pmod 3$
<p>1</p>  <p>$t=1$</p>	 <p>F</p>	 <p>F</p>	 <p>$g=1, v=2$ G</p>
<p>2</p>  <p>$t=2$</p>	 <p>$g=2, v=2$ G</p>	 <p>$g=1$</p>	 <p>$g=1, v=4$ G</p>
<p>3</p>  <p>$t=2$</p>	 <p>F</p>	 <p>F</p>	 <p>$g=2, v=2$ G</p>
<p>4</p>  <p>$t=3$</p>	 <p>$g=3, v=0$ G</p>	 <p>$g=3, v=0$ G</p>	 <p>$g=3, v=0$ G</p>

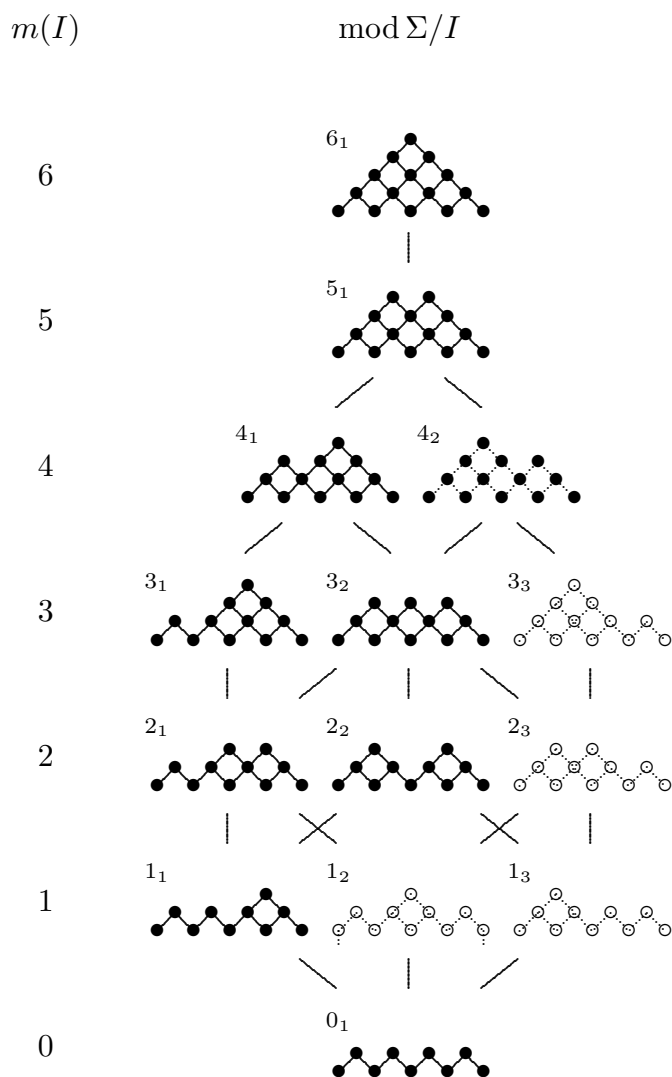
In addition of exhibiting the resolution quiver, we mention the type in question: type G means that we deal with a Gorenstein algebra (all cycles are black); the type F algebras are the algebras with no black cycle in R , thus the algebras of type F are CM-free. There also exist algebras which have cycles which are black as well as cycles which are not black (thus, for $p > s$, they are not Gorenstein algebras, but have non-projective Gorenstein projective modules: see the second row with $p \equiv 2 \pmod 3$). In the left column, we mention the number t of minimal projective modules, or, equivalently, the number of indecomposable projective-injective modules. For the algebras which are not of type F, we add the number g , this is the number of elementary Gorenstein projective modules, provided $p > s$. For the algebras of type G, the number v is the Gorenstein dimension (if d is the maximal distance between a vertex and a cyclically black vertex, then $v = 2d$). The dotted line separates the algebras Λ with loops in $R(\Lambda)$ from those without loops (as we have mentioned in the introduction, we will show in part II [R2] that for a connected Nakayama algebra Λ there is either no loop in $R(\Lambda)$, or else all cycles in $R(\Lambda)$ are loops).

In order to arrange the possible roofs for fixed s , we may proceed as follows: We consider the path algebra $\Sigma = \Sigma(s)$ of the linearly ordered quiver of type \mathbb{A}_{s+1} , and the

lattice of admissible ideals I of Σ , or, equivalently, the corresponding module categories $\text{mod } \Sigma/I$ (using as partial ordering the inclusion functors). For any ideal I we denote by $m(I)$ the number of isomorphism classes of indecomposable Σ/I -modules of length at least 3. Clearly, we have

$$0 \leq m(I) \leq \binom{s}{2}.$$

Here is the **roof diagram** in the case $s = 4$ (as we will remark below, the number of admissible ideals I in $\Sigma(s)$ is the Catalan number C_s and $C_4 = 14$).

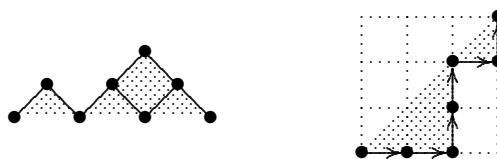


It is sufficient to look at the roofs drawn with black bullets, the remaining ones lead to algebras which are isomorphic to algebras already considered: the algebras with roof 3_3 or 2_3 are isomorphic to algebras with roof 3_1 or 2_1 , respectively; the algebras with roof 1_2 or 1_3 are isomorphic to algebras with roof 1_1 , always using rotations. We may add that the opposite algebra of a Nakayama algebra often is isomorphic to the given algebra —

the only exception for $s = 4$ are the cases 4_1 and 4_2 (the roof 4_2 is obtained from 4_1 by a reflection). Of course, the opposite algebra of a Gorenstein algebra is Gorenstein, the opposite of a CM-free algebra is CM-free. However, the examples 4_1 and 4_2 show that the resolution quiver of the opposite of an algebra Λ may be quite different from $R(\Lambda)$.

Remark. For any s , the number of admissible ideals I of $\Sigma(s)$ is the Catalan number $C_s = \frac{1}{s+1} \binom{2s}{s}$.

Proof: Let us rotate the Auslander-Reiten quiver of Σ/I by 135° . Then the arrows on the boundary of the Auslander-Reiten quiver yield a monotonic path along the edges of a grid with $(s \times s)$ square cells, starting from the lower left corner and ending in the upper right corner (here, monotonic means that we use only edges pointing rightwards or upwards). Here is an example with $s = 3$:



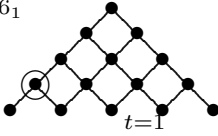
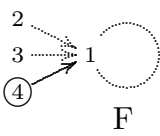
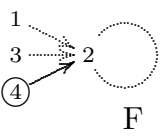
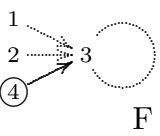
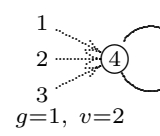
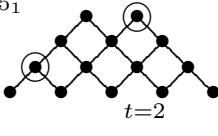
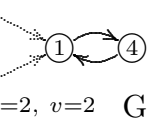
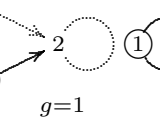
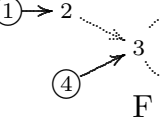
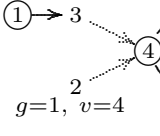
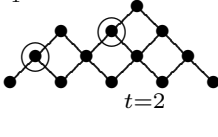
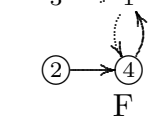
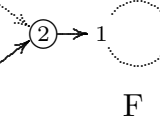
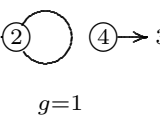
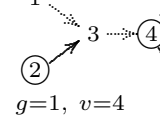
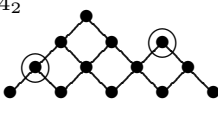
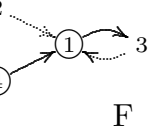
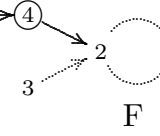
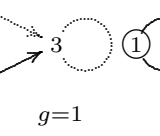
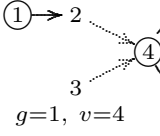
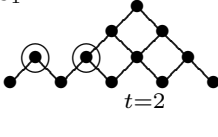
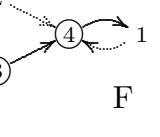
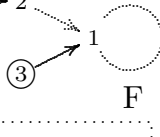
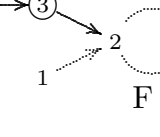
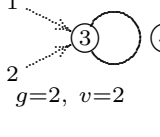
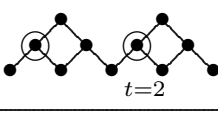
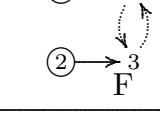
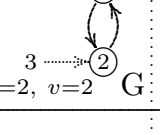
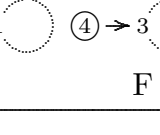
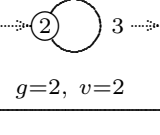
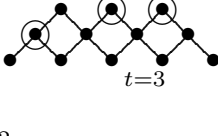
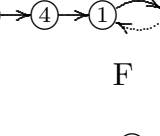
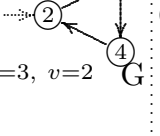
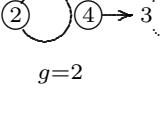
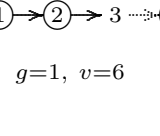
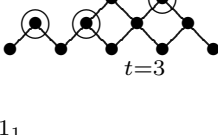
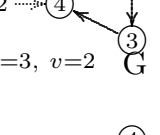
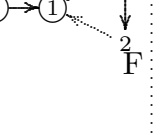
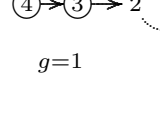
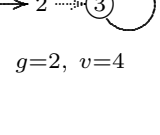
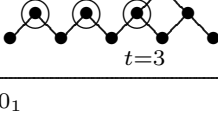
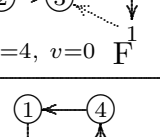
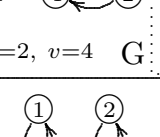
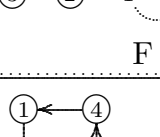
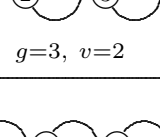

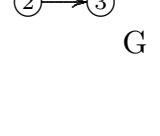
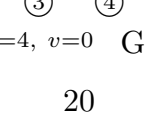
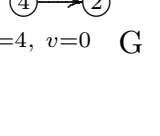
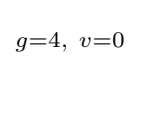
It is well-known that the number of such paths is just C_s .

Here are the Catalan numbers C_s with $1 \leq s \leq 10$.

s	1	2	3	4	5	6	7	8	9	10
C_s	1	2	5	14	42	132	429	1430	4862	16796

In the following table of the Nakayama algebras with $s = 4$, the roofs are ordered slightly different: namely, we use as first criterion the number t of minimal projective modules (or, equivalently, the number of indecomposable projective-injective modules). But then, the algebras with fixed t are presented in the order in which they are obtained in the roof diagram, reading row by row from left to right.

The cases $s = 4$.

	$p \equiv 1 \pmod{4}$	$p \equiv 2 \pmod{4}$	$p \equiv 3 \pmod{4}$	$p \equiv 0 \pmod{4}$
6_1 	 F	 F	 F	 $g=1, v=2$ G
5_1 	 $g=2, v=2$ G	 $g=1$	 F	 $g=1, v=4$ G
4_1 	 F	 F	 $g=1$	 $g=1, v=4$ G
4_2 	 F	 F	 $g=1$	 $g=1, v=4$ G
3_1 	 F	 F	 F	 $g=2, v=2$ G
2_2 	 F	 $g=2, v=2$ G	 F	 $g=2, v=2$ G
3_2 	 F	 $g=3, v=2$ G	 $g=2$	 $g=1, v=6$ G
2_1 	 $g=3, v=2$ G	 F	 $g=1$	 $g=2, v=4$ G
1_1 	 $g=4, v=0$ F	 $g=2, v=4$ G	 F	 $g=3, v=2$ G
0_1 	 G	 $g=4, v=0$ G	 $g=4, v=0$ G	 $g=4, v=0$ G

The cases $s = 5$: the cycles of the resolution quivers.

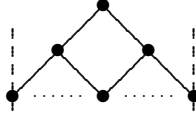
	$p \equiv 1$	$p \equiv 2$	$p \equiv 3$	$p \equiv 4$	$p \equiv 0$
10_1					
9_1					
8_1					
7_1					
6_1					
6_3					
4_2					
7_2					
6_2					
5_1					

	$p \equiv 1$	$p \equiv 2$	$p \equiv 3$	$p \equiv 4$	$p \equiv 0$
5_2 					
5_3 					
4_1 					
3_1 					
3_3 					
2_2 					
4_3 					
3_2 					
2_1 					
1_1 					
0_1 					

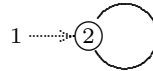
7. Nakayama algebras with $p \leq s$.

Some of our results need the condition $p > s$. Here are examples concerning the possible behavior of algebras with $p \leq s$ (see also [CY]).

Example 1. A Nakayama algebra with a black cycle (and no other cycles) in the resolution quiver which is CM-free: Let $s = p = 2$ with the following Auslander-Reiten quiver:

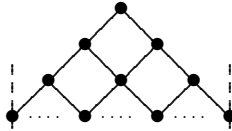


(thus with Kupisch series $(3, 2)$). The resolution quiver looks as follows:

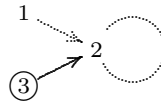


thus there is a black loop (thus a black cycle). However, Λ is CM-free, since it has finite global dimension (the global dimension is 2).

Example 2. A Nakayama algebra with a red loop and no other cycles in the resolution quiver which has finite global dimension: Consider the Kupisch series $(4, 3, 2)$:

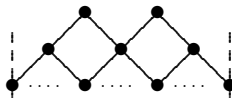


we have $p = 2$ and $s = 3$. The resolution quiver looks as follows:



now we deal with a red loop (thus with a cycle which is not black). As it should be, Λ is CM-free, but it is even of finite global dimension (the global dimension is again 2), thus all the modules have finite projective dimension.

Example 3. An algebra with black and red loops in the resolution quiver which is CM-free: take the Kupisch series $(3, 3, 2)$:



again, $p = 2$ and $s = 3$. The resolution quiver looks as follows:



We have both a red loop and a black loop. The algebra Λ has infinite global dimension but is CM-free.

These examples show that in Proposition 5 the assumption $p > s$ is necessary: Namely, Example 2 exhibits an algebra Λ of finite global dimension, thus a Gorenstein algebra, such that $R(\Lambda)$ has a red loop. Examples 1 and 3 are CM-free algebras with black cycles. Whereas Examples 1 and 2 are algebras of finite global dimension, we do not know any example Λ of finite global dimension such that $R(\Lambda)$ has both black cycles and cycles which are not black.

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