The shift orbits of the graded Kronecker modules.

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Abstract. Let k be a field. The Kronecker modules (or matrix pencils) are the representations of the n-Kronecker quiver $K(n)$; this is the quiver with two vertices, namely a sink and a source, and n arrows. The representations of $K(n)$ play an important role in many parts of mathematics. For $n = 2$, the indecomposable representations have been classified by Kronecker, but not much is known in case $n \geq 3$. In this paper, we usually will assume that $n \geq 3$.

The universal cover of $K(n)$ is the n-regular tree with bipartite orientation. Let $T(n)$ be the n-regular tree. We fix a bipartite orientation Ω of $T(n)$; the opposite orientation will be denoted by $\sigma\Omega$ (thus $\sigma^2\Omega = \Omega$). The k-representations of the quiver $(T(n), \Omega)$ can be considered as graded Kronecker modules and we denote by $\mathrm{mod}(T(n), \Omega)$ the category of these graded Kronecker modules. Only few Kronecker modules can be graded, but the graded Kronecker modules provide hints about the behavior of general Kronecker modules.

There is a reflection functor $\sigma\colon \mathrm{mod}(T(n), \Omega) \to \mathrm{mod}(T(n), \sigma\Omega)$ (the simultaneous Bernstein-Gelfand-Ponomarev reflection at all sinks); it will be called the *shift functor.* An indecomposable graded Kronecker module M is said to be *regular* provided $\sigma^t M \neq 0$ for all $t \in \mathbb{Z}$.

If p, q are vertices of $T(n)$, we denote by $d(p, q)$ their distance. Now, let M be an indecomposable regular representation of $(T(n), \Omega)$. We attach to M a positive integer $r_0(M)$ and a pair $p(M), q(M)$ of vertices of $T(n)$ with $0 \leq d(p(M), q(M)) \leq r_0(M)$ and such that $p(M)$ is a sink if and only if $r_0(M)$ is even. Here are the essential properties of the invariants $r_0(M), p(M), q(M)$. The σ-orbit of M contains a unique sink module M_0 with smallest possible radius, say with radius $r_0 = r_0(M)$. For $i \in \mathbb{Z}$, we write $M_i = \sigma^i M_0$ and call $i = \iota(M_i)$ the *index* of M_i. By duality, the σ-orbit of M contains a unique source module with radius r_0, say M_{b+1}, and we have $b \geq 0$. Let $p(M)$ be the center of M_0, let $q(M)$ be the center of M_{b+1}, and denote by $(p = a_0, a_1, \ldots, a_{b-1}, a_b = q)$ the unique path from p to q. For $i \geq 0$, the module M_{-i} is a sink module with center $p(M)$ and radius $r_0 + i$, whereas the module M_{b+1+i}

is a source module with center $q(M)$ and radius $r_0 + i$. The remaining modules M_i (with $1 \leq i \leq b$) are flow modules with radius $r_0 - 1$, and center $\{a_{i-1}, a_i\}$.

By construction, the triple $r_0(M)$, $p(M)$, $q(M)$ is invariant under the shift. We show that any triple r_0, p, q consisting of a positive integer r_0, and vertices p, q of $T(n)$ with $0 \leq d(p, q) \leq r_0$ and such that p a sink if and only if r_0 is even, arises in this way.

If M, M' are regular indecomposable modules with an irreducible map $M \to M'$, then we show that $\iota(M') = \iota(M) - 1$. In this way, we obtain a global way to index the regular indecomposable modules.

1. Introduction.

Let k be a field. We denote by $T(n)$ the n-regular tree and usually, we will assume that $n \geq 3$. If $(a_0, \ldots, a_t)$ is a path in $T(n)$ and $t = 2r$ is even, then a_r will be called its *center* and r its *radius*. If $t = 2r + 1$, then the pair $\{a_r, a_{r+1}\}$ is called its *center* and r its *radius*.

We fix a bipartite orientation Ω of $T(n)$ and denote by $\sigma\Omega$ the opposite orientation and we set $\sigma^2\Omega = \Omega$. (Recall that an orientation of a graph is bipartite provided any vertex is a sink or a source; there are just two bipartite orientations of $T(n)$.) The finite-dimensional k-representations of the quivers $(T(n), \Omega)$ and $(T(n), \sigma\Omega)$ will be called the *graded Kronecker modules* or just *modules* and we denote by $\mathrm{mod}(T(n), \Omega)$ the category of the k-representations of $(T(n), \Omega)$.

Let M be an indecomposable module. We denote by $T(M)$ its support, it is by definition the full subgraph of $T(n)$ given by all vertices a with $M_a \neq 0$. Any path in $T(M)$ of maximal length will be called a *diameter path*. The length of the diameter paths will be denoted by $d(M)$ and called the *diameter* of M. By definition, the *radius* $r(M)$ of M is the radius of a diameter path of M, thus $r(M) = \lfloor \frac{1}{2}d(M) \rfloor$. *All diameter paths of M have the same center* (see section 2), called the *center of M*.

The module M is called a *sink module* (or a *source module*) provided a diameter path (and hence all diameter paths) starts and ends in sinks (or in sources, respectively). Of course, the diameter of a sink module or a source module is even. If the diameter of M is odd (so that any diameter path for M connects a sink with a source), then M will be called a *flow module*. The center of a sink or a source module is a vertex, the center of a flow module is an edge.

There is a reflection functor $\sigma \colon \mathrm{mod}(T(n), \Omega) \to \mathrm{mod}(T(n), \sigma\Omega)$ (the simultaneous Bernstein-Gelfand-Ponomarev reflection at all sinks); it will be called the *shift functor*. An indecomposable module M is said to be *regular* provided $\sigma^t M \neq 0$ for all $t \in \mathbb{Z}$. If M is regular, we attach to M a positive integer $r_0(M)$ and a path $\pi(M) = (a_0, \ldots, a_b)$ in $T(n)$, such that a_0 is a sink if $r_0(M)$ is even and a source otherwise.

Theorem 1. *Let M be a regular indecomposable module. Then the shift orbit of M contains a unique sink module M_0 with smallest possible radius, say with radius $r_0 = r_0(M)$. Let $M_i = \sigma^i M_0$ for all $i \in \mathbb{Z}$. Then there is an integer $0 \leq b \leq r_0(M)$ and a path $(a_0, \ldots, a_b)$ in $T(n)$ with the following properties:*

(1) *For $i \geq 0$, the module M_{-i} is a sink module with radius $r_0 + i$ and center a_0.*
(2) *For $1 \leq i \leq b$, the module M_i is a flow module with radius $r_0 - 1$ and center $\{a_{i-1}, a_i\}$.*
(3) *For $i \geq 0$, the module M_{b+1+i} is a source module with radius $r_0 + i$ and center a_b.*
The integer $r_0(M)$ is positive. If $r_0(M)$ is even, then a_0 is a sink, otherwise a source.

We call $\pi(M) = (a_0, \ldots, a_b)$ the *center path* of M. and define $p(M) = a_0$, $q(M) = a_b$ and $b(M) = b = d(p(M), q(M))$. By construction, $r_0(M)$ and $\pi(M)$, thus also $p(M), q(M)$ and $b(M)$ are invariant under the shift.

Theorem 2. *Let r be a positive integer. Let $(a_0, \ldots, a_b)$ be a path in $T(n)$ of length $0 \leq b \leq r$. If r is even, we assume that a_0 is a sink, otherwise that a_0 is a source. Then there is a regular indecomposable module M with $r_0(M) = r$ and $\pi(M) = (a_0, \ldots, a_b)$.*

If M is a regular indecomposable module and $M_0 = \sigma^{-i}M$ is the sink module in the σ-orbit of M with smallest possible radius, we call $i = \iota(M)$ the *index* of M, thus $M = \sigma^{\iota(M)}M_0$.

Theorem 3. *If M, M' are regular indecomposable modules and $M \to M'$ is an irreducible map, then $\iota(M') = \iota(M) - 1$.*

Since the regular Auslander-Reiten components of the category $\mathrm{mod}(T(n), \Omega)$ are of the form $\mathbb{Z}\mathbb{A}_\infty$, we may define an operator η on the set of isomorphism classes of indecomposable regular modules as follows: if $0 \to X \to Y \oplus Y' \to Z \to 0$ is an Auslander-Reiten sequence with Y, Y' indecomposable, and $|Y| < |Y'|$, let $\eta Y = Y'$. By Theorem 3, Y and $\eta(Y)$ have the same index.

Theorem 4. *Let Y be an indecomposable regular module. Then Y is a sink module (or a flow module, or a source module), if and only if ηY is a sink module (or a flow module, or a source module, respectively). Also, ηY has the same center as Y and $r(\eta Y) = r(Y) + 2$.*

The proof of Theorem 1 will be given in section 3, the proof of Theorem 2 in section 5. Section 2 collects relevant definitions and some preliminary results. In section 4, we discuss in which way the diameter paths of M and σM are related.

According to Gabriel, the Auslander-Reiten translation τ in $\mathrm{mod}(T(n), \Omega)$ is nothing else than σ^2 (see [Gab]). Thus, Theorem 1 provides information about the τ-orbits of the regular indecomposable modules and on the shape of the components of the Auslander-Reiten quiver, see sections 6 and 7. In particular, section 7 will contain a proof of Theorem 3 and Theorem 4.

It will be convenient to call a module *even* or *odd,* provided it is indecomposable and regular and its index is even or odd, respectively.

2. Preliminaries.

(2.1) Trees. A *graph* G is given by a set G_0 (called the *vertices* of G) and a set G_1 of subsets of G_0 which have cardinality 2 (the elements of G_1 are called the *edges* of G; if $\{a, a'\}$ is an edge, then a, a' are also called *neighbors*). A *path* in G of length $t \geq 0$ is a sequence $(a_0, \ldots, a_t)$ of vertices such that for $1 \leq i \leq t$, the vertices a_{i-1}, a_i are neighbors, and for $1 \leq i < t$, we have $a_{i-1} \neq a_{i+1}$; we say that the path $(a_0, \ldots, a_t)$ connects a_0 and a_t, or also that it goes from a_0 to a_t. Instead of $\pi = (a_0, \ldots, a_t)$ we also will write $\pi = (q_0|\{a_0, a_1\}, \ldots, \{a_{t-1}, a_t\}|a_t)$. If $(a_0, \ldots, a_t)$ is a path of even length $t = 2r$, then a_r (or also $\{a_r\}$) is called its *center* and r its *radius*. If $(a_0, \ldots, a_t)$ is a path of odd length $t = 2r + 1$, then $\{a_r, a_{r+1}\}$ is called its *center* and r its *radius*. Thus, the center of a path is either a vertex or an edge.

A graph is *connected* provided for any pair a, a' of vertices, there is a path which connects a and a'. A graph G is a *tree* provided T is connected and the only paths which connect a vertex with itself are the paths of length 0. If T is a tree, for every pair a, a' of vertices, there is a unique path going from a to a'; the length of this path is called the *distance* between a, a' and is denoted by $d(a, a')$. If x is a vertex of T and A a subset of T, let $d(x, A) = \min\{d(x, a) \mid a \in A\}$.

If T is a finite tree, the paths of maximal possible length d are called the *diameter paths* and d is called the *diameter* of T.

Lemma 1. *Let T be a finite tree and π a diameter path in T with center $\{c, c'\}$ and radius r. If x is any vertex of T, then $d(x, \{c, c'\}) \leq r$.*

Proof. Let $\pi = (a_0, \ldots, a_d)$ and $A = \{a_0, \ldots, a_d\}$ Let x be a vertex of T. Let $j = d(x, A)$. Note that there is a path $(x = x_0, \ldots, x_j)$ with $x_j = a_i$ for some i and $x_{j-1} \neq a_{i-1}$, $x_{j-1} \neq a_{i+1}$.

First, assume that $i \leq r$. Since $x_{j-1} \neq a_{i+1}$, we see that $(x_0, \ldots, x_{j-1}, x_j = a_i, a_{i+1}, \ldots a_0)$ is a path. Its length is $j + (d - i)$. Since d is the maximal length of a path in T, it follows that $j \leq i$. Thus, $(x_0, \ldots, x_{j-1}, x_j = a_i, \ldots, a_r)$ is a path of length $j + (r - i) \leq r$. Since $a_r \in \{c, c'\}$, we have $d(x, \{c, c'\}) \leq r$.

In case $i > r$, we obtain in a similar way a path from x_0 via a_i to $c' = a_{d-r}$, and its length is again at most r. □

Corollary. *All diameter paths of a finite tree T have the same center* (and, of course, the same radius).

Proof: Let $\pi = (a_0, a_1, \ldots, a_d)$ be a diameter path with center $\{c, c'\}$ and radius r. Consider a diameter path π' which connects x with x'. Then $d(x, \{c, c'\}) \leq r$ and $d(x', \{c, c'\}) \leq r$. Thus there is a path from x say to c of length at most r. If there is also a path from x' to c of length at most r, then $d(x, x') \leq 2r$, thus $d = d(x, x') \leq 2r \leq d$. It follows that d is even, thus $c = c'$, and that the path from x to x' runs through c. But this means that c is the center of π'.

Thus, it remains to consider the case that $c \neq c'$ and that there is a path from x' to c' of length at most r. Then π' has to be the concatenation of the path from x to c with the edge $\{c, c'\}$ and the path from c' to x', thus the center of π' has to be $\{c, c'\}$. □

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By definition, the *center* $C(T)$ and the *radius* $r(T)$ of T are the center and the radius of the diameter paths of T.

If T is a finite tree, the *boundary* of T is defined to be the set of endpoints of the diameter paths (these are leaves of T; but usually, T will have additionsl leaves).

A vertex x of a graph is called a *leaf* provided x has at most one neighbor. Of course, the boundary vertices of a finite tree are leaves.

(2.2) Orientations. Given a graph G, an *orientation* Ω of G is a map $\Omega\colon G_1 \to G_0 \times G_0$ such that $\Omega(\{a, a'\})$ is either (a, a') or (a', a). If $\Omega(\{a, a'\}) = (a, a')$, we write $a \to a'$ and call a the *start* and a' the *target* of the arrow $a \to a'$. A vertex a of (G, Ω) is a *sink* (or a *source*) provided a is not the start (or the target, respectively) of any arrow. The orientation Ω is called *bipartite* provided any vertex is a sink or a source. If Ω is an orientation of G, then (G, Ω) is called an *oriented graph.*

Let (T, Ω) be a finite tree with bipartite orientation. If the diameter of T is odd, then any diameter path connects a sink with a source. If the diameter of T is even, and there is a diameter path which connects a sink with a sink, then all diameter paths connect sinks with sinks (since all diameter paths have the same center).

(2.3) Balls. Now we consider finite subgraphs of $T(n)$. If $c_1, \ldots, c_t$ are vertices of $T(n)$, let $B_r(c_1, \ldots, c_t)$ be the full subgraph of $T(n)$ given by all vertices a with $d(a, c_i) \leq r$ for some $1 \leq i \leq t$. We are only interested in the case $t = 1$ and in the case $t = 2$ with c_1, c_2 being neighbors. Of course, given a vertex c, then $B_r(c)$ has center c and radius r. Similarly, if c_1, c_2 are neighbors, then $B_r(c_1, c_2)$ has center $\{c_1, c_2\}$ and radius r. We call $B_r(c)$ and $B_r(c_1, c_2)$ the *ball* with radius r and with center c or $\{c_1, c_2\}$, respectively.

(2.4) Representations. If G is a graph and Ω an orientation of G, the oriented graph (G, Ω) is nothing also than a quiver without loops and multiple arrows, and we denote by $\operatorname{mod}(G, \Omega)$ the category of finite-dimensional k-representations of this quiver. The case we are interested in is $G = T(n)$ and Ω a fixed bipartite partition. The finite-dimensional k-representations of $(T(n), \Omega)$ are the *modules* we are dealing with.

If M is an indecomposable module, we denote by $T(M)$ its support; it is the full subgraph given by all vertices a of $T(n)$ with $M_a \neq 0$. Of course, this is a finite tree and we write $C(M) = C(T(M))$ and $r(M) = r(T(M))$ and call it the center and the radius of M, respectively.

An indecomposable module will be called a *sink* module provided any (and thus all) diameter paths in $T(M)$ start and end in sinks, and a *source* module provided any (and thus all) diameter paths in $T(M)$ start and end in sources. An indecomposable module with even diameter is either a sink module or a source module. An indecomposable module with odd diameter will be called a *flow* module. Note that any diameter path of a flow module connects a sink with a source. Recall that the center of an indecomposable module is either a vertex or an edge. For the sink and the source modules, the center is a vertex, for the flow modules, the center is an edge.

If M is an indecomposable module with center C and radius r, then we write $B(M) = B_r(C)$. We have $T(M) \subseteq B(M)$, and $T(M)$ and $B(M)$ have the same radius and the same

center. Note that the boundary vertices of $T(M)$ are just the leaves of $B(M)$ which belong to $T(M)$.

(2.5) Completeness of an indecomposable module. Let M be a sink or a source module with center C and radius r. The module M is said to be *complete* provided $r \geq 1$ and for any path $(x(0), x(1), \ldots, x(r))$ in $T(n)$ with $x(r) = p$ and such that all the vertices $x(i)$ with $1 \leq i \leq r$ belong to $T(M)$, one has $\dim M_{x(0)} = \dim M_{x(1)}$ (so that, in particular, also $x(0)$ belongs to $T(M)$).

Let M be a flow module with radius r. The module M is said to be *complete* provided $r \geq 1$ and such that for any path $(x(0), x(1), \ldots, x(r))$ in $T(n)$ such that $x(r)$ is a central vertex, but $x(r-1)$ is not a central vertex and such that all the vertices $x(i)$ with $1 \leq i \leq r$ belong to $T(M)$, one has $\dim M_{x(0)} = \dim M_{x(1)}$ (so that, in particular, also $x(0)$ belongs to $T(M)$).

A module is said to be *incomplete* provided it is indecomposable and not complete. Let us stress that by definition the simple modules and the indecomposable modules of length 2 (these are the indecomposable modules with radius 0) are incomplete.

(2.6) The shift functor. If x is a sink of a quiver Δ, we denote by $\sigma_x \Delta$ the quiver obtained from Δ by changing the orientation of all the arrows of Δ ending in x; in particular, x is a source of $\sigma_x \Delta$. Bernstein, Gelfand and Ponomarev ([BGP]) have defined a so-called reflection functor $\sigma_x : \mod \Delta \to \mod \sigma_x \Delta$; We denote by σ_x^- the left adjoint of σ_x. Note that σ_x sends $S(x)$ to zero and if M is an indecomposable representation of Δ different from $S(x)$, then $\sigma_x^- \sigma_x(M)$ is naturally isomorphic to M.

The reflection functors σ_x and $\sigma_{x'}$ for different sinks x, x' commute, thus the composition σ of the reflection functors σ_x for all sinks x of Δ is well-defined and independent of the order used, it is a functor $\mod \Delta$ to $\mod \sigma \Delta$, where $\sigma \Delta$ is the quiver obtained from Δ by changing the orientation of all the arrows of Δ ending in a sink. We denote by σ^- the left adjoint of σ. Note that σ sends $S(x)$ with x any sink to zero. If M is an indecomposable representation of Δ which is not simple projective, then $\sigma^- \sigma(M)$ is naturally isomorphic to M.

In particular, if $\Delta = (T(n), \Omega)$, then $\sigma \Delta = (T(n), \sigma \Omega)$, where $\sigma \Omega$ is the opposite orientation (the orientation of all arrows is changed). The reflection functor $\sigma \colon \mod(T(n), \Omega) \to \mod(T(n), \sigma \Omega)$ will be called the *shift functor*.

Of particular interest is the composition $\sigma^2 \colon \mod(T(n), \Omega) \to \mod(T(n), \Omega)$. According to Gabriel [Gab], this is just the Auslander-Reiten functor $\tau = D \operatorname{Tr}$ ("dual of transpose"); note that [Gab] clarified and corrected a previous assertion by Brenner and Butler [BB].

(2.7) From now on, we consider finite-dimensional k-representations of the quiver $(T(n), \Omega)$ (or also of $(T(n), \sigma \Omega)$), thus *graded Kronecker modules* and call them just *modules*.

An indecomposable module M will be said to be *preprojective* provided $\sigma^t M = 0$ for some $t \geq 1$. The preprojective modules are of the form $P_{2s}(x) = \sigma^{-2s} S(x)$ for some sink x in $(T(n), \Omega)$ or of the form $P_{2s+1}(y) = \sigma^{-2s-1} S(y)$ for some sink y in $(T(n), \sigma \Omega)$, with

$s \geq 0$. Note that the preprojective modules are sink modules (these modules have been considered in detail in [FR1]). Dually, an indecomposable module M will be said to be *preinjective* provided $\sigma^{-t} M = 0$ for some $t \geq 1$. The preinjective modules are of the form $Q_{2s}(y) = \sigma^{2s} S(y)$ for some source y in $(T(n), \Omega)$ or of the form $Q_{2s+1}(x) = \sigma^{2s+1} S(x)$ for some source x in $(T(n), \sigma\Omega)$, with $s \geq 0$. The preinjective modules are source modules. The preprojective and the preinjective modules are simple or complete.

An indecomposable module M will be called *regular* provided it is neither preprojective, nor preinjective, thus provided $\sigma^t M \neq 0$ for all $t \in \mathbb{Z}$.

3. Proof of Theorem 1.

Let M be a regular indecomposable module. We apply σ and describe the change of the diameter. Six different cases (1) ... (6) will occur.

Let M be a sink module with center c. Then:

$$ d(M) - 2 \leq d(\sigma M) \leq d(M). $$

There are the following three possibilities:

(1) $d(\sigma M) = d(M) - 2$ *if and only if σM is a sink module if and only if M is complete. In this case, the center of σM is c.*

(2) $d(\sigma M) = d(M) - 1$ *if and only if σM is a flow module. The center of M is of the form $\{c, c'\}$. with a path $(x_0, x_1, \ldots, x_r)$, where $x_0 = c, x_1 = c'$ such that x_r is a source for $\sigma\Omega$.*

(3) $d(\sigma M) = d(M)$ *if and only if σM is a source module. In this case, the center of σM is c again.*

Proof. Let M be a sink module with center c and radius r, thus $T(M) \subseteq B_r(c)$. Since the leaves of $B_r(c)$ are sinks, $\sigma(B_r(c)) \subseteq B_r(c)$. This shows that $d(\sigma M) \leq d(M)$. Given a diameter path π in $T(M)$ say connecting the leaves x and x', the vertices of π different from x and x' are in the support of σM. This shows that $d(\sigma M) \geq d(M) - 2$. This completes the proof of the inequalities.

For the proof of (1) we note that M is complete if and only if $T(\sigma M) \subseteq B_{r-1}(c)$, thus if and only if $d(\sigma M) = d(M) - 2$. In this case σM is a sink module. On the other hand, if there is a vertex $x \in T(\sigma M)$ which does not belong to $B_{r-1}(c)$, then there is a diameter walk in $T(\sigma M)$ which starts at x and this is a source for $\sigma\Omega$, thus σM is not a source module. This completes the proof of (1).

(3) If $d(\sigma M) = d(M)$, then there is a diameter walk for σM starting and ending in boundary vertices of $B_r(c)$, thus in sources of $\sigma Omega$. This shows that σM is a source module. On the other hand, if σM is a source module, consider a diameter walk π for σM. Its ends cannot lie in $B_{r-1}(c)$, thus its length is $2r = d$. In this way, we see (3).

The first part of assertion (2) follows from the inequalities and (1) and (3), since any indecomposable module is a sink module, a source module, or a flow module. It remains to describe in this case the center of σM. We start with a diameter path $(a_1, \ldots, a_{2r})$ of σM

and we may assume that a_1 is a sink and a_{2r} a source with respect to $\sigma\Omega$. Then the center of σM is $\{a_r, a_{r+1}\}$. Let a_0 be a neighbor of a_1 different from a_2. Then $(a_0, a_1, \ldots, a_{2r-1})$ is a path in the support of $M = \sigma^{-1}\sigma M$ and a_0 is a sink with respect to Ω. Since the support of M lies in $B_r(c)$, it follows that $a_r = c$. Thus, we obtain the required path $(x_0, \ldots, x_r)$ by looking at the vertices $x_i = a_{i+r}$ for $0 \leq i \leq r$. $\qquad\square$

Corollary. *If M is a regular indecomposable sink module with radius r. Then there is $1 \leq i \leq r$ such that $\sigma^i M$ is a flow module or a source module.*

Proof. We use induction on r. We have $r \geq 1$ since a sink module with radius $r = 0$ is simple, thus not regular. Thus, we start with a sink module M with radius $r \geq 1$. Either σM is a flow or a source module (then we take $i = 1$) or else σM is a regular sink module with radius $r - 1$. In the latter case, induction shows that $\sigma^j(\sigma M)$ is a flow or a source module, for some $1 \leq j \leq r - 1$. Thus $i = j + 1 \leq r$ and $\sigma^i M$ is a flow or a source module. $\square$

Let M be a flow module. We assume that $(a_0, \ldots, a_d)$ with $d = 2r + 1$ is a diameter path of M with a_0 a sink and a_d a source. Then:

$$d(M) \leq d(\sigma M) \leq d(M) + 1.$$

There are the following two possibilities:

(4) *$d(\sigma M) = d(M)$, thus σM is a flow module. Let a_{d+1} be a neighbor of a_d different from a_{d-1}. Then $(a_1, \ldots, a_{d+1})$ is a diameter path for σM. In particular, the center of σM is $\{a_{r+1}, a_{r+2}\}$ (and we note that a_1 is a sink for $\sigma\Omega$ and a_{d+1} is a source for $\sigma\Omega$.)*

(5) *$d(\sigma M) = d(M) + 1$, then σM is a source module and the center of σM is a_{r+1}.*

Proof: Let $(a_0, \ldots, a_d)$ be a diameter path of M with a_0 a sink and a_d a source. Let a_{d+1} be a neighbor of a_d different from a_{d-1}. Clearly, the path $(a_1, \ldots, a_d, a_{d+1})$ is in the support of σM. This shows that $d(\sigma M) \geq d(M)$. Also, if $d(\sigma M) = d(M)$, then $(a_1, \ldots, a_d, a_{d+1})$ is a diameter path for σM, thus the center of σM is $\{a_{r+1}, a_{r+2}\}$. This completes the proof of (4).

Now assume that $d(\sigma M) > d(M)$. The support of σM is contained in $B_{r+1}(a_{r+1})$, thus $d(\sigma M) \leq 2(r + 2) = d + 1$. It follows that $d(\sigma M) = d(M) + 1$ and that a_{r+1} is the center of σM. The path $(a_{r+1}, \ldots, a_d)$ in $(T(n), \Omega)$ ends in a source, thus the path $(a_{r+1}, \ldots, a_{d+1})$ in $(T(n), \sigma\Omega)$ ends also in a source. This shows that the boundary vertices of $B_{r+1}(a_{r+1})$ are sources, therefore σM is a source module. This completes the proof of (5). $\qquad\square$

(6) *Let M be a source module with center c and radius r. Then σM is a source module with center c and radius $r + 1$.*

Proof: The support of M is contained in $B_r(c)$, thus the support of σM is contained in $B_{r+1}(c)$. It follows that the radius of σM is at most $r + 1$. Any diameter walk in M starts and ends in a source. In σM such a walk is prolonged on both sides by an edge, thus we obtain in this way a walk of length $2r + 2$. As a consequence, the radius of σM is

precisely $r+1$. The boundary vertices of $B_{r+1}(c)$ are sources for $\sigma\Omega$, thus σM is a source module with center c. □

In order to see that any σ-orbit contains only finitely many flow modules, we need a new invariant. A path which connects two (not necessarily different) source leaves will be called a *source path*. Note that the length $|\pi|$ of a source path π is an even natural number (as before, we call $\frac{1}{2}|\pi|$ the *radius* of π).

Lemma 2. *Let M be indecomposable and regular with radius $r(M)$ such that $T(M)$ contains source leaves. Let $w(M)$ be the maximal radius of a source path of M. Then $w(M) \leq r(M)$ and $\sigma^{r(M)-w(M)+1}M$ is a source module.*

Proof. If M is a sink or a source module, then $2r(M)$ is the diameter of M, thus the maximal length of the paths in $T(M)$ and therefore $2w(M) \leq 2r(M)$, thus $w(M) \leq r(M)$. If M is a flow module with diameter d, then $2r(M) = d-1$ is the maximal length of a path which connects two sources, thus again $2w(M) \leq 2r(M)$.

Now assume that M is indecomposable and regular, with a source leaf x in its support. Then σM is again indecomposable and regular, with a source leaf in its support (namely the neighbors of x which are not vertices of $T(M)$ are source leaves in $T(\sigma M)$). By induction, all the modules $\sigma^i M$ with $i \geq 0$ are indecomposable and regular with source leaves in the support.

Now $T(M)$ has a source path of maximal length $2w(M)$, thus $T(\sigma M)$ has a source path of length $2w(M)+2$. Therefore $w(\sigma M) \geq w(M)+1$. Using induction, we see that

$$w(\sigma^i M) \geq w(M)+i$$

for all $i \geq 0$.

Let $t = r(M) - w(M) + 1$ and assume that $\sigma^t M$ is not a source module. Then, according to (6), none of the modules $M, \sigma M, \ldots, \sigma^t M$ is a source module. Thus all these modules are sink or flow modules, and therefore

$$r(M) \geq r(\sigma M) \geq r(\sigma^2 M) \geq \cdots \geq r(\sigma^t M),$$

according to (1), (2) and (4).

Altogether, we see that

$$\begin{aligned}
r(M) &\geq r(\sigma^{r(M)-w(M)+1}M) \\
&\geq w(\sigma^{r(M)-w(M)+1}M) \\
&\geq w(M) + (r(M)-w(M)+1) = r(M)+1,
\end{aligned}$$

a contradiction. □

Corollary 1. *The shift orbit of a regular indecomposable module contains source modules and sink modules.*

Proof. Consider the shift orbit $\mathcal{O}$ of the regular indecomposable module M.

First, assume that $\mathcal{O}$ contains a flow module M'. Then $T(M')$ has a source leaf, thus Lemma 2 asserts that the $\mathcal{O}$ contains a source module. By duality, $\mathcal{O}$ also contains a sink module.

Second, assume that the $\mathcal{O}$ contains a sink module M'. Corollary 1 asserts that one of the modules $\sigma^i M'$ with $i > 0$ has to be a flow module or a source module. But we know already that the existence of a flow module in $\mathcal{O}$ implies that there is also a source module in $\mathcal{O}$. Thus we see that $\mathcal{O}$ contains a source module.

By duality, the existence of a sink module in $\mathcal{O}$ implies the existence of a source module in $\mathcal{O}$. $\qquad\square$

We see: Let $\mathcal{O}$ be the shift orbit of a regular indecomposable module. Let M be a sink module in $\mathcal{O}$. According to corollary 1, there is some $i \geq 0$ such that $\sigma^{i+1} M$ is a flow or a source module. Choose i minimal with this property and let $M_0 = \sigma^i M$. Then M_0 is an incomplete sink module. We let $M_i = \sigma^i M_0$ for all $i \in \mathbb{Z}$.

Then all the module M_i with $i \leq 0$ are sink modules (by the dual assertion of (6)).

Let $b \geq 0$ be minimal such that M_{b+1} is a source module. Then the modules M_{b+i} with $i \geq 1$ are source module (by (6)). On the other hand, all the modules M_i with $1 \leq i \leq b$ have to be flow modules.

Corollary 2. *If M is a module with radius r, then the σ-orbit of M contains at most r flow modules.*

Proof. We can assume that the σ-orbit $\mathcal{O}$ of M contains at least one flow module M. According to corollary 1, the σ-orbit of M contains source and sink module. Thus the orbit $\mathcal{O}$ contains a source module M_0 such that $M_1 = \sigma M_0$ is a flow module. We apply Lemma 2 to M_1 and see that there is some $1 \leq b \leq r(M_1)$ such that the modules $M_1, \ldots, M_b$ are all the flow modules in the σ-orbit of M.

The assertions (1) to (6) show that $r(M_1) \leq r(M)$. According to the Lemma 2, there is some $0 \leq i \leq r(M_1)$ such that $\sigma^i M_1$ is a flow module, but $\sigma^{i+1} M_1$ is a source module. $\square$

Proof of Theorem 1. Consider a σ-orbit $\mathcal{O}$ of regular indecomposable modules. As we have seen, $\mathcal{O}$ contains both a sink module M and a source module, say $\sigma^s M$ for some $s \in \mathbb{Z}$. According to (6), s must be a positive integer. Thus there is some $0 \leq j < s$ such that $M_0 = \sigma^j M$ is a sink module, whereas σM_0 is not a sink module. Thus M_0 is incomplete. Let p be the center of M_0 and let r be its radius.

We write $M_i = \sigma^i M_0$ for all $i \in \mathbb{Z}$. According to the dual assertion of (6), the modules M_i with $i \leq 0$ all are sink modules with center p and radius $r + i$. According to (1), (2) and (4), the modules M_i with $i > 0$ are flow or source modules. It follows that M_0 is the only incomplete sink module in $\mathcal{O}$.

Since the number of flow modules in $\mathcal{O}$ is finite, there is a smallest number $b \geq 0$ such that M_{b+1} is a source module. Then the modules M_i with $1 \leq i \leq b$ are flow modules. According to (6), the modules M_i with $i \geq b + 1$ are source modules. This shows that $b \geq 0$ is the number of flow modules in $\mathcal{O}$.

According to (2) and (4), the modules M_i with $1 \leq i \leq b$ (the flow modules in $\mathcal{O}$) all have radius $r - 1$. According to (3), (5) and (6), the modules M_{b+1+i} with $i \geq 0$ (the source modules in $\mathcal{O}$) have radius $r + i$. Also, according to (6), all these source modules have the same center, say c_b. In case $b = 0$, we know from (3) that $c_0 = c_b$. In case $b > 0$, we use (2), (4) and (5) in order to see that the centers of the modules $M_0, M_1, \ldots, M_{b+1}$ are of the form $c_0, \{c_0, c_1\}, \ldots, \{c_{i-1}, c_i\}, \ldots, \{c_{b-1}, c_b\}, c_b$, where $(c_0, c_1, \ldots, c_b)$ is a path. $\square$

Thus, any regular σ-orbit looks as follows:

	sink modules				flow modules				source modules		
module:	$\cdots$ M_{-2}	M_{-1}	M_0	M_1	$\cdots$	M_i	$\cdots$	M_b	M_{b+1}	M_{b+2}	M_{b+3} $\cdots$
radius:	$\cdots$ $r{+}2$	$r{+}1$	r	$r{-}1$	$\cdots$	$r{-}1$	$\cdots$	$r{-}1$	r	$r{+}1$	$r{+}2$ $\cdots$
center:	$\cdots$ c_0	c_0	c_0	$\{c_0,c_1\}$	$\cdots$	$\{c_{i-1},c_i\}$	$\cdots$	$\{c_{b-1},c_b\}$	c_b	c_b	c_b $\cdots$

Let us stress that there may be no flow modules in $\mathcal{O}$, this is the case $b = 0$. In this case, $c_0 = c_b$ and the center path is a path of length 0.

Given an indecomposable regular module M, we define its *index* $\iota(M)$ as the integer t such that $\sigma^{-t} M$ is an incomplete sink module. Thus $\iota(M) = t$ means that $M = \sigma^t M_0$ for some incomplete sink module (namely for $M_0 = \sigma^{-t} M$).

There is the following Corollary to Theorem 1.

Corollary. *Let M be a sink module with center p, such that $\sigma^{b+1} M$ is a source module with center q. If $d(p, q) = b$, then $\iota(M) = 0$.*

4. Diameter paths and boundary vertices.

We are going to look in which way diameter paths of M and of σM are related. Given an indecomposable module M, we denote by $\gamma(M)$ the number of diameter paths for M.

The diameter paths may be quite different in case both M and σM are incomplete. Otherwise, the diameter paths of M and of σM are very similar — this concerns the cases where M and σM both are sink modules, or both are flow modules, or both are source modules, thus, the cases (1), (4) and (6).

Proposition 1. *Let M be a representation of $(T(n), \Omega)$.*
(a) Assume that M and σM both are sink modules. Let $(a_0, \ldots, a_d)$ be a path in $T(n)$. Then $(a_0, \ldots, a_d)$ is a diameter path for M if and only if $(a_1, \ldots, a_{d-1})$ is a diameter path for σM. Thus

$$\gamma(M) = (n - 1)^2 \gamma(\sigma M).$$

(b) Assume that M and σM both are flow modules. Let $(a_0, \ldots, a_d)$ be a path in $T(n)$ with a_0 (and also a_d) being sinks for Ω. Then $(a_0, \ldots, a_{d-1})$ is a diameter path for M if and only if $(a_1, \ldots, a_d)$ is a diameter path for σM. Thus

$$\gamma(M) = \gamma(\sigma M).$$

(c) *Assume that M and σM both are source modules. Let $(a_0, \ldots, a_d)$ be a path in $T(n)$. Then $(a_1, \ldots, a_{d-1})$ is a diameter path for M if and only if $(a_0, \ldots, a_d)$ is a diameter path for M. Thus*

$$\gamma(\sigma M) = (n-1)^2 \gamma(M).$$

Proof. This follows directly from the considerations in section 3. □

Example *of a sink module M and a source module $M' = \sigma M$ with arbitrarily large radius r such that no edge of $T(n)$ belongs both to a diameter of M and a diameter of M'.*

We need $n \geq 4$. For $r = 1$, let M be any 3-dimensional sink module. Then σM is an $(n-1)$-dimensional source module.

Now assume that $r \geq 2$. Let a_0 be a sink of $(T(n), \Omega)$ and $(a_0, a_1, \ldots, a_d)$ a path of length $d = 2r$. Since $n \geq 4$, there is a path $(b_0, \ldots, b_d)$ such that $b_r = a_r$ is the only common vertex of these paths. Let M be the thin indecomposable module with support the full subquiver with vertices a_j, b_j, where $1 \leq j \leq d-1$ and all the neighbors of a_1 and a_{d-1}. Then M is a sink module with center $c = a_r = b_r$. The diameter paths of M are the paths of the form $(a'_0, a'_1, \ldots, a'_{d-1}, a'_d)$, where $a'_j = a_j$ for $1 \leq j \leq d-1$ (the number of diameter paths is $(n-1)^2$). The diameter paths of $M' = \sigma M$ are all the paths of the form $(b'_0, \ldots, b'_d)$ with $b'_j = b_j$ for $1 \leq j \leq d-1$ (the number of diameter paths of M' is again $(n-1)^2$, but there is no natural correspondence between the diameter paths of M and of M').

For example, for $n = 4$ and $r = 2$, the support of M and M' looks as follows:

| with c a sink | with c a source |
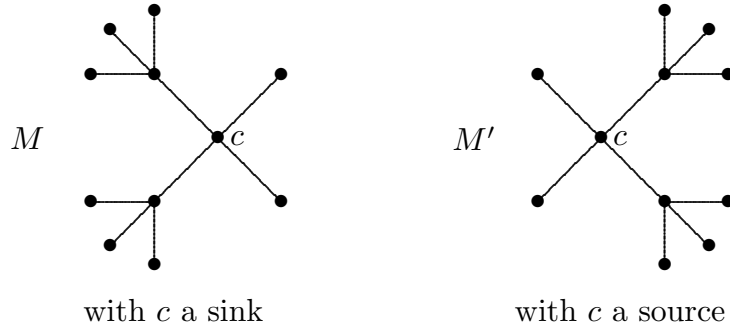
Let us add some comments concerning the number of boundary vertices of sink modules. If M be an indecomposable module, let $\beta(M)$ be the number of boundary vertices of $T(M)$. Note that any ball of radius r has precisely $\beta_r = n(n-1)^{r-1}$ boundary vertices. Thus, if M is a sink module (or a source) module with radius $r > 0$, then
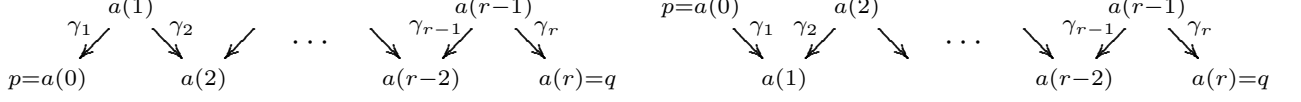
$$2 \leq \beta(M) \leq \beta_r.$$

If M is a source module, then for all $t \geq 0$

$$\beta(\sigma^t M) = (n-1)^t \beta(M).$$

This shows that the number of boundary vertices growths exponentially when we apply σ.

5. Examples, in particular proof of Theorem 2.

Let $(p|\gamma_1, \ldots, \gamma_r|q)$ be a path, say with arrows γ_i between $a(i-1)$ and $a(i)$. We assume that the path starts at the sink $p = a(0)$ in case r is even (see the left picture), and at the source $p = a(0)$ in case r is odd (see the right picture):

$$
\begin{array}{ccc}
& a(1) & & & a(r-1) & & p{=}a(0) & & a(2) & & & a(r-1) \\
\gamma_1 \nearrow & & \searrow \gamma_2 & \nearrow \cdots \searrow & \gamma_{r-1}\nearrow & \searrow \gamma_r & & \searrow \gamma_1 & \gamma_2 \nearrow & \searrow \cdots \searrow & \gamma_{r-1}\nearrow & \searrow \gamma_r \\
p{=}a(0) & & a(2) & & a(r{-}2) & a(r){=}q & & a(1) & & & a(r{-}2) & a(r){=}q
\end{array}
$$

In all cases, the vertex $a(r)$ is a sink.

The module $P_r(p)$ exists (since for r even, p is a sink, whereas for r odd, p is a source). Also, *the module $P_0(q)$ exists* (since $q = a(r)$ is a sink) *and it is a submodule of $P_r(p)$.*

I. Case $b = r$. We start with the path $\pi = (p|\gamma_1, \ldots, \gamma_r|q)$ and obtain the module $P_r(p)$ and its submodule $P_0(q)$. We define $M_0 = P_r(p)/P_0(q)$. This is obviously an incomplete sink module and we let $M_i = \sigma^i M_0$.

First, let us consider $i = 1$. We apply σ to the exact sequence

$$0 \to P_0(q) \to P_r(p) \xrightarrow{\epsilon} M_0 \to 0$$

and obtain an exact sequence

$$(*) \qquad\qquad 0 \to P_{r-1}(p) \xrightarrow{\sigma\epsilon} \sigma M_0 \to I_0(q) \to 0,$$

since $\sigma P_r(p) = P_{r-1}(p)$. The support of $P_{r-1}(p)$ are the vertices a of $T(n)$ with distance $d(p, a) \le r - 1$. In particular, the vertices $a(i)$ with $0 \le i < r$ belong to this support. On the other hand, the support of $I_0(q)$ is just the vertex q. The arrow γ_r is the only connection between the support of $P_{r-1}(p)$ and the support of $I_0(q)$. In particular, we have $\dim \mathrm{Ext}^1(I_0(q), P_{r-1}(p)) = 1$. This shows that σM_0 is the unique indecomposable extension of $P_{r-1}(p)$ by $I_0(q)$; it is given by replacing in the direct sum $P_{r-1}(p) \oplus I_0(q)$ the zero map at the arrow γ_r by the identity map

$$(P_{r-1}(p))_{a(r-1)} = k \xleftarrow{1} k = (I_0(q))_{a(r)}.$$

Next, we apply σ^{i-1} to $(*)$ with $1 \le i - 1 \le r - 1$ and we obtain the exact sequence

$$0 \to P_{r-i}(p) \xrightarrow{\sigma\epsilon} \sigma^i M_0 \to I_{i-1}(q) \to 0.$$

The arrow γ_{r-i+1} is the only connection between the support of $P_{r-i}(p)$ and the support of $I_{i-1}(q)$. In particular, we have $\dim \mathrm{Ext}^1(I_{i-1}(q), P_{r-i}(p)) = 1$.

support of $P_{r-i}(p)$ support of $I_{i-1}(q)$
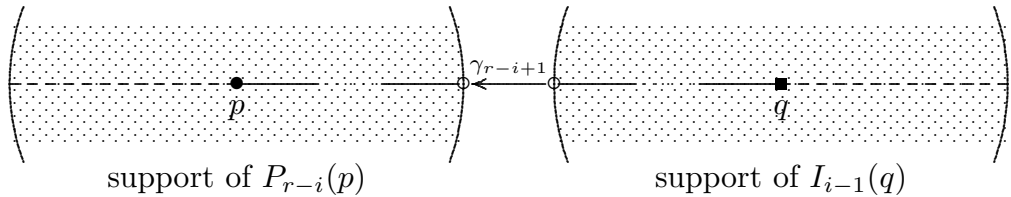
and $\sigma^i M_0$ is the unique indecomposable extension of $P_{r-i}(p)$ by $I_{i-1}(q)$; it is given by replacing in the direct sum $P_{r-i}(p) \oplus I_{i-1}(q)$ the zero map at the arrow γ_{r-i+1} by the identity map.

Finally, we apply σ to the exact sequence

$$0 \to P_0(p) \to M_r \xrightarrow{f} I_{r-1}(q) \to 0$$

and obtain the exact sequence

$$0 \to M_{r+1} \xrightarrow{\sigma f} I_r(q) \to I_0(p) \to 0,$$

in some sense dual to the exact sequence we started with. Of course, we see in this way that M_{r+1} is an incomplete source module.

Let us add how the modules M_{-i} and M_{r+1+i} with $i \geq 0$ look.

First, we consider the modules M_{-i} with $i \geq 0$. They are given by a projective presentation of the form

$$0 \to P_i(q) \to P_{r+i}(p) \to M_{-i} \to 0;$$

since $\dim \operatorname{Hom}(P_i(q), P_{r+i}(p)) = \dim \operatorname{Hom}(P_0(q), P_r(p)) = P_r(p)_q = 1$, the module M_{-i} is uniquely defined in this way. Clearly, the support of M_{-i} is contained in the support of $P_{r+i}(p)$, and this is the ball $B_{r+i}(p)$ with center p and radius $r + i$. If


$$B_{r+i}(p)$$
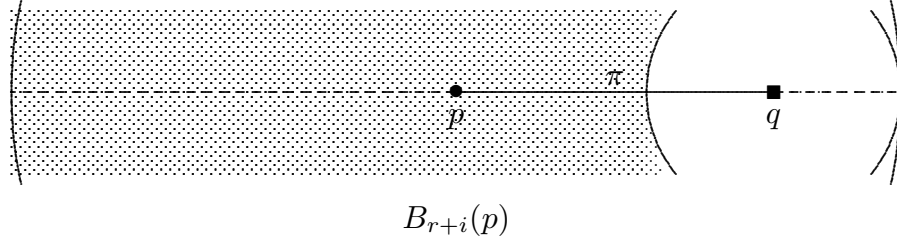
Any path in B_{r+i} starting at p, ending in a leaf and not using γ_1 belongs to the support of M_{-i}. Since $n \geq 3$, we obtain in this way diameter paths for M_{-i} of length $2(r + i)$. This shows that M_{-i} is a sink module.

Dually, the module M_{r+1+i} with $i \geq 0$ has an injective copresentation of the form
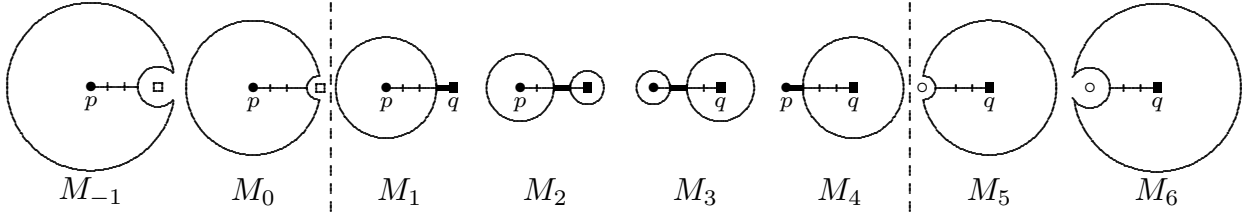
$$0 \to M_{r+1+i} \to I_{r+i}(q) \to I_i(p) \to 0,$$

thus its support is contained in the support of $I_{r+i}(q)$ and this is again is a ball with radius $r + i$, but now with center q.

$$B_{r+i}(q)$$

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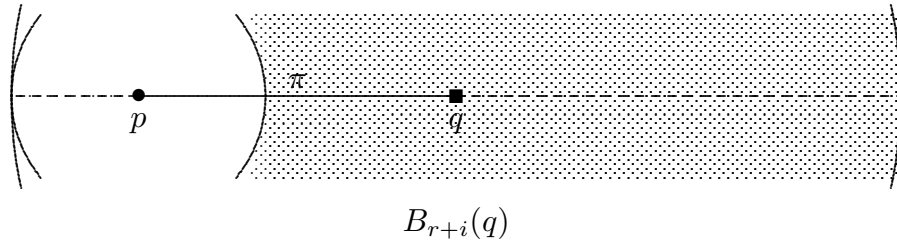
Clearly, M_{r+1+i} is a source module.

For $b = r = 4$, and $-1 \leq i \leq 6$, the modules M_i have the following shapes:

$$M_{-1} \qquad M_0 \qquad M_1 \qquad M_2 \qquad M_3 \qquad M_4 \qquad M_5 \qquad M_6$$

Note that for i even, the vertex p (drawn as a bullet or a small circle) is a sink for $T(M_i)$ and q (drawn as a small square) is a source, whereas for i odd, p is a source and q a sink.

II. Case $b < r$, with $(n, b, r) \neq (3, 0, 1)$.

Let $b < r$ and $(n, b, r) \neq (3, 0, 1)$. Here is the recipe for M_0. Let $(p | \gamma_1, \ldots, \gamma_b | q)$ be a path of length b and $(q | \delta_{b+1}, \ldots, \delta_r | x)$, $(q | \delta'_{b+1}, \ldots, \delta'_r | x')$ paths of length $r - b$ with $\delta_{b+1} \neq \delta'_{b+1}$:
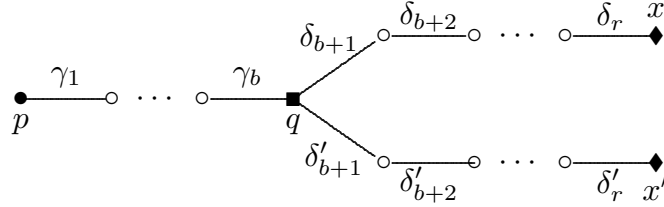
Let
$$M_0 = P_r(p)/(P_0(x) \oplus P_0(x')).$$

As we will see, this is the incomplete sink module we are looking for. (The module M_0 can be defined also in case $(n, b, r) = (3, 0, 1)$, but then M_0 is a flow module.)

We claim that M_0 is a module with center p and diameter $2r$ and that $M_{b+1} = \sigma^{b+1} M_0$ has center q and diameter $2r$. This then implies that the modules $M_1, \ldots, M_b$ have to be flow modules and that M_0 is the incomplete sink module, M_{b+1} the incomplete source module in the σ-orbit (see the Corollary at the end of section 3).

In order to see that M_0 is a (sink) module with center p and diameter $2r$, we have to exhibit a diameter path of the support $B_r(p)$ of $P_r(p)$ which does not start or end in x or x'. Then this is a diameter path for M_0, has center p and length $2r$. In case $n \geq 4$, or $b \geq 1$, we take a diameter path of $B_r(p)$ which does not involve γ_1. If $b = 0$ and $r \geq 2$, let z, z' be leaves of $B_r(p)$ with $d(x, z) = 2$ and $d(x', z') = 2$. Then the path of $P_r(p)$ which connects z with z' goes through p, thus is a diameter path of M_0.

Now let us consider the modules $M_i = \sigma^i M_0$ with $1 \leq i \leq b + 1$. The defining exact sequence
$$0 \to P_0(x) \oplus P_0(x') \to P_r(p) \to M_0 \to 0$$
yields an exact sequence

$(*)$ $\qquad\qquad\qquad 0 \to P_{r-1}(p) \to M_1 \to I_0(x) \oplus I_0(x') \to 0.$

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If we apply σ^b to this exact sequence $(*)$, we obtain a corresponding exact sequence

$$0 \to P_{r-b-1}(p) \to M_{b+1} \to I_b(x) \oplus I_b(x') \to 0$$

(here we use that $b < r$). This shows that the support of M_{b+1} is the union of the ball $B_{r-b-1}(p)$ and the balls $B_b(x)$ and $B_b(x')$.

First of all, let us construct a path of length $2r$ in the support of M_{b+1}. We choose a path in $B_b(x)$ from the center x' to a boundary vertex, say w, not using the arrow δ_r. Similarly, we choose a path in $B_b(x')$ from the center x to a boundary vertex, say w', not using the arrow δ_r'. Combining these two paths with the given path from x via q to x' (using the arrows δ_i and δ_i') we obtain a path of length $2b + 2(r - b) = 2r$ with center q.

On the other hand, we claim that $T(M_{b+1}) \subseteq B_r(q)$. Namely, if u belongs to $B_b(x)$, then $d(u, q) \le d(u, x) + d(x, q) \le b + (r - b) = r$. Similarly, for $u \in B_b(x')$, we have $d(u, q) \le r$. Finally, if $u \in B_{r-b-1}$, then $d(u, q) \le d(u, p) + d(p, q) \le (r - b - 1) + b = r - 1$. This completes the proof. Actually, the last calculations show that any boundary vertex of $T(M_{b+1})$ is a boundary vertex of $B_b(x)$ or of $B_b(x')$ (thus, in particular, a source).

It seems to be of interest to look at the module M_{b+1} in more detail. We distinguish two cases, the case $r > 2b$ and the case $r \le 2b$.

First, let us assume that $r > 2b$, thus $b < r - b$. In this case, the extension of the module $P_{r-b-1}(p)$ by the module $I_b(x)$ is furnished by the arrow δ_{r-b} (and similarly, the extension of the module $P_{r-b-1}(p)$ by the module $I_b(x')$ is furnished by the arrow δ_{r-b}'). In particular, the three balls $B_{r-b-1}(p), B_b(x)$ and $B_b(x')$ are pairwise disjoint.

On the other hand, if $r \le 2b$, thus $r - b \le b$, the extension of $P_{r-b-1}(p)$ by both modules $I_b(x)$ and $I_b(x')$ is furnished by the arrow γ_{r-b}. In this case, the vertex q belongs both to $B_b(x)$ and $B_b(x')$.

In general, the modules M_i with $1 \le i \le r$ are quite easy to visualize. We apply σ^{i-1} to the exact sequence $(*)$ and obtain the exact sequence:
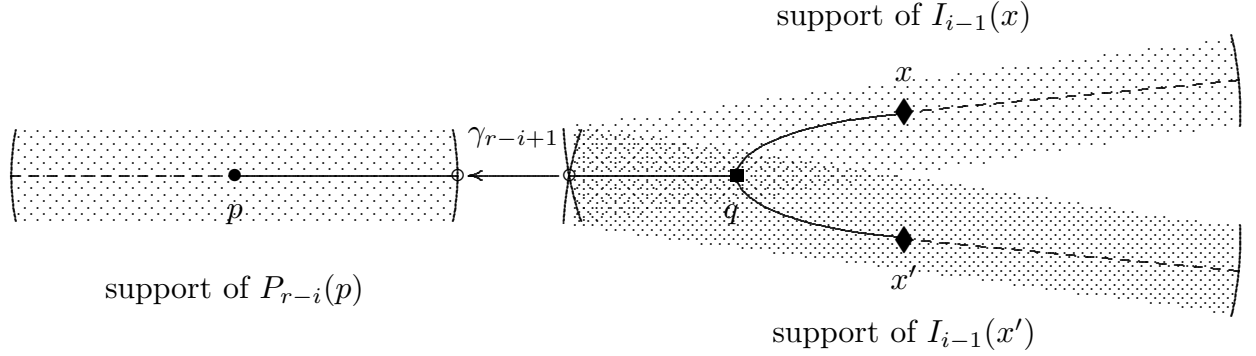
$$0 \to P_{r-i}(p) \to M_i \to I_{i-1}(x) \oplus I_{i-1}(x') \to 0.$$

First, assume that $i \le r - b$. Then the extension of $P_{r-i}(p)$ by $I_{i-1}(x)$ or by $I_{i-1}(x')$ is given by δ_{r-i+1} or δ_{r-i+1}', respectively:
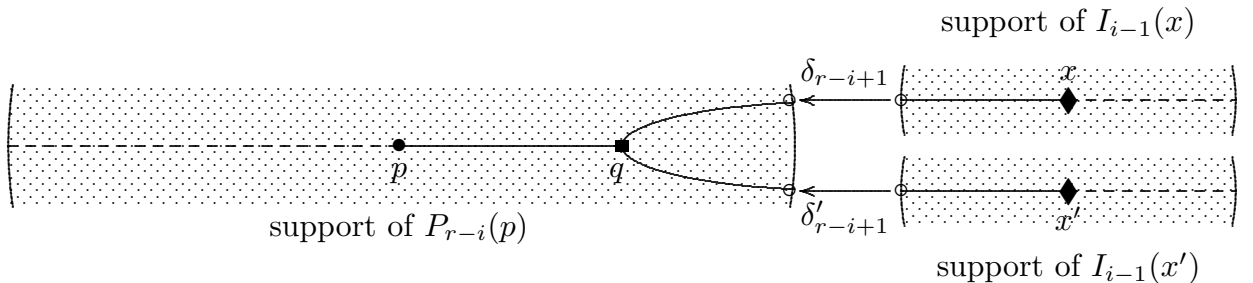
Second, assume that $i > r - b$. Then the extensions of $P_{r-i}(p)$ by $I_{i-1}(x)$ as well as

by $I_{i-1}(x')$ are given by γ_{r-i+1}:

We also want to describe the module M_{r+1}. The exact sequence

$$0 \to P_0(p) \to M_r \to I_{r-1}(x) \oplus I_{r-1}(x') \to 0$$

yields the exact sequence

$$0 \to M_{r+1} \to I_r(x) \oplus I_r(x') \to I_0(p) \to 0.$$

This shows that M_{r+1} is a maximal submodule of $I_r(x) \oplus I_r(x')$ with factor the simple module $I_0(p)$.

III. Case $b = 0, r = 1$. This concerns the remaining case $(n, b, r) = (3, 0, 1)$, but works for all $n \geq 3$.

We start with a source p and take an indecomposable module M with $\dim M_p = 2$ and $\dim M_x = 1$ for all the neighbors x of p, whereas $M_a = 0$ for all other vertices a of $T(n)$ (for $n = 3$, this concerns the representations of a quiver of type $\mathbb{D}_4$, thus there is a unique such module; for $n \geq 4$, there are many such modules). Since $T(M)$ is the full subquiver of $T(n)$ given by p and its neighbors, the module M is a sink module with radius 1. Since $\dim M_x = 1 < 2 = \dim M_p$, for the neighbors x of p, we see that M is incomplete. The vertices of the support of σM are again p (now a source) and its neighbors x (now sinks), and $\dim(\sigma M)_p = 2, \dim(\sigma M)_x = 1$. We see that $M_1 = \sigma M$ is an incomplete source module with center $q = p$ and radius 1. The center path for the shift orbit of M is the path from p to $q = p$ of length $b = 0$. $\qquad \square$

This completes the proof of Theorem 2. $\qquad \square$

Further examples. The examples constructed until now were individual modules. The same techniques allow to exhibit also families of modules. Let us construct a 1-parameter family of incomplete sink modules M_0 with a fixed dimension vector and equal radius and center path.

We start with a subquiver of $T(n)$ of the following kind:
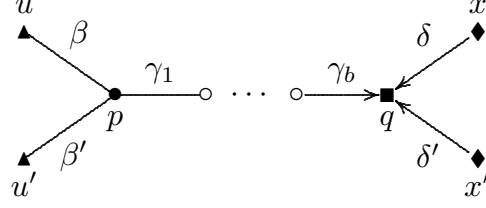
(with x, x' being sources). Let $\mathcal{A}$ be the set of the following four representations of $(T(n), \Omega)$

$$P_{b+1}(u), \ P_{b+1}(u'), \ I_0(x), \ I_0(x').$$

These are pairwise orthogonal bricks: one only has to verify that the first two modules are orthogonal bricks, however $P_{b+1}(u) = \sigma^{-b-1}P_0(u)$, $P_{b+1}(u') = \sigma^{-b-1}P_0(u')$, and $P_0(u)$, $P_0(u')$ are of course orthogonal bricks. Let us denote by $\mathcal{E}(\mathcal{A})$ the extension closure of $\mathcal{E}$. As one knows from [R1], $\mathcal{E}(\mathcal{A})$ is an exact abelian subcategory with simple objects the modules $\mathcal{A}$, and that this subcategory is equivalent to the representations of the Ext-quiver $\Delta(\mathcal{A})$ of $\mathcal{A}$: the vertices of $\Delta(\mathcal{A})$ are of the form $[A]$, where A is an object in $\mathcal{A}$, and there are t arrows $[A] \to [A']$ provided $\dim \mathrm{Ext}^1(A, A') = t$. In our case, $\Delta(\mathcal{A})$ is the following bipartite quiver

thus an affine quiver of type $\widetilde{\mathbb{A}}_{22}$.

Let $\mathcal{E}'(\mathcal{A})$ be the class of indecomposable objects in $\mathcal{E}(\mathcal{A})$ which correspond to sincere representations of the quiver $\Delta(\mathcal{A})$, thus they are the indecomposable representations M of $(T(n), \Omega)$ with an exact sequence of the form

$$(*) \qquad 0 \to P_{b+1}(u)^s \oplus P_{b+1}(u')^{s'} \to M \to I_0(x)^t \oplus I_0(x')^{t'} \to 0$$

with positive integers s, s', t, t'.

We claim that M is a sink module with center p (and radius $b+2$) and that $\sigma^{b+1}M$ is a source module with center q (and the same radius $b+2$). Since $d(p, q) = b$, it follows that M is an incomplete sink module with center path $(p|\gamma_1, \ldots, \gamma_b|q)$ (see the Corollary at the end of section 3).

Thus, let us analyze the diameter paths of M. It is obvious that $T(M) \subseteq B_{b+2}(p)$. Since $s \geq 1$, there is a path from the boundary of $B_{b+1}(u)$ to its center u which does not use β. Similarly, since $s' \geq 1$, there is a path from the boundary of $B_{b+1}(u')$ to its center u' which does not use β'. Combining these two paths with the arrows β and β', we obtain a path of length $2b+4$, thus a diameter path and its center is p. This shows that M is a sink module with center p and radius $b+2$.

If we apply σ^{b+1} to $(*)$, we obtain the exact sequence

$$0 \to P_0(u)^s \oplus P_0(u')^{s'} \to \sigma^{b+1}M \to I_{b+1}(x)^t \oplus I_{b+1}(x')^{t'} \to 0.$$

Thus, by duality we see that $\sigma^{b+1}M$ is a source module with center q and radius $b + 2$.

Let us add that the flow modules in the σ-orbit of M are the modules $\sigma^i M$ with $1 \le i \le b$, they are middle terms of exact sequences of the following form:

$$0 \to P_{b+1-i}(u)^s \oplus P_{b+1-i}(u')^{s'} \to \sigma^i M \to I_i(x)^t \oplus I_i(x')^{t'} \to 0$$

and the extension is furnished by the arrow γ_{b+1-i}.

6. The τ-orbits.

As we have mentioned in (2.6), the Auslander-Reiten translation τ in $\mathrm{mod}(T(n), \Omega)$ is nothing else than σ^2. Thus, Theorem 1 provides information on the τ-orbits of the regular indecomposable modules and on the shape of the components of the Auslander-Reiten quiver.

One should be aware that the very lucid behavior of the shift orbits looks more complicated when we deal with the τ-orbits. We use again the labeling M_i of the modules in a shift orbit, with M_0 the incomplete sink module and $M_i = \sigma^i M_0$. As we have mentioned in the introduction, the modules of the form M_i with i even will be called *even* modules, those with i odd will be called *odd* modules.

We obtain in this way two τ-orbits (but remember: in different categories — one τ-orbit consists of representations of $(T(n), \Omega)$, the other of representations of $(T(n), \sigma\Omega)$). Also, we have to stress that the labeling "even" and "odd" refers to our interest in sink modules. In a similar way, one may focus the attention to the unique incomplete source module in a given σ-orbit. Alternatively, we may concentrate on the invariant b which may be even or odd. It turns out that there are four different kinds of τ-orbits of regular indecomposable module. Here are these τ-orbits (below any module, we mention its radius).

The even modules, with b even:

	sink modules			flow modules			source modules		
$\cdots$	M_{-2}	M_0		M_2	$\cdots$	M_b	M_{b+2}	M_{b+4}	$\cdots$
$\cdots$	$r{+}2$	r		$r{-}1$	$\cdots$	$r{-}1$	$r{+}1$	$r{+}3$	$\cdots$

The even modules with b odd:

	sink modules			flow modules			source modules		
$\cdots$	M_{-2}	M_0		M_2	$\cdots$	M_{b-1}	M_{b+1}	M_{b+3}	$\cdots$
$\cdots$	$r{+}2$	r		$r{-}1$	$\cdots$	$r{-}1$	r	$r{+}2$	$\cdots$

The odd modules with b even:

sink modules			flow modules			source modules		
$\cdots$	M_{-3}	M_{-1}	M_1	$\cdots$	M_{b-1}	M_{b+1}	M_{b+3}	$\cdots$
$\cdots$	$r{+}3$	$r{+}1$	$r{-}1$	$\cdots$	$r{-}1$	r	$r{+}2$	$\cdots$

The odd modules with b odd:

sink modules			flow modules			source modules		
$\cdots$	M_{-3}	M_{-1}	M_1	$\cdots$	M_b	M_{b+2}	M_{b+4}	$\cdots$
$\cdots$	$r{+}3$	$r{+}1$	$r{-}1$	$\cdots$	$r{-}1$	$r{+}1$	$r{+}3$	$\cdots$

As we have seen in section 3, looking at a shift orbit, the change of the radius of the corresponding modules is given by a simple and uniform rule, in contrast to the four different rules which occur for τ-orbits.

Proposition 2. *Let $\mathcal{X}$ be a τ-orbit. We assume that the sink modules in $\mathcal{X}$ have center p, the source modules in $\mathcal{X}$ have center q and that $b = d(p, q)$.*
(a) If $b = 2s$ is even, then $\mathcal{X}$ contains precisely s flow modules.
(b) If $b = 2s + 1$ is odd, then the number of flow modules in $\mathcal{X}$ is s or $s + 1$. Let X be a sink module in $\mathcal{X}$ and that τX is not a sink module. Then $\iota(X) = 0$ in case the number of flow modules in $\mathcal{X}$ is s, otherwise $\iota(X) = -1$. $\square$

Of course, always we know: If X is a sink module and τX is not a sink module, then $\iota(X)$ is equal to -1 or 0.

Proposition 3. *Let $\mathcal{X}$ be a τ-orbit. We assume that the sink modules in $\mathcal{X}$ have center p, the source modules in $\mathcal{X}$ have center q and that $b = d(p, q)$. We assume that $b = 2s$ is even. Let X be a sink module in $\mathcal{X}$ such that τX is not a sink module, thus $\iota(X)$ is equal to 0 or -1. If $\iota(X) = 0$, then $r(X) < r(\tau^{s+1}X)$. If $\iota(X) = -1$, then $r(X) > r(\tau^{s+1}X)$.*

Proof. Let X_0 be the incomplete sink module in the σ-orbit of X. and r_0 the radius of X_0. By Theorem 1, the radius of X_{-1} is $r_0 + 1$, the flow modules $X_1, \ldots, X_{2s}$ have radius $r_0 - 1$, the module X_{2s+1} has radius r_0 and the module X_{2s+2} has radius $r_0 + 1$. For $\iota(X) = 0$, we have $\tau^{s+1}X = X_{2s+2}$, thus $r(X) = r_0 < r_0 + 1 = r(\tau^{s+1}X)$. Similarly, for $\iota(X) = -1$, we have $X = X_{-1}$ and $\tau^{s+1}X = X_{2s+1}$, thus $r(X) = r_0 + 1 > r_0 = r(X_{2s+1})$. $\square$

7. Auslander-Reiten components (Proof of Theorem 3 and Theorem 4).

It is well-known that the category $\mathrm{mod}(T(n), \Omega)$ has Auslander-Reiten sequences and that regular components are of the form $\mathbb{Z}\mathbb{A}_\infty$, thus the indecomposable regular modules

are quasi-serial (see [R]); we denote by $\mathrm{ql}(X)$ the quasi-length of X. We say that a regular component $\mathcal{C}$ of $\mathrm{mod}(T(n), \Omega)$ is *even* or *odd*, provided the quasi-simple modules in $\mathcal{C}$ are even or odd, respectively.

Given an indecomposable regular module X, we define

$$\bar{\iota}(X) = \frac{1}{\mathrm{ql}(X)} \sum_F \iota(F)$$

where we sum over all quasi-composition factors F of X and we call $\bar{\iota}(X)$ the *average index* of X. We will see that $\bar{\iota}(X) = \iota(X)$.

Lemma 3. *Let X be indecomposable and regular with $\bar{\iota}(X) \leq 0$. Then X is a sink module and $B(X) = B(X')$, where X' is the quasi-top of X.*

Proof. We assume that the quasi-composition factors F of X are

$$M_i, M_{i+2}, \ldots, M_{i+2(l-1)},$$

where l is the quasi-length of X (and where M_0 is an incomplete sink module and $M_i = \sigma^i M_0$). We assume that $\bar{\iota}(X) \leq 0$, thus $i = \iota(M_i) \leq 0$, thus M_i is a sink module, say with center p. Note that M_i is the quasi-top of X, thus $X' = M_i$. It is sufficient to show that $B(X) = B(M_i)$ (because this implies that X is a sink module). Thus, we have to show that $T(F) \subseteq B(M_i)$ for all quasi-composition factors F of X.

First, let us assume that we deal with an even component. Since M_i is a sink module, $i = -2t$ for some $t \geq 0$. Since $\bar{\iota}(X) \leq 0$, the number of sink factors is greater than the number of remaining factors, thus $i + 2(l-1) \leq 2t$. Thus, the quasi-length l of X is at most $2t + 1$ and the quasi-composition factors of X are

$$M_{-2t}, M_{-2t+2}, M_{-2t+4}, \ldots.$$

Let r be the radius of M_0. Then by Theorem 1, the radius of M_{-2s} with $0 \leq s \leq t$ is $r + 2s \leq r + 2t$; in particular: the radius of M_i is $r + 2t$. Thus, for $0 \leq s \leq t$, $T(M_{-2s}) \subseteq B_{r+2t}(p)$. In general, if $T(M) \subseteq B_u(p)$, then $T(\tau M) \subseteq B_{u+2}(p)$. Thus, since $T(M_0) \subseteq B_r(p)$ we see by induction that for $1 \leq s \leq t$ we have $T(M_{2s}) \subseteq B_{r+2s}(p) \subseteq B_{r+2t}(p) = B(M_i)$. This shows that $T(F) \subseteq B(M_i)$ for all factors F.

Second, assume that the component is odd. Since M_i is a sink module, $i = -2t - 1$ for some $t \geq 0$. Since $\bar{\iota}(X) \leq 0$, the number of sink factors is greater or equal the number of remaining factors, therefore $i + 2(l-1) \leq 2t + 1$. Thus, the quasi-length l of M is at most $2t + 2$ and the quasi-composition factors of M are

$$M_{-2t-1}, M_{-2t+1}, M_{-2t+3}, \ldots.$$

Again, let r be the radius of M_0. Then by Theorem 1, the radius of M_{-2s-1} with $0 \leq s \leq t$ is $r + 2s + 1 \leq r + 2t + 1$; in particular: $B(M_i) = B_{r+2t+1}(p)$. Again using Theorem 1, we know that $T(M_1) \subseteq B_r(p)$. It follows that $T(M_{2s+1}) \subseteq B_{r+2s}(p) \subseteq B_{r+2t+1}(p) = B(M_i)$. This shows that $T(F) \subseteq B(M_i)$ for all factors F. $\qquad\square$

We have found in this way many sink modules in a given regular component. Here is a visualization:

even component　　　　　　　　　　　odd component
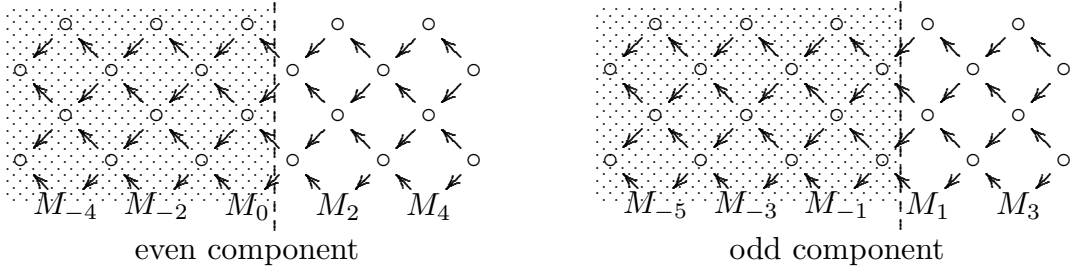
All the modules on the left of the dashes line are sink modules with center p.

Here is the dual assertion of Lemma 3. If M is indecomposable and regular, with a quasi-composition factor F. If the σ-orbit of F contains b flow modules, we write $b(X) = b$ (note that this does not depend on the choice of F).

Lemma 3*. *Let X be indecomposable and regular with $\bar{\iota}(X) \geq b(X) + 1$. Then X is a source module and $B(X) = B(X'')$, where X'' is the quasi-socle of X.* ☐

Here are the pictures dual to the previous ones; the modules on the right of the dashes line are source modules with a fixed center, say q.

even component　　　　　　　　　　　odd component
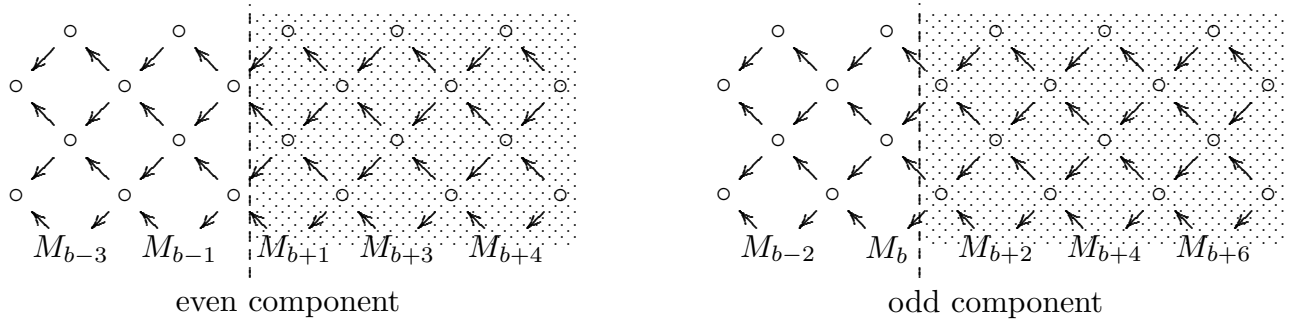
Now we apply Proposition 2 in order to see that the a module X with $1 \leq \bar{\iota}(X) \leq b(X)$ have to be flow modules.

Proposition 4. *For any indecomposable regular module X, we have $\bar{\iota}(X) = \iota(X)$.*

Proof. Let $\mathcal{X}$ be a τ-orbit. In order to show that $\bar{\iota} = \iota$ on $\mathcal{X}$, it is sufficient to to show that $\bar{\iota}(X) = \iota(X)$ for some X in $\mathcal{X}$. Let p be the center of the sink modules in $\mathcal{X}$ and q the center of the source modules in $\mathcal{X}$. Let $b = d(p, q)$.

First, assume that $b = 2s+1$ is odd. Then $\mathcal{X}$ contains either s or $s+1$ flow modules. If X belongs to $\mathcal{X}$ and $\bar{\iota}(X) = 0$, then X is a sink module, and the modules $\tau X, \ldots, \tau^s X$ are the flow modules in $\mathcal{X}$. Thus $\mathcal{X}$ has precisely s flow modules and according to Proposition 2, X has index 0. Thus $\bar{\iota}(X) = 0 = \iota X$.

If X belongs to $\mathcal{X}$ and $\bar{\iota}(X) = -1$, then X is a sink module, and the modules $\tau X, \ldots, \tau^{s+1} X$ are the flow modules in $\mathcal{X}$. Thus $\mathcal{X}$ has precisely $s + 1$ flow modules and again using Proposition 2, we see that X has index -1. Thus $\bar{\iota}(X) = -1 = \iota X$.

Second, assume that $b = 2s$ is even, thus $\mathcal{X}$ contains precisely s flow modules. We denote by M_i quasi-composition factors of the modules in $\mathcal{X}$, with $\iota(M_i) = i$. Let l be the quasi-length of the modules in $\mathcal{X}$. Assume that X belongs to $\mathcal{X}$ with $\bar{\iota}(X) = 0$. Then X is a sink module, the modules $\tau X, \ldots, \tau^s X$ are the flow modules in $\mathcal{X}$, and $\tau^{s+1} X$ is a source module. The quasi-top of X is M_{-l}, the quasi-socle of $\tau^{s+1} X$ is $M_{b+1+l+1}$. According to Lemma 3, $B(X) = B(M_{-l})$, thus $r(M) = r_0 + l$ According to Lemma 3*, $B(\tau M^{s+1}) = B(M_{b+1+l+1})$, thus $r(\tau M^{s+1}) = r_0 + l + 1$. This shows that $r(M) < r(\tau M^{s+1})$. Thus, Proposition 3 asserts that $\iota(X) = 0$. A similar calculation shows that for $\bar{\iota}(X) = -1$, we have $\iota(X) = -1$.

Thus, in all cases we have found a module $X \in \mathcal{X}$ with $\bar{\iota}(X) = \iota(X)$ and therefore $\bar{\iota}(X') = \iota(X')$ for all $X' \in \mathcal{X}$. $\qquad\square$

The distribution of the sink modules, flow modules and source modules in the regular Auslander-Reiten components is as follows:
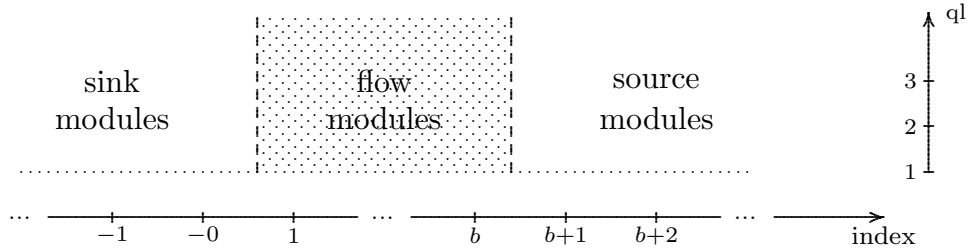
Of course, Proposition 4 implies (and actually is equivalent to) the assertion of Theorem 3.

Let us look now at Theorem 4. Actually, it remains to deal with the last assertion of Theorem 4. Clearly, it is sufficient to show this for one module in the τ-orbit of Y, thus we may assume that $\iota(Y) \leq 0$. Let M be the quasi-top of Y. According to Lemma 3, we have $B(Y) = B(M)$, thus $r(Y) = r(M)$. Recall that $\eta Y = Y'$, where $0 \to X \to Y \oplus Y' \to Z \to 0$ is an Auslander-Reiten sequence with $|Y| < |Y'|$. Then $\tau^- M$ is the quasi-top both of Y' and Z. Therefore $B(\eta Y) = B(Y') = B(\tau^- M)$ and we know that $r(\tau^- M) = r(M) + 2$, thus $r(\eta Y) = r(Y') = r(M) + 2 = r(Y) + 2$. Concerning the centers, the equalities $B(\eta Y) = B(\tau^- M)$ and $B(Y) = B(M)$ yield $C(\eta Y) = C(\tau^- M) = C(M) = C(Y)$. $\qquad\square$

We have seen: If M, M' belong to the same Auslander-Reiten component, then

$$p(M) = p(M'), \quad q(M) = q(M'), \quad r_0(M) - \mathrm{ql}(M) = r_0(M') - \mathrm{ql}(M').$$

Thus, if $\mathcal{C}$ is a regular component, and M belongs to $\mathcal{C}$, we may define

$$
\begin{aligned}
p(\mathcal{C}) &= p(M), \\
q(\mathcal{C}) &= q(M), \\
b(\mathcal{C}) &= d(p(M), q(M)), \\
r(\mathcal{C}) &= r_0(M) - \mathrm{ql}(M).
\end{aligned}
$$

The radius $r(M)$ of a module M in $\mathcal{C}$ is

$$r(\mathcal{C}) + R_{b(\mathcal{C})}(\iota(M), \mathrm{ql}(M)),$$

where $R_b : \mathbb{Z} \times \mathbb{N}_1 \to \mathbb{N}_0$ is the following function

$$R_b(i, l) = \begin{cases} -i + l & i \leq 0, \\ l - 1 & \text{for} \quad 1 \leq i \leq b, \\ i + l & b < i. \end{cases}$$

At the end of the paper, we exhibit the function $R_b(i, l)$ for some special values of b.

8. A warning, two questions and several remarks.

The Warning. Let M be an indecomposable regular module. Its center path $\pi(M)$ may not be contained in the support $T(M)$ of M (but it is always contained in the ball $B(M)$). Here is a typical example: Let $\pi = (q = a_0, a_1, \ldots, a_b = q)$ be a path with q a source. Then $P_0(p)$ is a submodule of $P_b(q)$ and q is not contained in the support of $M = P_b(p)/P_0(q)$. This is the module $M = M_1$ constructed as Case I in the proof of Theorem 2, see section 4. There, we have seen that $\pi(M)$ is the given path π.

Remark 1. The case $n = 2$. We have seen that for $n \geq 3$ (thus for the wild Kronecker algebras) almost all modules in the τ-orbit of a regular graded module are sink or source modules. In contrast, in the tame case $n = 2$, the sink modules are the preprojective modules, the source modules are the preinjective modules, thus all the indecomposable regular graded modules are flow modules. We should stress that also for $n = 2$, the regular components are of the form $\mathbb{Z}\mathbb{A}_\infty$ (and not proper quotients), so that any regular component contains infinitely many quasi-simple modules. To repeat: The number of quasi-simple flow modules in a regular component is finite, if $n \geq 3$, but it is infinite, if $n = 2$. As in our investigation of bristles see Appendix C of [R4], we encouter a finiteness result for wild cases which is not valid in the corresponding tame case.

Remark 2. When dealing with wild hereditary algebras, one knows that the regular modules have exponential growth when applying τ (or τ^-), see [R1,B,K1,K2]. This concerns the dimension vector (and all its coefficients). In contrast, the main observation of this paper asserts that the radius of an indecomposable regular module eventually grows linearly when we iterate the application of τ: it increases step by step by 2. Of course, we have seen also exponential growth (at the end of section 4): the number of boundary vertices in the support of a sink module increases exponentially, when we apply τ. Note that the number $\beta(M)$ of boundary vertices is a lower bound for the length of M.

Remark 3. Our main motivation for these investigations is our interest in (ungraded) Kronecker modules: to find invariants for regular Kronecker modules. The present paper provides such invariants for the gradable modules (those which are obtained from graded Kronecker modules by forgetting the grading, or, equivalently, for those representations of $K(n)$ which can be lifted to its universal covering $(T(n), \Omega)$). Given an indecomposable

gradable Kronecker module $\overline{M}$ with cover module M, the numerical invariants $r_0(M)$ and $b(M)$ are invariants of $\overline{M}$. Actually, it is r_0 and the path $\pi(M)$ (or better, its equivalence class under the covering group) which are the decisive invariants of $\overline{M}$.

Question 1. We have stressed in remark 2 that dealing with graded Kronecker modules, the radius of an indecomposable regular module eventually increases linearly when we iterate the application of τ. Is there a similar invariant for ungraded Kronecker modules with this growth behaviour?

Remark 4. If $\mathcal{C}$ is a regular component in $\mathrm{mod}(T(n), \Omega)$, iterated applications of τ to any module in $\mathcal{C}$ provide balls with a fixed center p (in particular, p is an invariant of $\mathcal{C}$). The preprojective and the preinjective component behave completely different, there is no focus on a special vertex of $T(n)$.

Remark 5. In this paper, when looking at a representations M of $(T(n), \Omega)$, we actually were dealing just with the dimension vector of M, not with M itself. Indecomposable representations of $(T(n), \Omega)$ with same dimension vectors have the same behavior with respect to the action of σ: If M, M' are indecomposable representations of $(T(n), \Omega)$ with $\mathbf{dim}\, M = \mathbf{dim}\, M'$, then $p(M) = p(M'), q(M) = q(M')$ and $r_0(M) = r_0(M')$.

Remark 6. Why is the index of a module defined in such a way that it decreases along irreducible maps? In general, we like to draw abelian categories by focusing the attention to the direction of maps, drawing arrows from left to right. According to Baer [B] and Kerner [K1,K2], there is a global direction of a hereditary module category, defined by the maps from the projective modules to the injective modules (going from left to right): for wild hereditary algebras, this global direction is opposite to the direction of the irreducible maps (thus, opposite to τ). To repeat, dealing with regular modules of wild hereditary algebras, the irreducible maps point in the opposite direction of the global direction of the category. Thus, it is reasonable to draw the regular Auslander-Reiten components for a hereditary algebra by using arrows from right to left.

Question 2. The definition of the index of an indecomposable representation of $(T(n), \Omega)$ relies not only on $\mathrm{mod}(T(n), \Omega)$, but also on $\mathrm{mod}(T(n), \sigma\Omega)$. Thus, let M be an indecomposable regular $(T(n), \Omega)$-module say with index i. Then τM is an indecomposable regular $(T(n), \Omega)$-module say with index $i - 2$. But attached to M are two modules with index $i - 1$, namely the $(T(n), \Omega)$-module $\mu(M)$ which arises as the middle term of the Auslander-Reiten sequence ending in M (this module is not necessarily indecomposable, it may be the direct sum of two indecomposable modules with index $i - 1$), and the $(T(n), \sigma\Omega)$-module σM. What is the precise relationship between $\mu(M)$ and σM?

Remark 7. The topic of this paper may be compared to the so-called Game of Life, as introduced by Conway, see [Gar], as a cellular automaton. Whereas the Game of Life is played on the grid with set of vertices $\mathbb{Z}^2$, here we start with the n-regular tree $T(n)$. If we want to increase the analogy, we should deal not with dimension vectors of modules but work modulo 2, thus dealing with functions $f : T(n)_0 \to \{0, 1\}$ with finite support.
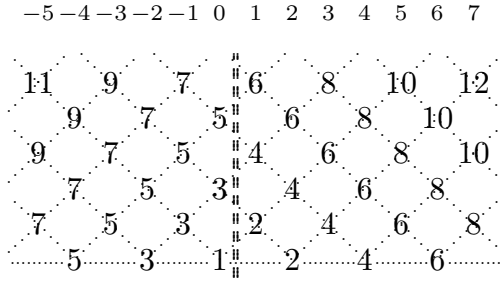
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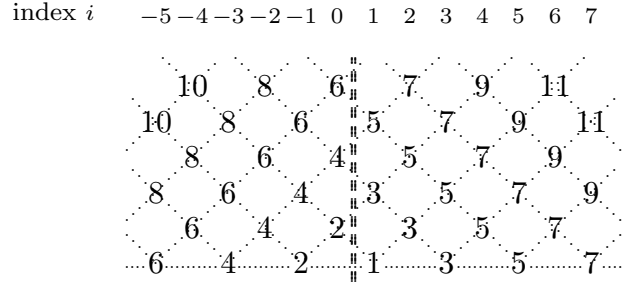
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The function $R_b(i,l)$ on even and an odd components, for some values b.

$b = 0$

<pre>
       −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7   index i   −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

       11   9   7   6   8   10   12   10   8   6   7   9   11
          9   7   5   6   8   10   10   8   6   5   7   9   11
       9   7   5   4   6   8   10   8   6   4   5   7   9
          7   5   3   4   6   8   8   6   4   3   5   7   9
       7   5   3   2   4   6   8   6   4   2   3   5   7
          5   3   1   2   4   6   6   4   2   1   3   5   7

                even component   odd component
</pre>

$b = 1$

<pre>
       −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7   index i   −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

       11   9   7   5   7   9   11   10   8   6   6   8   10
          9   7   5   5   7   9   10   8   6   4   6   8   10
       9   7   5   3   5   7   9   8   6   4   4   6   8
          7   5   3   3   5   7   8   6   4   2   4   6   8
       7   5   3   1   3   5   7   6   4   2   2   4   6
          5   3   1   1   3   5   6   4   2   0   2   4   6

                even component   odd component
</pre>

$b = 4$

<pre>
       −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7   index i   −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

       11   9   7   5   5   6   8   10   8   6   5   5   7
          9   7   5   4   4   6   10   8   6   4   4   5   7
       9   7   5   3   3   4   6   8   6   4   3   3   5
          7   5   3   2   2   4   8   6   4   2   2   3   5
       7   5   3   1   1   2   4   6   4   2   1   1   3
          5   3   1   0   0   2   6   4   2   0   0   1   3

                even component   odd component
</pre>

$b = 5$

<pre>
       −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7   index i   −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

       11   9   7   5   5   5   7   10   8   6   5   5   8
          9   7   5   4   4   5   10   8   6   4   4   4   8
       9   7   5   3   3   3   5   8   6   4   3   3   4
          7   5   3   2   2   3   8   6   4   2   2   2   4
       7   5   3   1   1   1   3   6   4   2   1   1   2
          5   3   1   0   0   1   6   4   2   0   0   0   2
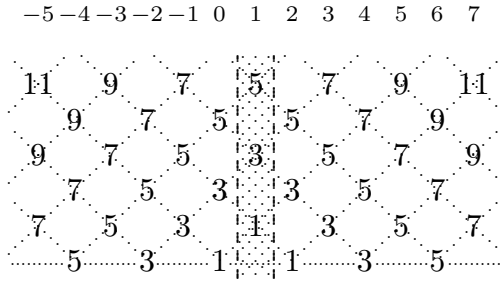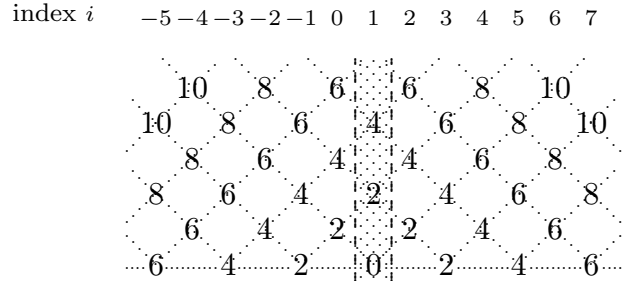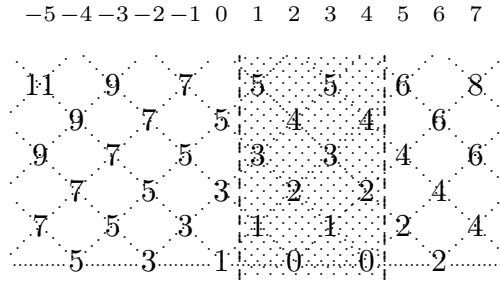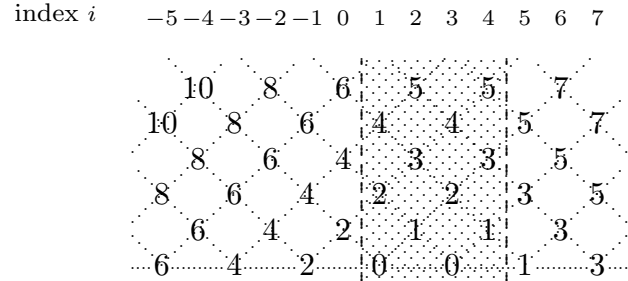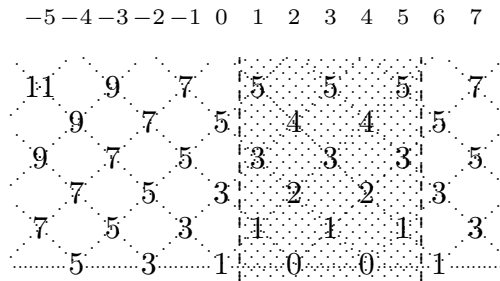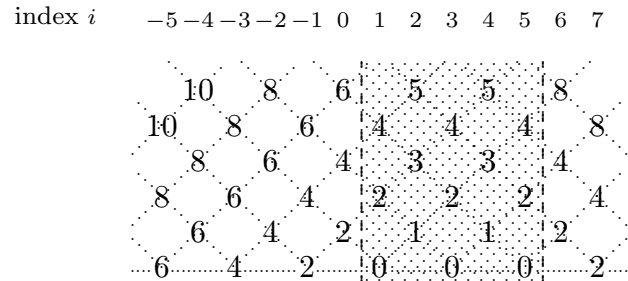
                even component   odd component
</pre>