SCIENCE CHINA
Mathematics

Progress of Projects Supported by NSFC

• ARTICLES •

February 2015 Vol. 58 No. 2: 221–232
doi: 10.1007/s11425-014-4954-4

Objective triangle functors

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Received May 16, 2014; accepted November 20, 2014; published online November 27, 2014

Abstract An additive functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is said to be objective, provided any morphism f in $\mathcal{A}$ with $F(f) = 0$ factors through an object K with $F(K) = 0$. We concentrate on triangle functors between triangulated categories. The first aim of this paper is to characterize objective triangle functors F in several ways. Second, we are interested in the corresponding Verdier quotient functors $V_F : \mathcal{A} \to \mathcal{A}/\mathrm{Ker}F$, in particular we want to know under what conditions V_F is full. The third question to be considered concerns the possibility to factorize a given triangle functor $F = F_2F_1$ with F_1 a full and dense triangle functor and F_2 a faithful triangle functor. It turns out that the behavior of splitting monomorphisms and splitting epimorphisms plays a decisive role.

Keywords triangulated category, triangle functor, objective functor, Verdier functor

MSC(2010) 16E30, 18A22, 16E35

Citation: Ringel C M, Zhang P. Objective triangle functors. Sci China Math, 2015, 58: 221–232, doi: 10.1007/s11425-014-4954-4

1 Introduction

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories (all functors considered in this paper are supposed to be covariant and additive). Following [9], we say that F is *objective*, provided any morphism $f : X \to Y$ in $\mathcal{A}$ with $F(f) = 0$ factors through an object K with $F(K) = 0$. We say that F is *sincere*, provided that F sends non-zero objects to non-zero objects. Clearly, a functor is faithful if and only if it is objective and sincere.

In this paper, we concentrate on triangle functors between triangulated categories. We will see that triangle functors behave quite different from general additive functors between additive categories, and also from exact functors between abelian categories. For examples, a full functor between additive categories may be not objective (see [9, Appendix]), but a full triangle functor between triangulated categories is objective (see Subsection 4.4); on the other hand, an exact functor between abelian categories is clearly objective (see Lemma 8.1), whereas there are sincere triangle functors between triangulated categories which are not objective (see Section 8).

If $\mathcal{I}$ is an ideal of an additive category $\mathcal{A}$, then we denote by $\mathcal{A}/\mathcal{I}$ the corresponding factor category: It has the same objects as $\mathcal{A}$ and

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{I}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)/\mathcal{I}(X, Y),$$

for any pair X, Y of objects in $\mathcal{A}$. We denote by $\pi_{\mathcal{I}} : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ the canonical projection functor, it sends an object X to itself, and a morphism f to its residue class modulo $\mathcal{I}$. Given a full subcategory $\mathcal{U}$ of $\mathcal{A}$, we denote by $\langle\mathcal{U}\rangle$ the ideal generated by $\mathcal{U}$, i.e., the class of morphisms of $\mathcal{A}$ which factor through a finite direct sum of objects in $\mathcal{U}$.

Assume now that $\mathcal{A}$ is a triangulated category and that $\mathcal{K}$ is a triangulated subcategory of $\mathcal{A}$ (triangulated subcategories are always assumed to be full subcategories). Then there is a triangulated category $\mathcal{A}/\mathcal{K}$ and a dense triangle functor $V_{\mathcal{K}}:\mathcal{A}\to\mathcal{A}/\mathcal{K}$ with the following universal property: $V_{\mathcal{K}}(\mathcal{K})=0$, and if $G:\mathcal{A}\to\mathcal{B}$ is a triangle functor with $G(\mathcal{K})=0$, then there is a unique triangle functor $G':\mathcal{A}/\mathcal{K}\to\mathcal{B}$ such that $G=G'V_{\mathcal{K}}$ (see Verdier [10], or also Neeman [7]). We call $V_{\mathcal{K}}$ the *Verdier quotient functor* for $\mathcal{K}$. There is no need to worry about a possible confusion using the same notation $\mathcal{A}/\mathcal{I}$ and $\mathcal{A}/\mathcal{K}$, for ideals $\mathcal{I}$ and triangulated subcategories $\mathcal{K}$, since a subcategory $\mathcal{U}$ of $\mathcal{A}$ is an ideal only in case $\mathcal{U}=\mathcal{A}$, see Lemma 2.1.

If $F:\mathcal{A}\to\mathcal{B}$ is a functor between additive categories, then we denote by $\ker(F)$ the class of morphisms f in $\mathcal{A}$ such that $F(f)=0$, and we denote by $\mathrm{Ker}(F)$ the full subcategory of $\mathcal{A}$ given by all objects X in $\mathcal{A}$ such that $F(X)=0$. Note that $\ker(F)$ is an ideal of $\mathcal{A}$, whereas $\mathrm{Ker}(F)$ is a subcategory, and we have $\langle\mathrm{Ker}(F)\rangle\subseteq\ker(F)$. It is easy to see Lemma 2.2 that a functor F is objective if and only if $\ker(F)=\langle\mathrm{Ker}(F)\rangle$. We will consider the factor category $\mathcal{A}/\ker(F)$ and we write $\pi_F:\mathcal{A}\to\mathcal{A}/\ker(F)$ instead of $\pi_{\ker(F)}$. If $F:\mathcal{A}\to\mathcal{B}$ is a triangle functor between triangulated categories, then the subcategory $\mathrm{Ker}F$ is a triangulated subcategory of $\mathcal{A}$, thus we may consider the Verdier quotient functor $V_{\mathrm{Ker}\,F}$, and we will denote it by $V_F:\mathcal{A}\to\mathcal{A}/\mathrm{Ker}\,F$. Since $F(\mathrm{Ker}F)=0$, the universal property of the Verdier quotient functor asserts that there exists a unique triangle functor $\widetilde{F}:\mathcal{A}/\mathrm{Ker}F\to\mathcal{B}$, such that $F=\widetilde{F}V_F$. The functor $\widetilde{F}$ is always sincere.

The first aim of this paper is to characterize objective triangle functors F in several ways. Second, we are interested in the corresponding Verdier quotient functors V_F, in particular we want to know under what conditions V_F is full. The third question to be considered concerns the possibility to factorize a given triangle functor $F=F_2F_1$ with F_1 a full and dense triangle functor and F_2 a faithful triangle functor.

Two conditions for a triangle functor F will play a decisive role, namely the weak splitting monomorphism condition (WSM) and the isomorphism condition (I). If F is faithful or full, then both conditions are satisfied (see Propositions 3.1–3.3).

(WSM) For each morphism $u:X\to Y$ in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism in $\mathcal{B}$, there exists a morphism $u':Y\to X'$ such that $F(u'u)$ is an isomorphism in $\mathcal{B}$.

(I) For each morphism $u:X\to Y$ in $\mathcal{A}$ such that $F(u)$ is an isomorphism in $\mathcal{B}$, there exists a morphism $u':Y\to X$ such that $F(u)^{-1}=F(u')$.

It is easy to see Proposition 3.1 that a functor F satisfies both conditions (WSM) and (I) if and only if it satisfies the splitting monomorphism condition (SM):

(SM) For each morphism $u:X\to Y$ in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism in $\mathcal{B}$, there exists a morphism $u':Y\to X$ such that $F(u'u)=1_{F(X)}$.

Here are the main results of the paper.

Theorem 1.1. *Let $F:\mathcal{A}\to\mathcal{B}$ be a triangle functor between triangulated categories. Then the following are equivalent*:

(i) *F satisfies the condition* (WSM);

(ii) *F is objective*;

(iii) *the induced functor $\widetilde{F}:\mathcal{A}/\mathrm{Ker}F\to\mathcal{B}$ is faithful.*

We say that an additive category $\mathcal{A}$ is a *Fitting category* provided for any endomorphism $a:X\to X$ in $\mathcal{A}$ there exists a direct decomposition $X=X'\oplus X''$ with $a(X')\subseteq X', a(X'')\subseteq X''$ such that the restriction of a to X' is an automorphism and the restriction of a to X'' is nilpotent. For example, if $\mathcal{A}$ is a Hom-finite k-category, where k is a field, and any object of $\mathcal{A}$ is a finite direct sum of objects with local endomorphism rings, then $\mathcal{A}$ is a Fitting category (see Subsection 5.5).

Theorem 1.2. *Let $F:\mathcal{A}\to\mathcal{B}$ be a triangle functor between triangulated categories.*

(1) *If V_F is full, then F satisfies the condition* (I).

(2) *Assume that F is objective or that $\mathcal{A}$ is a Fitting category. Then V_F is full if and only if F satisfies* (I).

Theorem 1.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then the following are equivalent*:

(i) *F satisfies the condition* (SM);

(ii) *F is objective and V_F is full*;

(iii) *There is an equivalence of additive categories $\Phi : \mathcal{A}/\ker(F) \to \mathcal{A}/\operatorname{Ker}(F)$ such that $V_F = \Phi\pi_F$*;

(iv) *There is factorization $F = F_2F_1$, where F_1 is a full and dense triangle functor and F_2 is a faithful triangle functor*;

(v) *There is factorization $F = F_2F_1$, where F_1 is a full triangle functor and F_2 is a faithful triangle functor.*

2 Preliminaries

Ideals and subcategories of additive categories.

Lemma 2.1. *A subcategory $\mathcal{U}$ of an additive category $\mathcal{A}$ is an ideal if and only if $\mathcal{U} = \mathcal{A}$.*

Proof. Let $\mathcal{U}$ be an ideal of $\mathcal{A}$ (this means that $\bigcup_{X,Y\in\operatorname{Obj}\mathcal{U}} \operatorname{Hom}_{\mathcal{U}}(X, Y)$ is an ideal of $\mathcal{A}$). For each object X in $\mathcal{A}$, by definition the zero morphism 0_X belongs to any ideal, thus to $\mathcal{U}$. This means X is an object of $\mathcal{U}$. But since $\mathcal{U}$ is a subcategory, 1_X is in $\mathcal{U}$. Thus any morphism of $\mathcal{A}$ is in $\mathcal{U}$. So $\mathcal{U} = \mathcal{A}$. □

Objective functors. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Recall that we denote by $\ker(F)$ the class of morphisms f in $\mathcal{A}$ such that $F(f) = 0$ (this is an ideal of the category $\mathcal{A}$) and by $\operatorname{Ker}(F)$ the full subcategory of $\mathcal{A}$ given by all objects X in $\mathcal{A}$ such that $F(X) = 0$. Clearly, $\langle\operatorname{Ker}(F)\rangle \subseteq \ker(F)$.

Lemma 2.2. *A functor F is objective if and only if* $\ker(F) = \langle\operatorname{Ker}(F)\rangle$.

Proof. Note that $\ker(F) = \langle\operatorname{Ker}(F)\rangle$ if and only if $\ker(F) \subseteq \langle\operatorname{Ker}(F)\rangle$, i.e., if and only if $F(f) = 0$ implies that f factors though an object K in $\operatorname{Ker}(F)$. So we see that $\ker(F) = \langle\operatorname{Ker}(F)\rangle$ if and only if F is objective. □

Triangulated categories and triangle functors. A triangulated category is of the form $\mathcal{T} = (\mathcal{T}, [1], \mathcal{E})$, where $\mathcal{T}$ is an additive category, [1] is an automorphism of $\mathcal{T}$, and $\mathcal{E}$ is a class of sixtuples of the form $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} X[1]$ with objects X, Y, Z and morphisms u, v, w (usually, we will denote such a sixtuple just by (X, Y, Z, u, v, w)) satisfying some well-known axioms. The sixtuples in $\mathcal{E}$ are said to be the *distinguished triangles*. If $\mathcal{A}$ and $\mathcal{B}$ are triangulated categories, a triangle functor from $\mathcal{A}$ to $\mathcal{B}$ is a pair $F = (F, \xi)$, where $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, and $\xi : F \circ [1] \to [1] \circ F$ is a natural isomorphism, such that if (X, Y, Z, u, v, w) is a distinguished triangle in $\mathcal{A}$, then $(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w))$ is a distinguished triangle in $\mathcal{B}$. We should stress that given a triangle functor (F, ξ), there may not exist a triangle functor (F', ξ') with ξ' the identity transformation such that F and F' are equivalent (as pointed out by Keller [5], see Bocklandt [1, Appendix]). Note that triangle functors are also called exact functors or triangulated functors (see [2–4, 7]), but we follow the terminology used for example by Keller [5].

The Verdier quotient functor. Let us recall some property of the Verdier quotient functor $V_{\mathcal{K}} : \mathcal{A} \to \mathcal{A}/\mathcal{K}$. Any morphism x in $\mathcal{A}/\mathcal{K}$ can be written as a right fraction $x = a/s = V_{\mathcal{K}}(a)V_{\mathcal{K}}(s)^{-1}$, where $a : X' \to Y$ and $s : X' \to X$ are morphisms in $\mathcal{A}$ such that there exists a distinguished triangle (X', X, K, s, v, w) in $\mathcal{A}$ with $K \in \mathcal{K}$. Note that x can also be written as a left fraction $x = s'\backslash a' = V_{\mathcal{K}}(s')^{-1}V_{\mathcal{K}}(a')$ for some morphisms $a' : X \to Y'$ and $s' : Y \to Y'$ in $\mathcal{A}$ with a distinguished triangle (Y, Y', K', s', v', w') in $\mathcal{A}$ such that K' belongs to $\mathcal{K}$. This follows directly from the construction of $\mathcal{A}/\mathcal{K}$ using the calculus of fractions.

Isomorphisms and splitting monomorphisms in triangulated categories. Recall that in a distinguished triangle (X, Y, Z, u, v, w), the morphism u is a splitting monomorphism if and only if v is a splitting epimorphism, and if and only if $w = 0$. Also, u is an isomorphism if and only if $Z = 0$. See Happel [3, Subsections I.1.4 and I.1.7].

3 Some conditions for triangle functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories.

Proposition 3.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Then F satisfies the conditions* (WSM) *and* (I) *if and only if it satisfies the condition* (SM).

Proof. Assume that F satisfies the condition (SM). Then clearly F satisfies the conditions (WSM). Let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is an isomorphism. Then $F(u)$ is a splitting monomorphism, and by (SM) there is u' in $\mathcal{A}$ such that $F(u'u) = 1_{F(X)}$. Thus $F(u')F(u) = 1_{F(X)}$ and therefore $F(u)^{-1} = F(u')$. This shows (I).

Conversely, assume that F satisfies the conditions (WSM) and (I). Let $F(u)$ be a splitting monomorphism with $u : X \to Y$. By (WSM) there exists a morphism $a : Y \to X'$ such that $F(au) = F(a)F(u)$ is an isomorphism in $\mathcal{B}$. By (I) there exists a morphism $b : X' \to X$ such that $F(b)F(au) = 1_{F(X)}$. Put $u' := ba : Y \to X$. Then $F(u')F(u) = F(b)F(au) = 1_{F(X)}$. This proves that F satisfies (SM). □

We need two further conditions for a functor $F : \mathcal{A} \to \mathcal{B}$.

(RSM) F reflects splitting monomorphisms (this means: If u is a morphism in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism in $\mathcal{B}$, then u is a splitting monomorphism in $\mathcal{A}$).

(RI) F reflects isomorphisms (this means: If u is a morphism in $\mathcal{A}$ such that $F(u)$ is an isomorphism in $\mathcal{B}$, then u is an isomorphism in $\mathcal{A}$).

Proposition 3.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then F is faithful if and only if F satisfies the conditions* (RSM).

Proof. Assume that $F = (F, \xi)$ is faithful. Let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism. Let (X, Y, Z, u, v, w) be a distinguished triangle in $\mathcal{A}$. Then

$$(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w))$$

is a distinguished triangle in $\mathcal{B}$. Since $F(u)$ is a splitting monomorphism, $\xi_X F(w) = 0$, thus also $F(w) = 0$. Since F is faithful, $w = 0$, thus u is a splitting monomorphism. Thus, the condition (RSM) is satisfied.

Conversely, assume that (RSM) holds. Let $w : Z \to X[1]$ be a morphism such that $F(w) = 0$. Then there is a distinguished triangle (X, Y, Z, u, v, w) in $\mathcal{A}$ and $(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w))$ is a distinguished triangle in $\mathcal{B}$. Since $F(w) = 0$, also $\xi_X F(w) = 0$, thus $F(u)$ is a splitting monomorphism. Since F satisfies the condition (RSM), u is a splitting monomorphism, thus $w = 0$. This shows that F is faithful. □

Proposition 3.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. If F is full or faithful, then it satisfies the condition* (SM).

Proof. Let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism. Thus there is $b : F(Y) \to F(X)$ such that $bF(u) = 1_X$.

If F is full, then $b = F(u')$ for some $u' : Y \to X$, and $F(u')F(u) = 1_{F(X)}$ shows that (SM) is satisfied.

If F is faithful, then by Proposition 3.2 F satisfies the condition (RSM). Thus there is u' in $\mathcal{A}$ such that $u'u = 1_X$, hence $F(u'u) = 1_{F(X)}$. □

Proposition 3.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then F is sincere if and only if F satisfies the condition (RI).

Proof. Assume that F is sincere. Let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is an isomorphism. Let (X, Y, Z, u, v, w) be a distinguished triangle in $\mathcal{A}$. Then $(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w))$ is a distinguished triangle in $\mathcal{B}$. Since $F(u)$ is an isomorphism, we have $F(Z) = 0$. Since F is sincere, this implies that $Z = 0$, thus u is an isomorphism. This shows that F reflects isomorphisms.

Conversely, assume that F reflects isomorphisms. In order to show that F is sincere, let Z be an object in $\mathcal{A}$ such that $F(Z) = 0$. Consider the map $u : Z \to 0$ in $\mathcal{A}$. If we apply F, we obtain a map $F(u) : F(Z) \to 0$. Since $F(Z) = 0$, the map $F(u)$ is an isomorphism. Since F reflects isomorphisms, we see that u itself is an isomorphism, but this means that $Z = 0$. □

4 Objectivity of triangle functors

The aim of this section is to prove Theorem 1.1 and to draw the attention to some consequences.

4.1 The proof of Theorem 1.1

Proof of Theorem 1.1. (i) ⇒ (ii). Assume that F satisfies the condition (WSM). Let $w : Z \to X[1]$ be a morphism in $\mathcal{A}$ with $F(w) = 0$. We take a distinguished triangle (X, Y, Z, u, v, w) in $\mathcal{A}$. Under F, we obtain the distinguished triangle

$$(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w)).$$

Since $F(w) = 0$, $F(u)$ is a split monomorphism. The condition (WSM) provides a morphism $u' : Y \to X'$ such that $F(u'u)$ is an isomorphism. We need a distinguished triangle involving $u'u$, say $(X, X', K, u'u, f, h)$. This yields the following commutative square on the left:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \| & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle g} & & \| \\ X & \xrightarrow{u'u} & X' & \xrightarrow{f} & K & \xrightarrow{h} & X[1]. \end{array}$$

So we get a morphism $g : Z \to K$ such that $w = hg$. Applying F to $(X, X', K, u'u, f, h)$, we obtain the distinguished triangle

$$(F(X), F(X'), F(K), F(u'u), F(f), \xi_X F(h)).$$

Since $F(u'u)$ is an isomorphism, $F(K) = 0$. Thus $w = hg$ is the required factorization.

(ii) ⇒ (iii). Assume that F is objective. In order to show that $\widetilde{F}$ is faithful, let a/s be a morphism in $\mathcal{A}/\operatorname{Ker} F$ with $\widetilde{F}(a/s) = 0$. Since $\widetilde{F}(a/s) = F(a)F(s)^{-1}$, we have $F(a) = 0$. Since F is objective, a factors through an object K with $F(K) = 0$. Thus $V_F(a)$ factors through $V_F(K) = 0$, it follows that $V_F(a) = 0$, therefore also $a/s = V_F(a)V_F(s)^{-1} = 0$.

(iii) ⇒ (i). Assume that $\widetilde{F}$ is faithful. Then $\widetilde{F}$ is objective. Let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism. Let (X, Y, Z, u, v, w) be a distinguished triangle in $\mathcal{A}$. Thus $(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w))$ is a distinguished triangle in $\mathcal{B}$. Since $F(u)$ is a splitting monomorphism, we know that $\xi_X F(w) = 0$, thus $F(w) = 0$. Since $F = \widetilde{F}V_F$ and $\widetilde{F}$ is faithful, we see that $V_F(w) = 0$. It follows that $V_F(u)$ is a splitting monomorphism, thus there is some x in $\mathcal{A}/\operatorname{Ker} F$ such that $xV_F(u) = 1_{V_F(X)}$. As we know, the morphism x can be written in the form $x = V_F(s)^{-1}V_F(u')$ for some morphisms $u' : Y \to X'$ and $s : X \to X'$ in $\mathcal{A}$ with $V_F(s)$ invertible. This implies that $V_F(u'u) = V_F(u')V_F(u) = V_F(s)$ is an isomorphism. If we now apply $\widetilde{F}$, we see that also

$$F(u'u) = \widetilde{F}V_F(u'u) = \widetilde{F}V_F(s)$$

is an isomorphism. □

4.2 The Verdier quotient functors are objective

As an immediate consequence of Theorem 1.1, we recover the following well-known result (see, for example, Krause [6, Proposition 4.6.2]):

Corollary. *Let $\mathcal{A}$ be a triangulated category and $\mathcal{K}$ a triangulated subcategory of $\mathcal{A}$. Then the Verdier quotient functor $V_{\mathcal{K}} : \mathcal{A} \to \mathcal{A}/\mathcal{K}$ is objective.*

Proof. Note that $\mathrm{Ker}(V_{\mathcal{K}}) = \overline{\mathcal{K}}$, where $\overline{\mathcal{K}}$ is the full subcategory of $\mathcal{A}$ consisting of direct summands of objects in $\mathcal{K}$. By the universal property we know that $\widetilde{V}_{\mathcal{K}} : \mathcal{A}/\overline{\mathcal{K}} \to \mathcal{A}/\mathcal{K}$ is an equivalence. By Theorem 1.1, $V_{\mathcal{K}}$ is objective. □

4.3 Sincere triangle functors

It is well known that a sincere triangle functor F which is full is also faithful, see Rickard [8, p. 446, Subsection l.1]. The previous discussions provide necessary and sufficient conditions for a sincere functor to be faithful: Namely, for any triangle functor F, there are the following implications:

$$\text{faithful} \iff \text{(RSM)} \implies \text{(SM)} \implies \text{(WSM)} \iff \text{objective}$$

(see Propositions 3.1–3.2 and Theorem 1.1). Since a sincere objective functor is of course faithful, all these conditions are equivalent in case F is sincere.

4.4 Full triangle functors are objective

Corollary. *A full triangle functor is objective*

Proof. According to Proposition 3.3, a full triangle functor satisfies the condition (SM), thus (WSM), and therefore F is objective by Theorem 1.1. □

5 Triangle functors F with V_F full

The aim of this section is to present the proof of Theorem 1.2.

5.1 If V_F is full, then (I) is satisfied

Assume that $F : \mathcal{A} \to \mathcal{B}$ is a triangle functor such that V_F is full. We want to show that F satisfies (I). Let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is an isomorphism, and (X, Y, Z, u, v, w) a distinguished triangle in $\mathcal{A}$. Thus,

$$(F(X), F(Y), F(Z), F(u), F(V), \xi_X F(w))$$

is a distinguished triangle in $\mathcal{B}$. Since $F(u)$ is an isomorphism, $F(Z) = 0$, thus Z belongs to $\mathrm{Ker}(F)$ and therefore $V_F(u)$ is invertible in $\mathcal{A}/\mathrm{Ker}(F)$. Since V_F is full, there is a map $u' : Y \to X$ such that

$$V_F(u') = (V_F(u))^{-1}.$$

If we apply the functor $\widetilde{F}$ (with $F = \widetilde{F}V_F$) we see that $F(u') = F(u)^{-1}$.

5.2 The Verdier quotient functor V_F for a functor F satisfying (I)

Assume that $F : \mathcal{A} \to \mathcal{B}$ is a triangle functor satisfying the condition (I). Note that the morphisms of $\mathcal{A}/\mathrm{Ker}(F)$ are of the form

$$a/s = V_F(a)(V_F(s))^{-1},$$

where a and s are morphisms in $\mathcal{A}$ with $V_F(s)$ being invertible. In order to show that V_F is full, it is sufficient to show that the morphisms of the form $(V_F(s))^{-1}$ are in the image of V_F.

5.3 The case when F is objective

Let $F : \mathcal{A} \to \mathcal{B}$ be an objective triangle functor which satisfies the condition (I). Let s be a morphism in $\mathcal{A}$ such that $V_F(s)$ is invertible. Applying $\widetilde{F}$ (with $\widetilde{F}V_F = F$) to $V_F(s)$ we see that $F(s) = \widetilde{F}V_F(s)$ is invertible in $\mathcal{B}$. Since F satisfies the condition (I), there is $s' : Y \to X$ such that $F(s') = (F(s))^{-1}$. Now F is objective, thus the functor $\widetilde{F}$ is faithful by Theorem 1.1. It follows from

$$\widetilde{F}V_F(s') = (\widetilde{F}V_F(s))^{-1}$$

that $V_F(s') = (V_F(s))^{-1}$. This proves that V_F is full.

5.4 The case when $\mathcal{A}$ is a Fitting category

Let us assume now that $\mathcal{A}$ is a Fitting category. If a is an endomorphism of $X = X' \oplus X''$ with $a(X') \subseteq X', a(X'') \subseteq X''$, then we write $a = a' \oplus a''$.

Consider a triangle functor F satisfying the condition (I) and let us write $V = V_F$. Let $s : X \to Y$ be a morphism in $\mathcal{A}$ such that $V(s)$ is invertible. Thus $F(s) = \widetilde{F}V(s)$ is invertible. By the condition (I) there is a morphism $t : Y \to X$ such that $F(s)^{-1} = F(t)$.

Put $a = ts : X \to X$. Since $\mathcal{A}$ is a Fitting category, there is a direct decomposition $X = X' \oplus X''$ with $a(X') \subseteq X', a(X'') \subseteq X''$ (thus $a = a' \oplus a''$) such that the restriction a' of a to X' is an automorphism, whereas the restriction a'' of a to X'' is nilpotent. Applying F, we see that

$$1_{F(X)} = F(a) = F(a') \oplus F(a'').$$

Since $F(a'')$ is nilpotent, it follows that $F(X'') = 0$ and therefore $V(X'') = 0$. We denote by $u' : X' \to X$ the canonical inclusion, by $p' : X \to X'$ the canonical projection, thus $p'u' = 1_{X'}$ and $p'au' = a'$. Now

$$(X', X, X'', u', p'', 0) \quad \text{and} \quad (X'', X, X', u'', p', 0)$$

are distinguished triangles in $\mathcal{A}$. It follows that $(V(X'), V(X), 0, V(u'), 0, 0)$ is a distinguished triangle in $\mathcal{A}/\operatorname{Ker} F$. It follows that $V(u')$ is an isomorphism, thus $V(p')V(u') = 1_{V(X')}$ implies that

$$V(u')V(p') = 1_{V(X)}.$$

Let $b' = (a')^{-1} : X' \to X'$. We consider the map

$$u'b'p'tsu'p' = u'b'p'au'p' = u'b'a'p' = u'p' : X \to X.$$

Applying V we get

$$V(u'b'p't)V(s) = V(u'b'p'ts) = V(u'b'p'ts)V(u'p') = V(u'b'p'tsu'p') = V(u'p') = 1_{V(X)}.$$

This shows that $V(s)^{-1} = V(u'b'p't)$.

This completes the proof of Theorem 1.2. □

5.5 Examples of Fitting categories

Let k be a field. A Hom-*finite* k-*category* $\mathcal{A}$ is an additive category such that $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is a finite-dimensional k-space for arbitrary objects X and Y of $\mathcal{A}$, such that the composition of morphisms is k-bilinear. A Hom-finite k-category $\mathcal{A}$ is called *a Krull-Remak-Schmidt category* provided all idempotents of $\mathcal{A}$ split. It is well known that *any* Hom-*finite Krull-Remak-Schmidt* k-*category is a Fitting category.*

Let us outline the proof. If Λ is a finite-dimensional k-algebra, the classical Fitting lemma asserts that the category $\operatorname{mod}\Lambda$ of all finite-dimensional Λ-modules is a Fitting category. Of course, also the full subcategory $\operatorname{proj}\Lambda$ of all projective Λ-modules is a Fitting category. Now assume that $\mathcal{A}$ is a Hom-finite Krull-Remak-Schmidt k-category. Let X be an object in $\mathcal{A}$ and $\operatorname{add}(X)$ the full subcategory of all direct summands of finite direct sums of copies of X. Let $\Gamma(X) = \operatorname{End}(X)^{\mathrm{op}}$. Then the category $\operatorname{add}(X)$ is equivalent to the category $\operatorname{proj}\Gamma(X)$ of all projective $\Gamma(X)$-modules, thus it is a Fitting category.

6 Proof of Theorem 1.3

Proof of Theorem 1.3. (i) $\Rightarrow$ (ii). By Proposition 3.1, the condition (SM) is equivalent to (WSM) and (I). By Theorem 1.1, the condition (WSM) implies that F is objective. Thus F is objective and satisfies (I). By Theorem 1.2(2), we see that V_F is full.

(ii) $\Rightarrow$ (iii). We assume that F is objective and V_F is full. By Lemma 2.2, $\ker(F) = \langle \operatorname{Ker}(F) \rangle$. Since V_F is always objective, we similarly have $\ker(V_F) = \langle \operatorname{Ker}(V_F) \rangle$. We always have $\operatorname{Ker}(F) = \operatorname{Ker}(V_F)$, and hence $\langle \mathrm{Ker}(F)\rangle = \langle \mathrm{Ker}(V_F)\rangle$. Thus $\ker(F) = \ker(V_F)$. It follows that there exists a faithful functor $\Phi : \mathcal{A}/\ker(F) \to \mathcal{A}/\mathrm{Ker}(F)$ such that $V_F = \Phi\pi_F$ (namely $\Phi(\overline{f}) = V_F(f)$, where $\overline{f}$ is the residue class of f modulo $\ker(F)$).

Since V_F is full and dense, the factorization $V_F = \Phi\pi_F$ shows that also Φ is full and dense. Altogether we see that Φ is an equivalence.

(iii) $\Rightarrow$ (iv). Let $\Phi : \mathcal{A}/\ker(F) \to \mathcal{A}/\mathrm{Ker}(F)$ be an equivalence with $V_F = \Phi\pi_F$. Since π_F is always full, V_F is full. Since V_F is always objective, $\pi_F = \Phi^{-1}V_F$ is objective. But this implies that F is objective. Namely, assume that $F(f) = 0$. Then $\pi_F(f) = 0$. Since π_F is objective, f factors through an object K with $\pi_F(K) = 0$. Thus 1_K belongs to $\ker(F)$, but this means that $F(K) = 0$.

Now in the factorization $F = \widetilde{F}V_F$, we know that $F_1 := V_F$ is full and dense, and by Theorem 1.1 $F_2 := \widetilde{F}$ is faithful.

(v) $\Rightarrow$ (i). Assume that $F = F_2F_1 : \mathcal{A} \to \mathcal{B}$ with F_1 a full triangle functor and F_2 a faithful triangle functor. In order to show (SM), let $u : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(u)$ is a splitting monomorphism. Consider the distinguished triangle (X, Y, Z, u, v, w) in $\mathcal{A}$. Thus $(F(X), F(Y), F(Z), F(u), F(v), \xi_X F(w))$ is a distinguished triangle in $\mathcal{B}$. Since $F(u)$ is a splitting monomorphism, we know that $\xi_X F(w) = 0$, thus also $F(w) = 0$. Since F_2 is faithful, it follows from $F_2F_1(w) = 0$ that $F_1(w) = 0$, and therefore $F_1(u)$ is a splitting monomorphism. Thus there is a morphism $c : F_1(Y) \to F_1(X)$ such that $cF_1(u) = 1_{F_1(X)}$. Since F_1 is full, there is $u' : Y \to X$ such that $F_1(u') = c$, thus $F_1(u'u) = 1_{F_1(X)}$. We apply F_2 to this equality in order to see that $F(u'u) = 1_{F(X)}$. □

7 Dual conditions

We also may consider dual conditions, in particular the following ones:

(WSE) For each morphism $v : Y \to Z$ in $\mathcal{A}$ such that $F(v)$ is a splitting epimorphism in $\mathcal{B}$, there exists a morphism $v' : Z' \to Y$ such that $F(vv')$ is an isomorphism in $\mathcal{B}$.

(SE) For each morphism $v : Y \to Z$ in $\mathcal{A}$ such that $F(v)$ is a splitting epimorphism in $\mathcal{B}$, there exists a morphism $v' : Z \to Y$ such that $F(vv') = 1_{F(Z)}$.

(RSE) For each morphism $v : Y \to Z$ in $\mathcal{A}$ such that $F(v)$ is a splitting epimorphism in $\mathcal{B}$, the morphism v is a splitting epimorphism in $\mathcal{A}$.

Of course, there are the following trivial implications: (RSE) $\Rightarrow$ (SE) $\Rightarrow$ (WSE). Note that most of the conditions considered in the paper are self-dual conditions: that a functor F is objective or that V_F is full, or that F is faithful, are self-dual conditions. Here, "duality" (or better: Left-right symmetry) refers to the procedure of looking at the opposite $\mathcal{A}^{\mathrm{op}}$ of a given category $\mathcal{A}$, and to consider a functor $F : \mathcal{A} \to \mathcal{B}$ as a functor $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$, with $F^{\mathrm{op}}(X) = F(X), F^{\mathrm{op}}(f) = F(f)$ for any object X and any morphism f in $\mathcal{A}^{\mathrm{op}}$. Since we assume that F is covariant, also F^{op} is covariant. For example, we see that F satisfies (SE) if and only if it satisfies both (WSE) and (I), this is the dual assertion of Proposition 3.1.

By duality, Theorems 1.1, 1.3 and Proposition 3.2 yield:

Theorem 1.1′. *Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then F is objective if and only if F satisfies the condition* (WSE).

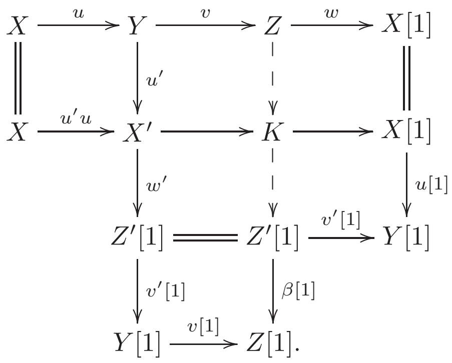
Theorem 1.3′. *Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then F satisfies the condition* (SE) *if and only if F is objective and V_F is full.*

Proposition 3.2′. *Let $F : \mathcal{A} \to \mathcal{B}$ be a triangle functor between triangulated categories. Then F is faithful if and only if F satisfies the condition* (RSE).

In particular, we see that F satisfies the condition (WSM) if and only if it satisfies (WSE), that F satisfies the condition (SM) if and only if it satisfies (SE), and that F satisfies the condition (RSM) if and only if it satisfies (RSE).

As a bonus for the reader, let us insert a direct proof that the condition (WSM) for a triangle functor F implies the condition (WSE):

Proof. Assume that F is a triangle functor which satisfies the condition (WSM). Given a morphism $v: Y \to Z$ such that $F(v)$ is a splitting epimorphism, consider a distinguished (X, Y, Z, u, v, w). Applying F we know that $F(v)$ is a splitting epimorphism, thus $F(u)$ is a splitting monomorphism. By (WSM), there exists a morphism $u' : Y \to X'$ such that $F(u')F(u)$ is an isomorphism in $\mathcal{B}$. We embed u' into a distinguished triangle $(Y, X', Z'[1], u', w', v'[1])$. By the octahedral axiom we get the following commutative diagram:



Since $F(u'u)$ is an isomorphism, it follows that $F(K) = 0$, and hence $F(\beta[1])$ is an isomorphism. Thus $F(vv') = F(v)F(v') = F(\beta)$ is an isomorphism. This shows that F satisfies the condition (WSE). □

8 Examples of triangle functors which are not objective

We present examples of triangle functors which are not objective (but sincere).

Lemma 8.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then F is objective. Thus, an exact sincere functor between abelian categories is faithful.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{A}$ such that $F(f) = 0$. Let I be the image of f, say $f = hg$ with $g : X \to I$ an epimorphism and $h : I \to Y$ a monomorphism. Then

$$F(f) = F(hg) = F(h)F(g).$$

Since F is exact, $F(g)$ is epic and $F(h)$ is monic. Thus $F(I)$ is the image of $F(f)$. Since $F(f) = 0$, it follows that $F(I) = 0$. By definition F is objective. □

Proposition 8.2. *Let $F_0 : \mathcal{A} \to \mathcal{B}$ be an exact sincere functor between abelian categories. Then F_0 induces a sincere triangle functor $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$, where $D^b(\mathcal{A})$ is the bounded derived category of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is not semi-simple whereas $\mathcal{B}$ is semi-simple, then F is not objective.*

Let us add that such a functor F always satisfies the condition (I). Namely, since F is sincere, the Verdier quotient functor V_F is the identity functor, in particular V_F is full. Thus, according to Theorem 1.2, F satisfies the condition (I).

Proof. Since $F_0 : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories, it induces a triangle functor $F = (F, \mathrm{Id}) : D^b(\mathcal{A}) \to D^b(\mathcal{B})$, which maps a complex C with cohomology $\boldsymbol{H}^n(C)$ to the complex $F(C)$ with cohomology $F_0(\boldsymbol{H}^n(C)) = \boldsymbol{H}^n(F(C))$.

Assume that $F(C) = 0$. Then $F_0(\boldsymbol{H}^n(C)) = 0$, for all $n \in \mathbb{Z}$. Since F_0 is sincere, it follows that $\boldsymbol{H}^n(C) = 0$, for all $n \in \mathbb{Z}$, thus C is acyclic and therefore $C = 0$ in $D^b(\mathcal{A})$. This shows that F is sincere.

Since $\mathcal{A}$ is not semi-simple, there exist objects X and Y in $\mathcal{A}$ such that $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) \neq 0$. Since $\mathcal{B}$ is semi-simple, we have

$$\mathrm{Ext}^1_{\mathcal{B}}(F(X), F(Y)) = 0.$$

Thus $\mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[1]) \neq 0$, but

$$\mathrm{Hom}_{D^b(\mathcal{B})}(F(X), F(Y)[1]) = 0,$$

i.e., F is not faithful. It follows that F cannot be objective, since sincere objective functors are faithful. □

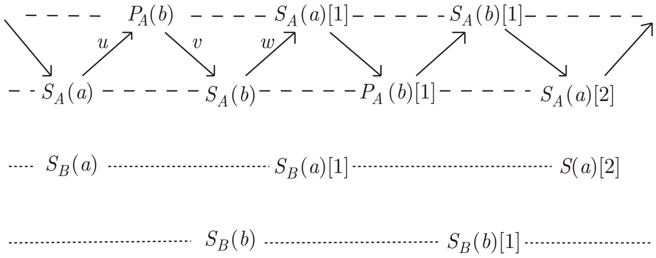


Figure 1 Auslander-Reiten quivers of $D^b(A)$ and Auslander-Reiten quivers of $D^b(B)$

Example. Let us consider an example in detail. Let A be the path algebra of the quiver $b \to a$ over the field k and B the semisimple algebra given by the quiver with the two vertices a, b and no arrow. Then B is a subalgebra of A and we consider the forgetful functor $F_0 : A\text{-mod} \to B\text{-mod}$, given by the inclusion $B \hookrightarrow A$.

Given a vertex x, we denote by $S_A(x)$ the simple A-module of B-module, and by $P_A(x)$ the indecomposable projective A-module, corresponding to x. The functor F_0 sends $S_A(x)$ to $S_B(x)$ for $x = a, b$, and it sends $P_A(b)$ to $S_B(a) \oplus S_B(b)$. Clearly, F_0 is an exact and faithful functor.

The upper part of the following picture (see Figure 1) shows the Auslander-Reiten quiver of $D^b(A)$, the dashed lines indicate the mesh relations. The lower part is the Auslander-Reiten quiver of $D^b(B)$ (it just consists of isolated vertices), and here we use dotted lines to indicate the two shift orbits in $D^b(B)$, see Figure 1.

The induced functor

$$F : D^b(A) \to D^b(B)$$

sends $S_A(x)[i]$ to $S_B(x)[i]$ for $x = a, b$ and all $i \in \mathbb{Z}$, and it sends $P_A(b)[i]$ to $S_A(a)[i] \oplus S_A(b)[i]$. In $D^b(A)$ we have labeled three arrows u, v, w, they form a distinguished triangle

$$(S(a), P(b), S(b), u, v, w).$$

Consider the map

$$w : S_A(b) \to S_A(a)[1].$$

Since

$$\mathrm{Hom}_{D^b(B)}(S_B(b), S_B(a)[1]) = 0,$$

we have $F(w) = 0$. Thus, we see that F is not faithful.

On the other hand, consider the map

$$u : S_A(a) \to P_A(b).$$

Applying the functor F, we obtain the inclusion map

$$S_B(a) \to S_B(a) \oplus S_B(b),$$

which is splitting mono: There is a projection map

$$u' : S_B(a) \oplus S_B(b) \to S_B(a)$$

with $u'F(u) = 1_{S_B(a)}$. Since there is no non-zero map $P_A(b) \to S_A(a)$, such a map u' is not in the image of F. This shows that the condition (SM) is not satisfied. Thus, Theorem 1.1 asserts that F is not objective.

9 Overlook

9.1 The main conditions

We consider any triangle functor $F : \mathcal{A} \to \mathcal{B}$ (see Figure 2). The conditions in any box, line by line, are equivalent; note that in the central box which mentions the conditions (SM) and (SE), the dots indicate that there are several further equivalent conditions, namely the conditions (iii)–(v) mentioned in Theorem 1.2 as well as the conjunction of the conditions (WSM) and (I), see Proposition 3.1, and dually also the conjunction of (WEM) and (I).

The arrows show the relevant implications between the boxes. The dashed implication with the label $(*)$ is valid under the assumption that F is objective or that $\mathcal{A}$ is a Fitting category.

9.2 References

Two implications are trivial: a faithful functor is of course sincere. And if F is a sincere triangle functor, then V_F is the identity functor, thus full. Here are the references for the remaining implications mentioned in Figure 3.

9.3 Examples

Finally, let us outline typical examples in order to see that the implications (A)–(F) cannot be reversed (see Figure 4):

(A) Take any faithful functor which is not full.

(B) Take any full functor which is not faithful.

(C) Take any sincere functor F which is not objective as presented in Section 8. Such a functor is of course not faithful.

(D) In order to find an objective functor F such that V_F is not full, consider a Verdier quotient functor $V_{\mathcal{K}}$, these functors are very seldom full!

For example, let A be an Artin algebra, $\operatorname{mod} A$ the category of finitely generated A-modules, $K^-(A)$ the homotopy category of the upper bounded complexes over $\operatorname{mod} A$, $\mathcal{E}$ the full subcategory of $K^-(A)$ consisting of the upper bounded acyclic complexes, and $D^-(A)$ the derived category of the upper bounded complexes over $\operatorname{mod} A$. Then we have the Verdier quotient functor $V_{\mathcal{E}} : K^-(A) \to K^-(A)/\mathcal{E} = D^-(A)$. It is well known that $V_{\mathcal{E}}$ is full if and only if A is semi-simple.

(E) Take any sincere functor F, which is not objective as presented in Section 8. Since F is sincere, it satisfies the condition (I) but it cannot satisfy the condition (SM), since otherwise it would be objective.

(F) Take a functor F which is not sincere, such that V_F is full, for example the zero functor $\mathcal{A} \to 0$, where $\mathcal{A}$ is a non-zero triangulated category.

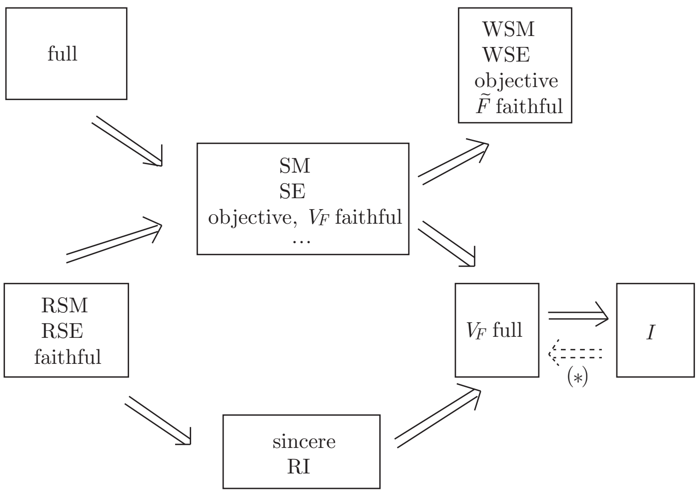


Figure 2 Implications of the conditions in this paper

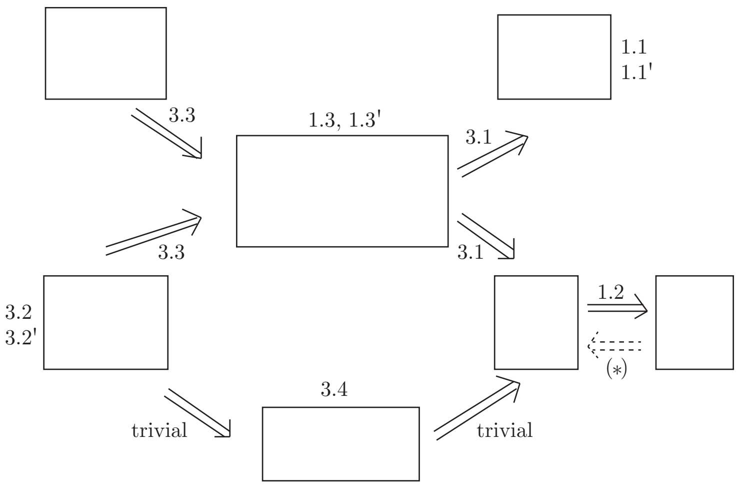


Figure 3 The places where the implications can be founded

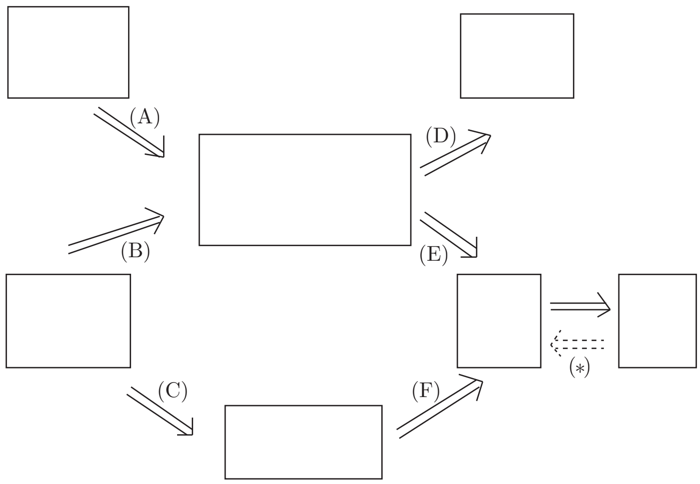


Figure 4 How about the converses of the implications

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11271251 and 11431010), and Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20120073110058). The authors thank the anonymous referees for their carefully reading the manuscript and helpful suggestions.

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