Gorenstein-projective and semi-Gorenstein-projective modules. II

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Abstract: Let $k$ be a field and $q$ a non-zero element of $k$. In Part I, we have exhibited a 6-dimensional $k$-algebra $\Lambda = \Lambda(q)$ and we have shown that if $q$ has infinite multiplicative order, then $\Lambda$ has a 3-dimensional local module which is semi-Gorenstein-projective, but not torsionless, thus not Gorenstein-projective. This Part II is devoted to a detailed study of all the 3-dimensional local $\Lambda$-modules for this particular algebra $\Lambda$. If $q$ has infinite multiplicative order, we will encounter a whole family of 3-dimensional local modules which are semi-Gorenstein-projective, but not torsionless.

Key words and phrases. Gorenstein-projective module, semi-Gorenstein-projective module, torsionless module, extensionless module, reflexive module, $t$-torsionfree module, $t$-quiver.

2010 Math. Subject Classification. Primary 16G10, 16G50. Secondary 16E05, 16E65, 20G42.

Supported by NSFC 11431010

1. Introduction.

(1.1) We refer to our previous paper [RZ1] as Part I. As in Part I, let $k$ be a field, and $q$ a non-zero element of $k$. We consider again the $k$-algebra $\Lambda = \Lambda(q)$ generated by $x, y, z$ with relations

$$x^2, \quad y^2, \quad z^2, \quad yz, \quad xy + qyx, \quad xz - zx, \quad zy - zx.$$ 

The algebra $\Lambda$ is a 6-dimensional local algebra with basis $1, x, y, z, yx, zx$. Its socle is $\text{soc} \Lambda = \text{rad}^2 \Lambda = \Lambda yx \oplus \Lambda zx$. If not otherwise stated, all the modules considered will be left $\Lambda$-modules.

We follow the terminology used in Part I. In particular, we denote by $\partial M$ the cokernel of a minimal left $\text{add}(\Lambda)$-approximation of $M$. In addition, we introduce the following definitions. We say that a module $M$ is extensionless if $\text{Ext}^1(M, \Lambda) = 0$. An indecomposable semi-Gorenstein-projective module will be said to be pivotal provided it is not torsionless. An indecomposable $\infty$-torsionfree module will be said to be pivotal provided it is not extensionless. Thus, a module $M$ is semi-Gorenstein-projective if and only if $\Omega^t M$ is extensionless for all $t \geq 0$; a torsionless module $M$ is reflexive if and only if $\partial M$ is torsionless (see Part I (2.4)); a module $M$ is $\infty$-torsionfree if and only if $\Omega^t M$ is reflexive for all $t \geq 0$; and $M$ is Gorenstein-projective if and only if $M$ is both semi-Gorenstein-projective and $\infty$-torsionfree.

(1.2) We are interested in the semi-Gorenstein-projective and the $\infty$-torsionfree modules and will exhibit those which are 3-dimensional. We recall that a finite length module is said to be local provided its top is simple. Thus, a local module is indecomposable; and if $R$ is a left artinian ring, then a left $R$-module $M$ is local if and only if $M$ is a quotient of an indecomposable projective module. A consequence of our study is the following assertion
Proposition. Let $M$ be a non-zero module of dimension at most 3. If $M$ is semi-Gorenstein-projective, then all the modules $\Omega^t M$ with $t \geq 0$ are 3-dimensional and local. If $M$ is $\infty$-torsionfree, then all the modules $\Omega^t M$ with $t \geq 0$ are 3-dimensional and local. In particular, if $M$ is Gorenstein-projective, then all the modules $\Omega^t M$ and $\Omega^t M$ with $t \geq 0$ are 3-dimensional and local.

(1.3) The text restricts the attention to the 3-dimensional local modules. We recall that if $A$ is a finite-dimensional algebra, an $A$-module $M$ is said to have Loewy length at most $t$ provided $\text{rad}^t M = 0$. The starting point of our investigation are two observations. The first one:

Proposition 1. A module of dimension at most 3 has Loewy length at most 2.

The second observation is:

Proposition 2. An indecomposable 3-dimensional torsionless module is local.

The proof of Proposition 1 will be given in (2.6), the proof of Proposition 2 in (2.7).

(1.4) The 3-dimensional local modules. We identify $(a, b, c) \in k^3 \setminus \{0\}$ with $ax + by + cz$ and denote by $(a:b:c)$ the 1-dimensional subspace of $k^3$ generated by $(a,b,c)$. The left ideal $U(a,b,c) = U(a:b:c) = \Lambda(a,b,c) + \text{soc} \Lambda$ has dimension 3, and we obtain the left $\Lambda$-module

$$M(a, b, c) = M(a:b:c) = \Lambda \Lambda / U(a, b, c).$$

Clearly, $M(a, b, c)$ is a 3-dimensional local module and the modules $M(a, b, c)$, $M(a', b', c')$ are isomorphic if and only if $(a:b:c) = (a':b':c')$. Let us add that the definition of $M(a,b,c)$ implies that $\Omega M(a, b, c) \simeq U(a, b, c)$, this will be used throughout the text.

Conversely, any 3-dimensional local module is isomorphic to a module of the form $M(a, b, c)$. In order to see this, one should look at the factor algebra $\overline{\Lambda}$ of $\Lambda$ modulo $\text{soc} \Lambda = \text{rad}^2 \Lambda$, thus $\overline{\Lambda}$ is the $k$-algebra generated by $x, y, z$ with relations all monomials of length 2. The $\Lambda$-modules of Loewy length at most 2 are just the modules annihilated by all monomials of length 2, thus the $\overline{\Lambda}$-modules. It is clear that the modules $M(a, b, c) = \overline{\Lambda} / (a:b:c)$ are representatives of the 3-dimensional local $\overline{\Lambda}$-modules. According to Proposition 1, all the 3-dimensional $\Lambda$-modules are $\overline{\Lambda}$-modules, thus the modules $M(a, b, c)$ are representatives of the 3-dimensional local $\Lambda$-modules.

(1.5) The following theorem characterizes the modules of dimension at most 3 which have some relevant properties. We write $o(q)$ for the multiplicative order of $q$.

Theorem. An indecomposable module $M$ of dimension at most 3 is

- torsionless if and only if $M$ is simple or isomorphic to $\Lambda(x - y)$, to $\Lambda z$, to a module $M(1, b, c)$ with $b \neq -q$, to $M(0, 1, 0)$ or to $M(0, 0, 1)$;
- extensionless if and only if $M$ is isomorphic to a module $M(1, b, c)$ with $b \neq -1$;
- reflexive if and only if $M$ is isomorphic to a module $M(1, b, c)$ with $b \neq -q^i$ for $i = 1, 2$;
- Gorenstein-projective if and only if $M$ is isomorphic to a module $M(1, b, c)$ with $b \neq -q^i$ for $i \in \mathbb{Z}$;
• semi-Gorenstein-projective if and only if $M$ is isomorphic to a module $M(1, b, c)$ with $b \neq -q^i$ for $i \leq 0$;
• $\infty$-torsionfree if and only if $M$ is isomorphic to a module $M(1, b, c)$ with $b \neq -q^i$ for $i \geq 1$;
• pivotal semi-Gorenstein-projective if and only if $o(q) = \infty$ and $M$ is isomorphic to a module $M(1, -q, c)$;
• pivotal $\infty$-torsionfree if and only if $o(q) = \infty$ and $M$ is isomorphic to a module $M(1, -1, c)$.

For the proof of the Theorem, see (7.9).

Looking at the Theorem, the reader will be aware that in the context considered here, the relevant modules of dimension at most 3 are of the form $M(1, b, c)$ with $b, c \in k$. Nearly all the modules mentioned in Theorem are of this kind, the only exceptions are four isomorphism classes of torsionless modules, namely the 2-dimensional left ideals $\Lambda(x - y)$ and $\Lambda z$, as well as the 3-dimensional modules $M(0, 1, 0)$ and $M(0, 0, 1)$.

(1.6) As we have seen in (1.4), the set of isomorphism classes of the 3-dimensional local modules can be identified in a natural way with the projective plane $\mathbb{P}^2 = \mathbb{P}(\text{rad} \Lambda/\text{rad}^2 \Lambda)$, with the element $(a : b : c) \in \mathbb{P}^2$ corresponding to the module $M(a, b, c)$.

We use homogeneous coordinates in order to highlight elements and subsets of $\mathbb{P}^2$ (or the corresponding modules):

Let $H$ be the affine subspace of $\mathbb{P}^2$ given by the points $(1 : b : c)$ with $b, c \in k$. As we have mentioned already, Theorem (1.5) shows that it is this subset $H$ which is of special interest. We will see in section 7 that $H$ is a union of $U$-components, and that the set of 3-dimensional Gorenstein-projective modules is always a (proper) subset of $H$. A module $M$ in $H$ is torsionless if and only if it does not belong to the line $T = \{(1 : (-q) : c) \mid c \in k\}$, and is extensionless if and only if it does not belong to the line $E = \{(1 : (-1) : c) \mid c \in k\}$ (see (6.1) and (5.1), respectively):

In case the multiplicative order $o(q)$ of $q$ is infinite, $H$ is the set of the 3-dimensional modules which are semi-Gorenstein-projective or $\infty$-torsionfree; the line $E$ consists of the pivotal semi-Gorenstein-projective modules in $H$; the line $T$ of the pivotal $\infty$-torsionfree modules in $H$. 

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Let us emphasize: There are 3-dimensional pivotal semi-Gorenstein-projective modules if and only if there are 3-dimensional pivotal \(\infty\)-torsionfree modules if and only if the multiplicative order of \(q\) is infinite.

Note that a 3-dimensional local module \(M\) belongs to \(H\) if and only if \(\text{soc } M = \text{Ker}(y) = \text{Ker}(z) = yM \oplus zM\), see A.4 in the appendix.

(1.7) The algebra \(\Lambda = \Lambda(q)\) with \(o(q) = \infty\) was exhibited in Part I in order to present a module \(\Lambda\) which is not reflexive, such that both \(\Lambda\) and its \(\Lambda\)-dual \(\Lambda^*\) are semi-Gorenstein-projective: namely the module \(\Lambda = M(1, -q, 0)\). Now we see:

Let \(o(q) = \infty\) and assume that \(\Lambda\) is a module of dimension at most 3. Then both \(\Lambda\) and \(\Lambda^*\) are semi-Gorenstein-projective, whereas \(\Lambda\) is not reflexive, if and only if \(\Lambda\) is isomorphic to a module of the form \(M(1, -q, c)\) with \(c \in k\). In this case \(\Lambda\) is not even torsionless and all the modules \(M(1, -q, c)^*\) with \(c \in k\) are isomorphic, see (9.5). Thus, we encounter a 1-parameter family of pairwise non-isomorphic semi-Gorenstein-projective left modules \(\Lambda\) such that their \(\Lambda\)-dual modules \(\Lambda^*\) are isomorphic and semi-Gorenstein-projective.

Also, we see that for all \(c \neq 0\), the modules \(M(1, -1, c)\) are pairwise non-isomorphic \(\infty\)-torsionfree modules with a fixed module \(\Omega M(1, -1, c) = M(0, 0, 1)\), see (4.1). Thus, we encounter non-isomorphic \(\infty\)-torsionfree modules with isomorphic first syzygy module (of course, in this situation, the syzygy module cannot be \(\infty\)-torsionfree).

(1.8) The modules \(M(1, b, 0)\) with \(b \in k\) have been studied already in Part I (there, they have been denoted by \(M(-b)\)). Theorem (1.5) shows that these modules are quite typical for the behavior of the modules \(M(1, b, c)\). Namely: The module \(M(1, b, c)\) is Gorenstein-projective (or semi-Gorenstein-projective, or \(\infty\)-torsionfree, or torsionless, or extensionless) if and only if \(M(1, b, 0)\) has this property.

(1.9) Outline of the paper. Section 2 provides some preliminary results. Here, the main target is to show that any module of length at most 3 has Loewy length at most 2. In section 3 we collect some formulae which show that certain products of elements in \(\Lambda\) are zero. Sections 4 to 7 deal with the 3-dimensional local left \(\Lambda\)-modules, section 8 with the 3-dimensional local right \(\Lambda\)-modules. Section 9 discusses the \(\Lambda\)-duality. The final section 10 provides an outline of the general frame for this investigation: the study of semi-Gorenstein-projective and \(\infty\)-torsionfree modules over local algebras with radical cube zero. There is an appendix which provides a diagrammatic description of the 3-dimensional indecomposable left \(\Lambda\)-modules.

2. Some left ideals and some right ideals of \(\Lambda\).

(2.1) Lemma. The left ideal \(\Lambda(a, b, c)\) is 2-dimensional if and only if \(a + b = 0\) and \(ac = 0\). We have \(\text{soc } \Lambda(1, -1, 0) = \Lambda yx\) and \(\text{soc } \Lambda(0, 0, 1) = \Lambda zz\).

Proof. An easy calculation shows that \(\text{soc } \Lambda(1, -1, 0) = \Lambda yx\) and \(\text{soc } \Lambda(0, 0, 1) = \Lambda zz\). Thus, the left ideals \(\Lambda(0, 0, 1)\) and \(\Lambda(1, -1, 0)\) are 2-dimensional.

Now, let \(L = \Lambda(a, b, c)\) be any left ideal. If \(a \neq 0\), then \(yx \in L\) since \(y(a, b, c) = ayx\).

First, assume that \(a + b \neq 0\). Then \(z(a, b, c) = (a + b)zx\) shows that \(zx \in L\). We know already that for \(a \neq 0\), also \(yx \in L\). If \(a = 0\), then \(b \neq 0\). Thus \(x(a, b, c) = -qbyx + czx\) shows that also in this case \(yx \in L\). Thus \(L\) cannot be 2-dimensional.
Next, assume that \( ac \neq 0 \). Since \( a \neq 0 \), we know that \( yx \in L \). Since \( c \neq 0 \), we use
\[
x(a, b, c) = -qbyx + czx
\]
in order to see that \( zx \in L \). Again, \( L \) cannot be 2-dimensional. \( \square \)

\textbf{(2.2)} Let \( L \) be a 2-dimensional left ideal, different from \( \text{soc} \Lambda \). Then either \( L \subseteq U(1, -1, 0) \) and then \( \text{soc} L = \Lambda yx \) and \( L \) is isomorphic to \( \Lambda(x - y) \) or else \( L \subseteq U(0, 0, 1) \) and then \( \text{soc} L = \Lambda zx \) and \( L \) is isomorphic to \( \Lambda z \).

Proof: There is an element \( (a, b, c) + w \in L \), with \( (a, b, c) \neq 0 \) and \( w \in \text{soc} \Lambda \). Since \( \text{rad} \Lambda((a, b, c) + w) = \text{rad} \Lambda(a, b, c) \), also \( L' = \Lambda(a, b, c) \) is 2-dimensional and \( L \subseteq L' + \text{soc} \Lambda = U(a, b, c) \). According to (2.1), \( (a : b : c) \) is equal to \((1 : (-1) : 0)\) or to \((0 : 0 : 1)\). Of course, \( L \) and \( L' \) are isomorphic as (left) modules. \( \square \)

\textbf{(2.3) Lemma.} There is no 3-dimensional torsionless module with simple socle.

Proof. Assume that \( U \) is a 3-dimensional torsionless module with simple socle. Then \( U \) is a submodule of \( \Lambda \). It is a proper submodule, thus of Loewy length at most 2. Therefore, \( U \) is the sum of two 2-dimensional left ideals \( L \neq L' \) with \( \text{soc} L = \text{soc} L' \). Now we use (2.2). If \( L, L' \) have socle equal to \( \Lambda yx \), then \( U = L + L' = U(1, -1, 0) \). If \( L, L' \) have socle equal to \( \Lambda zx \), then also \( U = L + L' = U(0, 0, 1) \). In both cases \( \text{soc} \Lambda \subseteq U \), a contradiction. \( \square \)

\textbf{(2.4) Any 3-dimensional left ideal contains soc} \( \Lambda \).

\textbf{(2.5) The 3-dimensional left ideals are the subspaces} \( U(a, b, c) \). They have the following structure: \( U(1, -1, 0) = \Lambda(1, -1, 0) \oplus \Lambda zx \); \( U(0, 0, 1) = \Lambda(0, 0, 1) \oplus \Lambda yx \); and if \( a + b \neq 0 \) or \( ac \neq 0 \), then \( U(a, b, c) = \Lambda(a, b, c) \) is a local module (in particular, indecomposable).

Proof. The left ideals \( U(a, b, c) \) are 3-dimensional. Conversely, let \( U \) be a 3-dimensional left ideal of \( \Lambda \). Since \( \text{soc} \Lambda \) is contained in \( U \), there is an element \( (a, b, c) \neq 0 \) with \( (a, b, c) \in U \), thus \( U = U(a, b, c) \).

If \( a + b = 0 \) and \( ac = 0 \), then \( (a : b : c) \) is equal to \((1 : (-1) : 0)\) or to \((0 : 0 : 1)\). By (2.1), we have \( U(1, -1, 0) = \Lambda(1, -1, 0) \oplus \Lambda zx \) and \( U(0, 0, 1) = \Lambda(0, 0, 1) \oplus \Lambda yx \). If \( a + b \neq 0 \) or \( ac \neq 0 \), then \( U(a, b, c) = \Lambda(a, b, c) \) is a local module, thus indecomposable. \( \square \?

\textbf{(2.6) Proposition.} Any module of dimension at most 3 has Loewy length at most 2.

Proof. Let \( M \) be a module of dimension at most 3. If \( M \) is not local, then clearly \( M \) has Loewy length at most 2. If \( \dim M \leq 2 \), then again \( M \) has Loewy length at most 2. Thus, we can assume that \( M \) is 3-dimensional and local and therefore a factor module of \( \Lambda \), say \( M = \Lambda / U \). According to (2.4), \( \text{soc} \Lambda \subseteq U \), thus \( M \) is annihilated by \( \text{soc} \Lambda \), and therefore \( M \) has Loewy length at most 2. \( \square \)

\textbf{(2.7) Lemma.} Any indecomposable torsionless module \( M \) of dimension at most 3 is local and isomorphic to a left ideal of \( \Lambda \). If \( \dim M = 3 \), then \( M \) is of the form \( U(a, b, c) \).

Proof. Let \( M \) be indecomposable and torsionless. If \( \dim M \leq 2 \), then \( M \) is of course local and isomorphic to a left ideal. Thus we can assume that \( \dim M = 3 \).

Since \( M \) is torsionless, there is a set of non-zero maps \( u_i : M \to \Lambda \Lambda \) (say with index set \( I \)) such that \( \bigcap_{i \in I} K_i = 0 \), where \( K_i \) is the kernel of \( u_i \).

If \( K_i = 0 \) for some \( i \), then already \( u_i \) is an embedding (thus \( M \) is isomorphic to a left ideal). In particular, if the socle of \( M \) is simple, then we must have \( K_i = 0 \) for some \( i \).
Thus, we can assume that the socle of \( M \) is not simple. Therefore \( M \) has to be a local module and we have a surjective map \( \pi : \Lambda \Lambda \to M \).

It remains to look at the case where \( \dim K_i = 1 \) or 2 for all \( i \). Since the only 2-dimensional submodule of \( M \) is its radical, we have \( \bigcap_{i \in I'} K_i = 0 \), where \( I' \) is the set of indices \( i \) with \( \dim K_i = 1 \). But then \( K_i \cap K_j = 0 \) for some \( i \neq j \) in \( I' \). This shows that we can assume that \( I = \{1, 2\} \) and that \( K_1, K_2 \) are different 1-dimensional submodules of \( M \).

Now \( u_i \) provides an isomorphism from \( M/K_i \) onto a (2-dimensional) left ideal of \( \Lambda \). Since \( M/K_i \) is indecomposable, (2.2) shows that \( M/K_i \) is isomorphic to \( \Lambda(1, -1, 0) \) or to \( \Lambda(0, 0, 1) \). Let \( K'_i = \ker(u_i \pi) \) for \( i = 1, 2 \).

If \( M/K_i \cong \Lambda(1, -1, 0) \), then \( K'_i \) is equal to \( \Lambda(x + qy) + \Lambda z \), since \( \Lambda(0, 0, 1) \) is annihilated by \( x + qy \) and by \( z \). Similarly, if \( M/K_i \cong \Lambda(0, 0, 1) \), then \( K'_i \) is equal to \( \Lambda(x + qy) + \Lambda z \). Thus one of \( M/K_i \) has to be isomorphic to \( \Lambda(1, -1, 0) \), the other one to \( \Lambda(0, 0, 1) \) and \( \ker(\pi) = K'_1 \cap K'_2 = U(0, 0, 1) \). It follows that \( M \cong \Lambda \Lambda/\ker(\pi) = \Lambda \Lambda/U(0, 0, 1) \). But \( \Lambda \Lambda/U(0, 0, 1) \) is isomorphic to the left ideal \( \Lambda(1, -1, 1) = U(1, -1, 1) \).

We have shown that \( M \) is isomorphic to a left ideal, thus of the form \( U(a, b, c) \), see (2.5). Since we assume that \( M \) is indecomposable, (2.5) asserts that \( M \) is local.

We need to know also the right ideals \( (a, b, c) \Lambda \). Note that \( U(a, b, c) \) is always a twosided ideal and it will be pertinent to denote \( U(a, b, c) \) by \( U'(a, b, c) \), if we consider it as a right ideal (thus as a right module).

(2.8) The right ideals \( (a, b, c) \Lambda \). If \( a \neq 0 \) or \( bc \neq 0 \), then \( (a, b, c) \Lambda = U(a, b, c) \) is 3-dimensional. The right ideals \( (0, 1, 0) \Lambda \) and \( (0, 0, 1) \Lambda \) are 2-dimensional with soc \( (0, 1, 0) \Lambda = yx \Lambda \) and soc \( (0, 0, 1) \Lambda = zx \Lambda \).

Proof: Let \( V = (a, b, c) \Lambda \). First, let \( a \neq 0 \). Then \( zx \) belongs to \( V \), since \((a, b, c)z = axz \). Also \( yx \in V \), since \((a, b, c)y = -qaxz + czx \). Second, assume that \( a = 0 \) and \( bc \neq 0 \). Then \((0, b, c)y = czx \) shows that \( zx \in V \), and \((0, b, c)x = byx + czx \) shows that also \( yx \in V \).

(2.9) If a 3-dimensional indecomposable right module \( N \) is torsionless, then it is isomorphic to a right ideal, thus to \( U'(a, b, c) \) for some \( (a, b, c) \neq 0 \).

Proof. Let \( N \) be a 3-dimensional indecomposable torsionless right module. As in (2.7) one shows that \( N \) is isomorphic to a right ideal, using (2.8) instead of (2.2). It remains to see that all 3-dimensional right ideals are of the form \( U'(a, b, c) \). Here, one has to copy the proof of (2.5).

3. The transformations \( \omega \) and \( \omega' \).

If \((a : b : c)\) is different from \((1 : (−1) : 0)\) and \((0 : 0 : 1)\), then (2.5) shows that \( U(a, b, c) \) is a 3-dimensional local module, thus of the form \( M(a' : b' : c') \). In order to describe in which way \((a' : b' : c')\) depends on \((a : b : c)\), we will need the transformations \( \omega \) and \( \omega' \). We start with some equalities in \( \Lambda \).

(3.1) Formulae. Let \( a, b, c \in k \). Then

\[
\begin{align*}
(1) & \quad (ax + by - \frac{a}{a+b}cz)(ax + by + cz) = 0 \quad \text{if} \quad a + b \neq 0 \\
(2) & \quad z(ax - ay + cz) = 0 \\
(3) & \quad (ax + by + cz)(ax + c^{-1}by - \frac{a + c^{-1}b}{a}cz) = 0 \quad \text{if} \quad a \neq 0 \\
(4) & \quad (by + cz)z = 0 
\end{align*}
\]
Proof of the equality (1):

\[(a + qby - \frac{a}{a+b}cz)(a + by + cz)\]

\[= abxy + acxz + qabxy - \frac{a}{a+b}aczx - \frac{a}{a+b}bczy\]

\[= ab(xy + qyx) + \left(1 - \frac{a}{a+b} - \frac{b}{a+b}\right)aczx = 0.\]

The proof of the remaining equalities is similar. \(\square\)

(3.2) In case \(a + b \neq 0\), let \(\omega(a, b, c) = (a, qb, -\frac{a}{a+b}c)\). In case \(a' \neq 0\), let \(\omega'(a', b', c') = (a', q^{-1}b', -\frac{a'+q^{-1}b'}{a'}c').\)

Proposition. The transformation \(\omega\) provides a bijection from the set \(\{(a, b, c) \in k^3 | a(a + b) \neq 0\}\) onto the set \(\{(a', b', c') \in k^3 | a'(a' + q^{-1}b') \neq 0\}\), with inverse \(\omega'\).

Proof. Let \(a(a + b) \neq 0\). Then \((a', b', c') = \omega(a, b, c)\) is defined and \(a' = a \neq 0\), and \(a' + q^{-1}b' = a + q^{-1}b = a + b \neq 0\). Thus \(\omega\) maps \(\{(a, b, c) \in k^3 | a(a + b) \neq 0\}\) onto \(\{(a, b, c) \in k^3 | a(a + q^{-1}b) \neq 0\}\). Similarly, \(\omega'\) maps \(\{(a', b', c') \in k^3 | a'(a' + q^{-1}b') \neq 0\}\) onto \(\{(a, b, c) \in k^3 | a(a + b) \neq 0\}\). It is easy to check that \(\omega'(\omega(a, b, c)) = (a, b, c)\) for \(a(a + b) \neq 0\) and that \(\omega'(a', b', c') = (a', b', c')\) for \(a'(a' + q^{-1}b') \neq 0\). \(\square\)

4. The isomorphism class of \(U(a, b, c) \simeq \Omega M(a, b, c)\).

(4.1) Proposition. Let \((a, b, c) \neq 0\). Then

\[
\begin{align*}
\Omega M(a, b, c) & \simeq \begin{cases} 
M(\omega(a, b, c)) & \text{if } a \neq 0, a + b \neq 0, \quad (1) \\
M(0, 0, 1) & \text{if } a \neq 0, a + b = 0, c \neq 0, \quad (2) \\
\Lambda(x - y) \oplus \Lambda z x & \text{if } a \neq 0, a + b = 0, c = 0, \quad (3) \\
M(0, 1, 0) & \text{if } a = 0, b \neq 0, \quad (4) \\
\Lambda z \oplus \Lambda y x & \text{if } a = 0, b = 0. \quad (5)
\end{cases}
\end{align*}
\]

Proof. If \(a = 0\) and \(b = 0\), then \(U(a, b, c) = U(0, 0, 1)\). If \(a + b = 0\) and \(c = 0\), then \(U(a, b, c) = U(1, -1, 0)\). According to (2.3), \(U(0, 0, 1) = \Lambda z \oplus \Lambda y x\) and \(U(1, -1, 0) = \Lambda(x - y) \oplus \Lambda z x\). This shows (5) and (3). In this way, we have considered all triples \((a, b, c)\) with \(a + b = 0\) and \(ac = 0\).

Thus, let \(a + b \neq 0\) or \(ac \neq 0\). By (2.5), \(U(a, b, c) = \Lambda(a, b, c)\) is local and we look at the surjective map \(\phi : \Lambda \Lambda \to U(a, b, c)\) which sends 1 to \((a, b, c)\).

Let \(a + b \neq 0\). According to formula (1) of (3.1), \(\Lambda(a, b, c)\) is annihilated by \(\omega(a, b, c)\), thus \(M(\omega(a, b, c)) = \Lambda \Lambda / \Lambda(\omega(a, b, c))\) maps onto \(\Lambda(a, b, c)\). Since the modules \(M(\omega(a, b, c))\) and \(\Lambda(a, b, c)\) both have dimension 3, we see that \(U(a, b, c) = \Lambda(a, b, c)\) is isomorphic to \(M(\omega(a, b, c))\). This yields (1) and (4) (namely, if \(a = 0\), and \(b \neq 0\), we have \(\omega(0, 0, c) = (0, qb, 0)\)).

Finally, we show (2). For \(c \neq 0\), the module \(U(1, -1, c)\) is isomorphic to \(M(0, 0, 1)\). Now we use in the same way formula (2) of (3.1). \(\square\)
The subspace also as a right $\Lambda$-module. 

\((w, \text{and (iii) asserts that soc } \Lambda. \text{ Since } \Lambda \subseteq \mathfrak{M}(\omega(a, b, c))\) for \(a(a+b) \neq 0\). 

Proof. This follows directly from Propositions (3.2) and (4.1). \(\square\)

5. The extensionless modules \(M(a, b, c)\).

(5.1) Proposition. The module \(M(a, b, c)\) is extensionless if and only if \(a(a+b) \neq 0\). 

For the proof, we need some preparations.

(5.2) Lemma. The following conditions are equivalent:

(i) The module \(M(a, b, c)\) is extensionless.

(ii) The inclusion map \(ι: U(a, b, c) \rightarrow _Λ \Lambda\) is a left \(\text{add}(Λ)\)-approximation.

(iii) \(U(a, b, c) = Λ(a, b, c)\) and the inclusion map \(ι: Λ(a, b, c) \rightarrow _Λ \Lambda\) is a left \(\text{add}(Λ)\)-approximation.

(iv) The subspace \(U(a, b, c)\) is indecomposable both as a left module and as a right module, and the image of every homomorphism \(\lambda U(a, b, c) \rightarrow _Λ \Lambda\) is contained in \(U(a, b, c)\).

Proof. The equivalence of (i) and (ii) follows from Part 1, Lemma 2.1.

(ii) \(\Rightarrow\) (iii): We assume (ii). If \(U(a, b, c) = U_1 \oplus U_2\) with \(U_1, U_2\) both non-zero, then a minimal left \(\text{add}(Λ)\)-approximation \(U(a, b, c) \rightarrow Λ^t\) is the direct sum of minimal left \(\text{add}(Λ)\)-approximations \(U_1 \rightarrow Λ^{t_1}\) and \(U_2 \rightarrow Λ^{t_2}\), thus \(t = t_1 + t_2 \geq 2\). This shows that \(U(a, b, c)\) is indecomposable. According to (2.5), this means that \(U(a, b, c) = Λ(a, b, c)\).

(iii) \(\Rightarrow\) (iv). Since \(Λ(a, b, c)\) is a local module, it is indecomposable. Thus \(U(a, b, c) = Λ(a, b, c)\) implies that \(U(a, b, c)\) considered as a left module is indecomposable. Given any homomorphism \(ϕ: U(a, b, c) \rightarrow _Λ Λ\), (iii) provides \(λ \in Λ\) with \(ϕ(a, b, c) = (a, b, c)λ \in (a, b, c)Λ \subseteq U(a, b, c)\). Now assume that \((a, b, c)Λ\) is a proper subset of \(U(a, b, c)\). Let \(w \in \text{soc } Λ\). Since \(Λw\) is simple, there is a homomorphism \(ϕ: Λ(a, b, c) \rightarrow Λ\) with \(ϕ(a, b, c) = w\) and (iii) asserts that \(w = ϕ(a, b, c) = (a, b, c)λ\) for some \(λ \in Λ\). This shows that \(\text{soc } Λ \subseteq (a, b, c)Λ\) and therefore \(U(a, b, c) = (a, b, c)Λ\). In particular, \(U(a, b, c)\) is indecomposable also as a right \(Λ\)-module.

(iv) \(\Rightarrow\) (ii). Let \(ϕ: U(a, b, c) \rightarrow _Λ Λ\) be a homomorphism. Since \(U(a, b, c)\) is indecomposable as a left module, we have \(U(a, b, c) = Λ(a, b, c)\). Since \(U(a, b, c)\) is indecomposable as a right module, we have \(U(a, b, c) = (a, b, c)Λ\). According to (iv), \(ϕ(a, b, c) \in U(a, b, c) = (a, b, c)Λ\), thus \(ϕ(a, b, c) = (a, b, c)λ = r_λ(a, b, c)\) for some \(λ \in Λ\), where \(r_λ: Λ \rightarrow _Λ Λ\) is the right multiplication by \(λ\). Since the left module \(U(a, b, c) = Λ(a, b, c)\) is generated by \((a, b, c)\), the equality \(ϕ(a, b, c) = r_λ(a, b, c)\) implies that \(ϕ = r_λ\). \(\square\)
(5.3) Lemma. Let $R$ be a ring and $X$ a left $R$-module. If $\phi: R \rightarrow X$ is an $R$-module homomorphism and $w \in R$ annihilates $X$, then $Rw \subseteq \text{Ker}\phi$.

Corollary. Let $L$ be a left ideal of $R$ and $X$ an $R$-module annihilated by $w_1, \ldots, w_t \in R$. The image of any map $R/L \rightarrow X$ is a factor module of $R/(L + Rw_1 + \cdots + Rw_t)$.

Proof. Let $\phi: R/L \rightarrow X$ be a homomorphism. Let $\pi: R \rightarrow R/L$ be the canonical projection. By construction, $L$ is contained in $\text{Ker}(\phi\pi)$. By the lemma, also the left ideals $Rw_i$ are contained in $\text{Ker}(\phi\pi)$. Thus $L + Rw_1 + \cdots + Rw_t \subseteq \text{Ker}(\phi\pi)$.

(5.4) Proof of Proposition (5.1). According to (5.2), $M(a, b, c)$ is extensionless if and only if condition (iv) is satisfied. We look at all the elements $(a: b: c) \in \mathbb{P}^2$, using the partition of $\mathbb{P}^2$ into the subsets (1) to (5) as in (4.1).

The cases (3) and (5): Both $U(1, -1, 0)$ and $U(0, 0, 1)$ are decomposable as left modules, see (2.5). Case (4): According to (4.1), $U(0, 1, c) \simeq M(0, 1, 0)$. Obviously, $M(0, 1, 0)$ has $\Lambda z$ as a factor module, thus there is a homomorphism $U(0, 1, c) \rightarrow \Lambda$ with image $\Lambda z$ and $\Lambda z \not\subseteq U(0, 1, c)$. The case (2) is similar: (4.1) shows that $U(1, -1, c) \simeq M(0, 0, 1)$, and $M(0, 0, 1)$ maps onto $\Lambda z$; thus there is a homomorphism $U(1, -1, c) \rightarrow \Lambda$ with image $\Lambda z$ and $\Lambda z \not\subseteq U(1, -1, c)$. This shows that none of the modules $M(a, b, c)$ with $a(a + b) = 0$ is extensionless.

It remains to consider the case (1). Thus, assume that $a(a + b) \neq 0$. Let $(1, b', c') = \omega(1, b, c)$, thus $b' = qb$. We want to show that the conditions (iv) of (5.2) are satisfied. According to (2.5) and (2.8), $U(a, b, c)$ is indecomposable both as a left module and as a right module. It remains to show that the image of every homomorphism $\Lambda U(a, b, c) \rightarrow \Lambda$ is contained in $U(a, b, c)$.

(a) The only left ideal isomorphic to $U(1, b, c)$ is $U(1, b, c)$ itself. Proof. The 3-dimensional left ideals are of the form $U(a'', b'', c'')$, for some $(a'', b'', c'') \neq 0$, see (2.5). Assume that $U(1, b, c) \simeq U(a'', b'', c'')$. We have $U(a'', b'', c'') \simeq \Omega M(a'', b'', c'')$ and by (4.1) we must be in case (1), namely $a'' \neq 0$ and $a'' + b'' \neq 0$. In particular, we may assume that $a'' = 1$ and (4.1)(1) asserts that $\Omega M(1, b'', c'') = M(\omega(1, b'', c''))$. The isomorphism $M(\omega(1, b, c)) \simeq M(\omega(1, b'', c''))$ implies that the triples $\omega(1, b, c)$ and $\omega(1, b'', c'')$ yield the same element in $\mathbb{P}^2$, and since the first coordinate of both triples is equal to 1, we have $\omega(1, b, c) = \omega(1, b'', c'')$. Since $1 + b \neq 0$ and $1 + b'' \neq 0$, we use (3.2) in order to conclude that $(1, b, c) = (1, b'', c'')$.

(b) The left ideal $\Lambda z$ is not a factor module of $U(1, b, c)$. The proof uses Corollary (5.3) for the left ideal $L = U(1, b', c')$ and the module $X = \Lambda z$ which is annihilated by $y$ and $z$. Namely, on the one hand, we have $U(1, b, c) \simeq \Omega M(1, b, c) \simeq M(\omega(1, b, c)) = M(1, b', c') = \Lambda/U(1, b', c') = \Lambda/L$. On the other hand, rad $\Lambda = \Lambda(x + b'y + c'z) + \Lambda y + \Lambda z \subseteq U(1, b', c') + \Lambda y + \Lambda z \subseteq \text{rad}\Lambda$ shows that $L + \Lambda y + \Lambda z = \text{rad}\Lambda$. Therefore, (5.3) asserts that the image of any homomorphism $U(1, b, c) \rightarrow \Lambda z$ is a factor module of $\Lambda/L$, thus simple or zero.

(c) The left ideal $\Lambda(x - y)$ is not a factor module of $U(1, b, c)$. Again, we use Corollary (5.3) for $L = U(1, b', c')$ and now for $X = \Lambda(x - y)$. Note that $\Lambda(x - y)$ is annihilated by $x - qy$ and $z$. We recall from (b) that $U(1, b, c) \simeq \Lambda/L$. And we have rad $\Lambda = \Lambda(x + b'y + c'z) + \Lambda(x - qy) + \Lambda z = \text{rad}\Lambda$, and (5.3) asserts that the image of any homomorphism $U(1, b, c) \rightarrow \Lambda z$ is simple or zero.
Any homomorphism $\phi: U(1, b, c) \to \Lambda \Lambda$ maps into $U(1, b, c)$. Proof. According to (b) and (c), the image $I$ of $\phi$ is not of dimension 2. If the image $I$ is of dimension 3, then (a) shows that $I$ is equal to $U(1, b, c)$. Of course, if $I$ is of dimension at most 1, then $I \subseteq \text{soc} \Lambda \subseteq U(1, b, c)$. \hfill \Box

(5.5) Corollary. If $M(a, b, c)$ is extensionless, then $\Omega M(a, b, c) \simeq M(\omega(a, b, c))$.

Proof. This follows directly from (5.1) and the case (1) of (4.1). \hfill \Box.

6. The torsionless modules $M(a, b, c)$.

(6.1) Proposition. The module $M(a, b, c)$ is torsionless if and only if either $a(a + q^{-1}b) \neq 0$ or else $a = 0$ and $bc = 0$ (so that $(a:b:c)$ is equal to $(0:1:0)$ or to $(0:0:1)$).

In order to prove (6.1), we consider the possible cases separately. First, we consider the modules $M(a, b, c)$ with $a \neq 0$. In section 5 we have seen that $M(1, b, c)$ is extensionless if and only if $b \neq -1$, and then $\Omega M(1, b, c) \simeq M(\omega(1, b, c))$. There is the following corresponding assertion concerning the torsionless modules (see also (7.1)).

(6.2) The module $M(1, b, c)$ is torsionless if and only if $b \neq -q$, and in this case $\Omega M(1, b, c) \simeq M(\omega(1, b, c))$.

Proof. Let $b \neq -q$. Then $\omega(1, b, c) = (1, q^{-1}b, c')$ for some $c'$. According to (5.1) and (5.5), $M(1, q^{-1}b, c')$ is extensionless and $\Omega M(1, q^{-1}b, c') \simeq M(1, b, c)$, since $\omega(1, q^{-1}b, c') = \omega \omega'(1, b, c) = (1, b, c)$. This shows that $M(1, b, c)$ is torsionless and that $\Omega M(1, b, c) \simeq M(\omega'(1, b, c))$.

Conversely, we consider $M(1, -q, c)$ and assume, for the contrary, that $M(1, -q, c)$ is torsionless. According to (2.7), this means that $M(1, -q, c)$ is isomorphic to a left ideal $U(a', b', c') = \Omega M(a', b', c')$. According to (4.1), we must be in the case $a' + b' \neq 0$ and $a' \neq 0$. We can assume that $a' = 1$, thus $1 + b' \neq 0$. We have $\Omega M(1, b', c') \simeq M(\omega(1, b', c')) = M(1, qb', c'')$ for some $c''$. Since $M(1, -q, c) \simeq \Omega M(1, b', c') \simeq M(1, qb', c'')$, we see that $(1, -q, c) = (1, qb', c'')$, thus $b' = -1$. But this is a contradiction to $1 + b' \neq 0$. \hfill \Box

(6.3) For $M = M(0, 1, 0)$ and $M(0, 0, 1)$, there is no monomorphism $M \to \Lambda \Lambda$ which is an add($\Lambda$)-approximation.

Proof. Let $M$ be equal to $M(0, 1, 0)$ or to $M(0, 0, 1)$. Assume that there is a monomorphism $u: M \to \Lambda \Lambda$ which is an add($\Lambda$)-approximation. The image $u(M)$ is a 3-dimensional left ideal, thus of the form $U(a, b, c)$ for some $(a, b, c) \neq 0$, see (2.7). The implication (ii) $\implies$ (iv) in (5.2) asserts that any homomorphism $U(a, b, c) \to \Lambda \Lambda$ maps into $U(a, b, c)$.

Obviously, both modules $M(0, 1, 0)$ and $M(0, 0, 1)$ have a factor module isomorphic to $\Lambda z$, thus there is a surjective homomorphism $U(a, b, c) \to \Lambda z$, and therefore $\Lambda z \subseteq U(a, b, c)$. But $\Lambda z$ is an indecomposable module of length 2, and $U(a, b, c) \simeq M$ is a local module of length 3 with socle of length 2. A local module of length 3 with socle of length 2 has no indecomposable submodule of length 2, thus we obtain a contradiction. \hfill \Box

(6.4) Proposition. The modules $M(0, b, c)$ with $bc \neq 0$ are not torsionless.

Proof. Let $M = M(0, b, c)$ with $bc \neq 0$ and assume that $M$ is torsionless. According to (2.7), this means that $M \simeq U(a', b', c') \simeq \Omega M(a', b', c')$ for some triple $(a', b', c')$, and (2.5) asserts that $a' + b' \neq 0$ or $a'c' \neq 0$. Now we use (4.1) and have to distinguish
the three cases (1), (2) and (4). Case (1) means that \( a' + b' \neq 0 \) and \( a' \neq 0 \), then \( \Omega M(a', b', c') \simeq M(\omega(a', b', c')) \) and the first component of \( \omega(a', b', c') \) is \( a' \), thus non-zero. But then \( M(\omega(a', b', c')) \) cannot be isomorphic to \( M(0, b, c) \). Case (4) means that \( a' = 0 \) and \( b' \neq 0 \). Then \( \Omega M(a', b', c') \simeq M(0, 1, 0) \), thus not isomorphic to \( M(0, b, c) \) with \( bc \neq 0 \). Finally, there is the case (2) with \( a' + b' = 0 \) and \( a'c' \neq 0 \). Then \( \Omega M(a', b', c') \simeq M(0, 0, 1) \), again not isomorphic to \( M(0, b, c) \) with \( bc \neq 0 \). In all cases, we get a contradiction. \( \square \)

(6.5) Proposition. If \( M \) is equal to \( M(0, 1, 0) \) or \( M(0, 0, 1) \), then \( M \) is torsionless and the module \( \Omega M \) has Loewy length 3. Since \( \Omega M \) is indecomposable and non-projective, it is not torsionless.

Proof. The modules \( M \) of the form \( M(0, 1, 0) \) and \( M(0, 0, 1) \) are torsionless, since (4.1), (4) and (2) assert that \( M(0, 1, 0) \simeq \Omega M(0, 1, 0) \) and that \( M(0, 0, 1) \simeq \Omega M(1, -1, 1) \). According to (5.2), in both cases there is no inclusion map \( M \rightarrow \Lambda \) which is an \( \text{add}(\Lambda) \)-approximation. Thus, a minimal left \( \text{add}(\Lambda) \)-approximation of \( M \) is an injective map \( M \rightarrow \Lambda^t \) with \( t \geq 2 \). This shows that \( \Omega M \) has dimension \( 6t - 3 \) and its top has dimension \( t \). According to Part I (3.2), \( \Omega M \) is indecomposable and not projective. The Loewy length of \( \Omega M \) has to be 3. [Namely, an indecomposable module with Loewy length at most 2 and top of dimension \( t \geq 2 \) has dimension at most \( 4t - 1 \), since it is a proper factor module of \( \Lambda^t \). But \( 6t - 3 \leq 4t - 1 \) implies \( t \leq 1 \), a contradiction.] An indecomposable non-projective module of Loewy length 3 cannot be torsionless. \( \square \)

(6.6) We finish this section by reformulating the results concerning the modules of the form \( M(0, b, c) \) in terms of \( \Omega \)-components. Here, we will exhibit the structure of all the \( \Omega \)-components containing modules of the form \( M(0, b, c) \). We have to distinguish between the modules \( M(0, 1, 0) \) and \( M(0, 0, 1) \) and the modules \( M(0, b, c) \) with \( bc \neq 0 \), thus lying on the dashed line \( A' = \{(0: b: c) \mid bc \neq 0\} \):

The modules in \( A' \) are singletons (that is, components of type \( \text{I}_1 \)) in the \( \Omega \)-quiver. And, there are the following two \( \Omega \)-components of the form \( \text{I}_2 \):

\[
\begin{array}{ccc}
M(0, 0, 1) & \dashv & \Omega M(0, 0, 1) \\
M(0, 1, 0) & \dashv & \Omega M(0, 1, 0)
\end{array}
\]

(If \( M \) is an indecomposable module, then we represent \([M]\) in the \( \Omega \)-quiver usually just by a circle \( \bullet \). We use a bullet \( \bullet \) in case we know that \( M \) is torsionless and extensionless, a black square \( \blacksquare \) in case we know that \( M \) is extensionless, but not torsionless; and a black lozenge \( \blacklozenge \) in case we know that \( M \) is torsionless, but not extensionless.)

7. The modules \( M(1, b, c) \) and proof of Theorem (1.5).

We consider now the affine subspace \( H \) of \( \mathbb{P}^2 \) given by the points \((1: b: c)\) with \( b, c \in k \) and the corresponding modules \( M(1, b, c) \). We recall that \( o(q) \) denotes the multiplicative order of \( q \).
(7.1) We have seen in (4.2) that \( \Omega \) provides a bijection from the set of modules \( M(1, b, c) \) with \( b \neq -1 \) onto the set of modules \( M(1, b', c') \) with \( b' \neq -q \). The sections 5 and 6 strengthen this bijection as follows:

*If \( b \neq -1 \), then the exact sequence*

\[
0 \to M(1, b', c') \to \Lambda \to M(1, b, c) \to 0
\]

*with \( (1, b', c') = \omega(1, b, c) \) is an \( \mathcal{U} \)-sequences (here, \( (1, b', c') \) is an arbitrary triple with \( b' \neq -q \), and \( (1, b, c) = \omega'(1, b', c') \)). We obtain in this way all the \( \mathcal{U} \)-sequences involving modules of the form \( M(1, b, c) \).

(7.2) **Reformulation.** The neighborhood of \( M(1, b, c) \) in the \( \mathcal{U} \)-quiver looks like:

\[
\begin{array}{ccc}
\ldots & \infty & \ldots \\
M(\omega(1, b, c)) & M(1, b, c) & M(\omega'(1, b, c)) \\
\ldots & \infty & \ldots \\
M(\omega(1, -q, c)) & M(-q, c) & \\
& \infty & \ldots \\
M(1, -1, c) & M(\omega'(1, -1, c)) & \\
\end{array}
\]

\( b \notin \{-1, -q\} \)

\( b = -q \neq -1 \)

\( b = -1 \neq -q \)

and \( M(1, b, c) \) is a singleton in the \( \mathcal{U} \)-quiver if \( q = 1 \) and \( b = -1 \).

(7.3) *The module \( M(1, b, c) \) is semi-Gorenstein-projective if and only if \( b \neq -q^t \) for all \( t \leq 0 \). The module \( M(1, b, c) \) is \( \infty \)-torsionfree if and only if \( b \neq -q^t \) for all \( t \geq 1 \).

Proof: \( M(1, b, c) \) is semi-Gorenstein-projective if and only if \( \omega^s(1, b, c) \notin E \) for all \( s \geq 0 \). Since \( \omega^s(1, b, c) = (1, q^s b, c_s) \) for some \( c_s \in k \), we see that \( M(1, b, c) \) is semi-Gorenstein-projective if and only if \( 1 + q^s \neq 0 \) for all \( s \geq 0 \), thus if and only if \( q^{-s} \neq -b \) for all \( s \geq 0 \). Write \( t = s + 1 \).

Similarly, \( M(1, b, c) \) is \( \infty \)-torsionfree if and only if \( \omega^{-s}(1, b, c) \notin T \) for all \( s \geq 0 \), thus if and only if \( 1 + q^{-1} q^{-s} b \neq 0 \) for all \( s \geq 0 \), if and only if \( -b \neq q^{s+1} \) for all \( s \geq 0 \). Write \( t = s + 1 \).

\[ \square \]

**Corollary.** The module \( M(1, b, c) \) is Gorenstein-projective if and only if \( b \neq -q^t \) for all \( t \in \mathbb{Z} \).

(7.4) *Any module \( M(1, 0, c) \) with \( c \in k \) is Gorenstein-projective with \( \Omega \)-period 1 or 2.*

Proof. According to (6.2), the modules \( M(1, 0, c) \) are extensionless and torsionless. Since \( \omega(1, 0, c) = (1, 0, -c) \), we see that \( M(1, 0, 0) \) has \( \Omega \)-period 1, and \( M(1, 0, c) \) with \( c \neq 0 \) has \( \Omega \)-period 2 in case the characteristic of \( k \) is different from 2, otherwise its \( \Omega \)-period is also 1.

\[ \square \]

(7.5) **Proposition.** If \( o(q) = \infty \), then any module of the form \( M(1, b, c) \) is semi-Gorenstein-projective or \( \infty \)-torsionfree (whereas the modules of the form \( M(0, b, c) \) are never semi-Gorenstein-projective nor \( \infty \)-torsionfree).

Proof. The first assertion follows immediately from (7.3), the additional assertion in the bracket is a consequence of (5.1), (6.4) and (6.5).

\[ \square \]
(7.6) Proposition. If \( M(1, b, c) \) belongs to an \( \tilde{U} \)-component of the form \( \mathbb{A}_n \), then \( o(q) = n \).

Proof. We consider an \( \tilde{U} \)-component of type \( \mathbb{A}_n \), say containing a module \( M \) which is not torsionless. Since \( M \) belongs to \( T \), we have \( M = M(1, -q, c) \) and the component consists of the modules \( M, \Omega M, \ldots, \Omega^{n-1} M \). In particular, \( \omega^{n-1}(1, -q, c) \) belongs to \( E \). Now \( \Omega^{n-1} M = M(\omega^{n-1}(1, -q, c)) = M(1, -q^n, c') \) for some \( c' \). Since \( \Omega^{n-1} M \) is not extensionless, \( (1, -q^n, c') \) belongs to \( E \), thus \( -q^n = -1 \). This shows that \( q^n = 1 \). Finally, for \( 1 \leq t < n \), we have \( q^t \neq 1 \), since otherwise \( \omega^{t-1}(1, -q, c) \) would belong to \( E \). \( \square \)

Corollary. If \( o(q) = \infty \), then all the \( \tilde{U} \)-components in \( H \) are cycles or of type \( \mathbb{Z} \), or \(-N, or \( N \). Thus, any module in \( H \) is semi-Gorenstein-projective or \( \infty \)-torsionfree.

For \( o(q) = \infty \), there are the following \( \tilde{U} \)-components of the form \(-N \) and \( N \):

\[
\begin{array}{cccc}
& \bullet & \leftarrow & \bullet \leftarrow \\
M(1, -q^3, c_3) & M(1, -q^2, c_2) & M(1, -q, c_1) \\
& \downarrow & \leftarrow & \downarrow \\
M(1, -1, d_0) & M(1, -q^{-1}, d_1) & M(1, -q^{-2}, d_2) & \ldots
\end{array}
\]

with arbitrary elements \( c_0, d_1 \in k \) and \( c_{t+1} = -\frac{1}{q^t} c_t \) for \( t \geq 1 \), whereas \( d_{t+1} = -(1 - q^{-1})d_t \) for \( t \geq 0 \). Of course, \((1, -q, c_1) \in T \) and \((1, -1, d_0) \in E \), thus the module \( M(1, -q, c_1) \) is pivotal semi-Gorenstein-projective, whereas \( M(1, -1, d_0) \) is pivotal \( \infty \)-torsionfree.

(7.7) The case that \( q \) has finite multiplicative order. Now let \( o(q) = n < \infty \). Then the modules \( M(1, -q^t, c) \) with \( 0 \leq t < n \) and \( c \in k \) belong to \( \tilde{U} \)-components of the form \( \mathbb{A}_n \). These \( \tilde{U} \)-components look as follows:

\[
\begin{array}{cccc}
\bullet & \leftarrow & \bullet & \leftarrow \\
M(1, -1, c_n) & M(1, -q^{n-1}, c_{n-1}) & \ldots & M(1, -q^2, c_2) & M(1, -q, c_1)
\end{array}
\]

with an arbitrary element \( c_1 \in k \) and \( c_{t+1} = -\frac{1}{q^t} c_t \) for \( 1 \leq t < n \) (of course, \((1, -1, c_n) \in E \) and \((1, -q, c_1) \in T \)).

Corollary (7.3) asserts that the remaining modules \( M(1, b, c) \) (those with \( -b \notin q \mathbb{Z} \)) are Gorenstein-projective.

(7.9) Proof of Theorem (1.5).

Torsionless modules: According to (2.7), an indecomposable torsionless module is isomorphic to a left ideal. Of course, \( k \) is torsionless. According to (2.2), a 2-dimensional indecomposable left ideal is isomorphic to \( \Lambda(x - y) \) or \( \Lambda z \). According to (2.3), a 3-dimensional indecomposable torsionless module has to be local, thus it is of the form \( M(a, b, c) \), and (6.1) says that \( a(a + q^{-1}b) \neq 0 \) or else \( M(a, b, c) \) is equal to \( M(0, 1, 0) \) or to \( M(0, 0, 1) \).

Extensionless modules: We show: An indecomposable module \( M \) of dimension at most 3 with simple socle is not extensionless.

Of course, \( \text{Ext}^1(k, \Lambda) \neq 0 \), since otherwise we would have \( \text{Ext}^1(X, \Lambda) = 0 \) for all modules \( X \).
Let \( I \) be an indecomposable module of length 2. A projective cover of \( I \) as an \( \overline{\Lambda} \)-module provides an exact sequence \( 0 \to k^2 \to \overline{\Lambda} \to I \to 0 \). We apply \( \text{Hom}_{\overline{\Lambda}}(-, J) \), where \( J = \text{rad} \Lambda \). We obtain the exact sequence

\[
0 \to \text{Hom}_{\overline{\Lambda}}(I, J) \to \text{Hom}_{\overline{\Lambda}}(\overline{\Lambda}, J) \to \text{Hom}_{\overline{\Lambda}}(k^2, J) \to \text{Ext}^1_{\overline{\Lambda}}(I, J) \to 0.
\]

Now, \( \dim \text{Hom}_{\overline{\Lambda}}(I, J) \geq \dim \text{Hom}_{\overline{\Lambda}}(k, J) = 2 \), \( \dim \text{Hom}_{\overline{\Lambda}}(\overline{\Lambda}, J) = \dim J = 5 \), and finally \( \dim \text{Hom}_{\overline{\Lambda}}(k^2, J) = 4 \), thus \( \dim \text{Ext}^1_{\overline{\Lambda}}(I, J) \geq 1 \). This shows that there exists a non-split exact sequence \( \epsilon : 0 \to J \to E \to I \to 0 \) with some \( \overline{\Lambda} \)-module \( E \). The inclusion map \( \iota : J \to \Lambda \) yields an induced exact sequence \( \epsilon' : 0 \to \Lambda \to E' \to I \to 0 \). Assume that \( \epsilon' \) splits. Then we obtain a map \( v : E \to \Lambda \) such that \( vu = \iota \). Now \( E \) is an \( \overline{\Lambda} \)-module, thus of Loewy length at most 2. Therefore \( v : E \to \Lambda \) maps into \( \text{rad} \Lambda = J \), thus \( v = vv' \) for some \( v' : E \to J \). But \( vv'u = vu = \iota \) implies that \( v'u \) is the identity map of \( E \), thus \( \epsilon \) splits, a contradiction. The exact sequence \( \epsilon' \) shows that \( \text{Ext}^1_{\overline{\Lambda}}(I, \Lambda) \neq 0 \). Thus \( I \) is not extensionless.

A similar proof shows that \( \text{Ext}^1(V, \Lambda) \neq 0 \) for any 3-dimensional module \( V \) with simple socle. Again, we use that \( V \) is an \( \overline{\Lambda} \)-module (see (1.3) Proposition 1), thus we start with an exact sequence \( 0 \to k^3 \to \overline{\Lambda}^2 \to V \to 0 \).

This completes the proof that an indecomposable module \( M \) of dimension at most 3 with simple socle is not extensionless. The remaining indecomposable modules of dimension at most 3 are the modules of the form \( M(1, b, c) \). According to (5.1) \( M(1, b, c) \) is extensionless if and only if \( b \neq -1 \).

**Reflexive modules:** We recall from Part I that a module \( M \) is reflexive if and only if both \( M \) and \( \overline{\text{U}}M \) are torsionless. We show: A module \( M \) with simple socle is not reflexive. Assume that \( M \) has simple socle and is torsionless. Since \( M \) has simple socle, there is an embedding \( M \to \Lambda \Lambda \), say with cokernel \( Q \). The elements \( yx \) and \( zx \) cannot both belong to \( u(M) \), since the socle of \( u(M) \) is simple. If \( yx \notin u(M) \), then \( yxQ \neq 0 \), otherwise \( zxQ \neq 0 \). Let \( f : M \to \Lambda \Lambda \) be a minimal left add(\( \Lambda \))-approximation; its cokernel is \( \overline{\text{U}}M \). There is \( u' : \Lambda \Lambda \to \Lambda \) with \( u'f = u \). The map \( u' \) has to be surjective, since otherwise \( u' \) would vanish on the socle of \( \Lambda \Lambda \). This implies that the map \( \overline{\text{U}}M \to Q \) induced by \( u' \) is also surjective. Since \( \overline{\text{U}}M \) is indecomposable, non-projective and not annihilated by \( \text{rad}^2 \Lambda \), \( \overline{\text{U}}M \) cannot be torsionless.

Let us assume that \( M \) is reflexive and \( \dim M \leq 3 \). It follows that \( M \) has to be a torsionless module with \( \dim M = 3 \). Since also \( \overline{\text{U}}M \) has to be torsionless, (6.5) shows that the cases \( M(0, 1, 0) \) and \( M(0, 0, 1) \) are not possible, thus \( M \) is of the form \( M(1, b, c) \) with \( b \neq -q \). Using (6.2) and (6.1), we see that we also must have \( b \neq -q^2 \). Conversely, the same references show that all the modules \( M(1, b, c) \) with \( b \neq -q^i \) for \( i = 1, 2 \) are reflexive.

**Semi-Gorenstein-projective and \( \infty \)-torsionfree modules.** The semi-Gorenstein-projective modules are extensionless, the \( \infty \)-torsionfree modules are reflexive. The previous considerations therefore show that we only have to consider the modules of the form \( M(1, b, c) \). (7.3) provides the conditions on \( b \) so that \( M(1, b, c) \) is semi-Gorenstein-projective, \( \infty \)-torsionfree, or Gorenstein-projective.

If \( M(1, b, c) \) is pivotal semi-Gorenstein-projective, then \( M(1, b, c) \) is not torsionless, thus \( b = -q \). If \( M(1, -q, c) \) is semi-Gorenstein-projective, then \( -q \neq -q^{-s} \) for all \( s \geq 0 \),
Since this is a map between 3-dimensional modules, it has to be an isomorphism. $t \geq t_f$ torsionless, $\phi$.

Remark. It seems worthwhile to note that the set of modules $M(1, b, c)$ with $b, c \in k$ is a union of $\mathcal{U}$-components.

8. Right modules.

Recall that we write $U'(a, b, c)$ instead of $U(a, b, c)$, if we consider $U(a, b, c)$ as a right ideal. Let $M'(a, b, c) = \Lambda \Lambda / U'(a, b, c)$, this is a right module (of course, the sets $M(a, b, c)$ and $M'(a, b, c)$ are the same, but we use the notation $M'(a, b, c)$ if we want to stress that we deal with a right module).

(8.1) Proposition. Let $(a, b, c) \neq 0$. Then

$$\Omega M'(a, b, c) \simeq \begin{cases} M'(\omega'(a, b, c)) & \text{if } a \neq 0, \\ M'(0, 0, 1) & \text{if } a = 0, bc \neq 0, \\ y \Lambda \oplus zx \Lambda & \text{if } a = 0, c = 0, \\ z \Lambda \oplus yx \Lambda & \text{if } a = 0, b = 0. \end{cases}$$

Proof. We have $\Omega M'(a, b, c) = U'(a, b, c)_\Lambda$. According to (2.8), $U'(a, b, c)_\Lambda = (a, b, c)\Lambda$ if $a \neq 0$ or $bc \neq 0$, and $U'(0, 1, 0) = y \Lambda \oplus zx \Lambda, U'(0, 0, 1) = z \Lambda \oplus yx \Lambda$.

Consider the map $\pi: \Lambda \Lambda \rightarrow U'(a, b, c)$ defined by $\pi(1) = (a, b, c)$. We assume that $a \neq 0$ or $bc \neq 0$, thus $\pi$ is surjective. If $a \neq 0$, the formula (3.1) (3) asserts that $\omega'(a, b, c)$ is in the kernel of $\pi$, thus $\pi$ yields an epimorphism $M'(\omega'(a, b, c)) = \Lambda \Lambda / \omega'(a, b, c) \Lambda \rightarrow U'(a, b, c)$.

Since this is a map between 3-dimensional modules, it has to be an isomorphism.

If $a = 0$ and $bc \neq 0$, we use formula (3.1) (4) in order to get similarly an isomorphism $M'(0, 0, 1) = \Lambda \Lambda / (0, 0, 1) \Lambda \rightarrow U'(0, b, c).$ \hfill \Box

(8.2) If a 3-dimensional indecomposable right module $N$ is torsionless and no embedding $N \rightarrow \Lambda \Lambda$ is a left add$(\Lambda \Lambda)$-approximation, then $\Omega N$ has Loewy length 3 and is not torsionless.

Proof. Let $\phi: N \rightarrow \Lambda \Lambda$ be a minimal left add$(\Lambda \Lambda)$-approximation of $N$. Since $N$ is torsionless, $\phi$ is a monomorphism. By assumption, we must have $t \geq 2$. It follows that the cokernel $\overline{\Omega N}$ of $\phi$ is an indecomposable right $\Lambda$-module of length $6t - 3$ with top of length $t$. But an indecomposable right $\Lambda$-module of Loewy length at most 2 with top of length $t \geq 2$ is a right $\Lambda \Lambda$-module of length at most $4t - 1$. Thus $6t - 3 \leq 4t - 1$, therefore $2t \leq 2$, thus $t \leq 1$, a contradiction. This shows that $\Omega N$ has Loewy length equal to 3. Of course, $\Omega N$ is not projective. Since an indecomposable non-projective torsionless right $\Lambda$-module has Loewy length at most 2, we see that $\Omega N$ cannot be torsionless. \hfill \Box

(8.3) The right modules $M'(0, b, c)$. The only right module of the form $M'(0, b, c)$ which is torsionless is $M'(0, 0, 1)$. The right module $\Omega M'(0, 0, 1)$ has Loewy length 3 and thus it is not torsionless. No right module of the form $M'(0, b, c)$ is extensionless.

Proof. Let $N = M'(0, b, c)$. 

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(a) If $N$ is torsionless, then $b = 0$ (thus $0:b:c = (0:0:1)$). Namely, According to (2.9), $M'(0,b,c)$ arises as a right ideal and (8.1) shows that this happens only for $b = 0$.

(b) No embedding $M'(0,0,1) \to \Lambda$ is a left add$(\Lambda_A)$-approximation. Proof. Let $\phi : M'(0,0,1) \to \Lambda$ be an embedding. According to (2.9), the image of $\phi$ is of the form $U'(0,b,c)$ with $bc \neq 0$. Now $M'(0,0,1)$ has a factor module isomorphic to $(0,0,1)\Lambda$, thus there is $f : M'(0,0,1) \to \Lambda$ with image $(0,0,1)\Lambda$. If $\phi$ is a left add$(\Lambda_A)$-approximation, then there exists $f' : \Lambda \to \Lambda$ with $f = f'\phi$. The homomorphism $f'$ is the left multiplication by some element $\lambda$ in $\Lambda$. If $\lambda$ belongs to $\text{rad} \, \Lambda$, then the image of $f'\phi$ is contained in $\text{rad}^2 \Lambda = \text{soc} \, \Lambda$. If $\lambda$ is invertible, then the image of $f'\phi$ is 3-dimensional. In both cases, we get a contradiction, since the image of $f$ is $(0,0,1)\Lambda$, thus 2-dimensional and not contained in soc$\, \Lambda$.

(c) It follows from (8.2) that $\Omega M'(0,0,1)$ has Loewy length 3 and is not torsionless.

(d) A right module of the form $M'(0,b,c)$ is decomposable, or else $\Omega M'(0,b,c) = M'(0,0,1)$ and according to (b), no embedding $M'(0,0,1) \to \Lambda$ is a left add$(\Lambda_A)$-approximation.

Reformulation. The right modules $M'(0,1,c)$ are singletons in the $\Omega$-quiver. The right module $M'(0,0,1)$ belongs to an $\Omega$-component of the form $k_2$:

\[
\begin{array}{c}
\bigstar \\
\leftarrow\ldots\rightarrow\\
\text{M'}(0,0,1)\quad \Omega \text{M'}(0,0,1)
\end{array}
\]

(8.4) The right modules $M'(1,b,c)$ with $c \neq 0$.

Proposition. Let $c \neq 0$. The right module $M'(1,b,c)$ is torsionless if and only if $b \neq -1$, and then $\Omega M'(1,b,c) = M'(\omega(1,b,c))$. Let $c' \neq 0$. The right module $M'(1,b',c')$ is extensionless if and only if $b' \neq -q$, and then $\Omega M'(1,b',c') = M'(\omega'(1,b',c'))$.

Remark. If $b \neq -1$ and $c \neq 0$, then $\omega(1,b,c) = (1,b',c')$ with $b' \neq -q$ and some $c' \neq 0$. If $b' \neq -q$, then $\omega'(1,b',c') = (1,b,c)$ with $b \neq -1$ and some $c \neq 0$. Thus, the proposition provides $\Omega$-sequences

\[
0 \to M'(1,b,c) \to \Lambda \to M'(1,b',c') \to 0
\]

with $b \neq -1$ and $b' \neq -q$ (and both $c, c'$ being non-zero). Any triple $(1,b,c)$ with $b \neq -1$ and $c \neq 0$ occurs on the left and given $(1,b,c)$, then we have $(1,b',c') = \omega(1,b,c)$ on the right. Any triple $(1,b',c')$ with $b' \neq -q$ and $c' \neq 0$ occurs on the right and given $(1,b',c')$, then we have $(1,b,c) = \omega'(1,b',c')$ on the left.

Proof of Proposition. We follow closely the proof of (5.1) and (6.1). We always assume that $c \neq 0$. As in (5.2) one sees that $M'(1,b,c)$ is extensionless if and only if the image of every homomorphism $U'(1,b,c) \to \Lambda$ is contained in $U'(1,b,c)$.

(a) The module $M'(1,-q,c)$ is not extensionless. Proof. According to (8.1), we have $U'(1,-q,c') \simeq \Omega M'(1,-q,c') \simeq M'(\omega'(1,-q,c')) = M'(1,-1,0)$ for all $c' \in k$. Thus, there is a homomorphism $U'(1,-q,0) \to \Lambda$ with image $U'(1,-q,0)$ and this image $U'(1,-q,0)$ is not contained in $U'(1,-q,c)$.

(b) If $b \neq -q$, then the module $M'(1,b,c)$ is extensionless. For the proof, we need three assertions (b1), (b2) (b3). Note that (8.1) asserts that $U'(1,b,c) \simeq \Omega M'(1,b,c) \simeq M'(\omega'(1,b,c)) = M'(1,q^{-1}b,c')$, where $\omega'(1,b,c) = (1,q^{-1}b,c')$. 

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(b1) The only right ideal isomorphic to \( U'(1, b, c) \) is \( U'(1, b, c) \) itself. Proof. Let \( V \) be a right ideal of \( \Lambda \) which is isomorphic to \( U'(1, b, c) \), say \( V = U'(a'', b'', c'') \) for some triple \((a'', b'', c'')\). By (8.1), we have \( U(a'', b'', c'') \simeq \Omega M'(a'', b'', c'') = M'(a'', x^{-1}b'', d) \), where \( \omega'(a'', b', c') = (a'', q^{-1}b'', d) \) for some \( d \). We must have \( a'' \neq 0 \), since \( M(a'', x^{-1}b'', d) \simeq U'(1, b, c) \simeq M'(1, q^{-1}b', c') \). Thus, we may assume that \( a'' = 1 \) and then \( M'(1, q^{-1}b', c') \) implies that \((1, q^{-1}b', d) = (1, q^{-1}b', c')\). In particular, we have \( b' = b \neq -q \). The equality \( \omega'(1, b', c') = \omega'(1, b', c') = (1, b', c') \) yields \((1, b', c') = (1, b, c)\), see Proposition (3.2). Therefore \( V = U(1, b'', c'') = U(1, b, c) \).

(b2) The right ideal \( y \Lambda \) is not a factor module of \( U'(1, b, c) \). Proof. The right ideal \( z \Lambda \) is annihilated by \( x - y \) and \( z \), thus Corollary (5.3) asserts that the image \( I \) of any homomorphism \( M'(1, b', c') \to z \Lambda \) is a factor module of \( \Lambda / ((1, b, c) \Lambda + (x - y) \Lambda + z \Lambda) \). Now \((x + by + cz) \Lambda + (x - y) \Lambda + z \Lambda = \text{rad} \Lambda \), since \( b \neq -1 \), thus \( I \) is simple or zero.

(b3) The right ideal \( y \Lambda \) is not a factor module of \( U'(1, b, c) \). Proof. The right ideal \( y \Lambda \) is annihilated by \( y \) and \( z \), thus Corollary (5.3) asserts that the image \( I \) of any homomorphism \( M'(1, b', c') \to y \Lambda \) is a factor module of \( \Lambda / ((1, b, c) \Lambda + y \Lambda + z \Lambda) \). Now \((x + by + cz) \Lambda + y \Lambda + z \Lambda = \text{rad} \Lambda \), since \( b \neq -1 \), thus \( I \) is simple or zero.

The assertions (b1), (b2) and (b3) show: if \( \phi \) is any homomorphism \( U'(1, b, c) \to \Lambda \) and its image \( I \) is of dimension at least 2, then \( I \) is contained in \( U'(1, b, c) \). Of course, if \( I \) is 1-dimensional, then \( I \) is contained in \( \text{soc} \Lambda \) and \( \text{soc} \Lambda \subseteq U'(1, b, c) \). Thus, we have obtained a proof of (b). In addition, (8.1) asserts that \( \Omega M'(1, b, c) \cong M'(\omega'(1, b, c)) \).

(c) If \( b \neq -1 \), then \( M'(1, b, c) \) is torsionless and \( \bar{\Omega} M'(1, b, c) = M'(\omega(1, b, c)) \). Proof. Let \( \omega(1, b, c) = (1, b', c') \). Then \( b' = q b \neq -q \), and \( \omega'(1, b', c') = \omega(1, b, c) = (1, b, c) \) by Proposition (3.2). According to (8.1), we have \( \Omega M'(1, b', c') \cong M'(\omega'(1, b', c')) \) and \( M'(1, b, c) \). This shows that \( M'(1, b, c) \) is torsionless. According to (b), the module \( M'(\omega(1, b, c)) \) is extensionless, thus \( \bar{\Omega} M'(1, b, c) = M'(1, b', c') = M'(\omega(1, b, c)) \).

(d) The modules \( M'(1, -1, c) \) are not torsionless. Proof. Assume, for the contrary, that \( M'(1, -1, c) \) is torsionless, thus isomorphic to \( U'(a', b', c') \) for some \((a', b', c')\). According to (8.1), we must have \( a' \neq 0 \), thus we can assume that \( a' = 1 \), and \((1, -1, c) = \omega(1, b', c') = (1, q^{-1}b', -(1 + q^{-1}b')c') \). It follows that \( b' = -q \) and therefore \( c = -(1 + q^{-1}b')c' = 0 \), a contradiction.

This completes the proof of (8.4).

Reformulation. The neighborhood of \( M'(1, b, c) \) with \( c \neq 0 \) in the \( \bar{\Omega} \)-quiver looks as follows:

\[
\begin{array}{c}
\cdots \circ \longrightarrow \bullet \longrightarrow \circ \cdots \\
M'(\omega(1, b, c)) \quad M'(1, b, c) \quad M'(\omega(1, b, c)) \\
\cdots \circ \longrightarrow \bullet \quad M'(1, -q, c) \quad b = -q \neq -1 \\
\bullet \longrightarrow \circ \cdots \\
M'(1, -1, c) \quad M'(\omega(1, -1, c)) \\
\end{array}
\]

and \( M'(1, b, c) \) is a singleton in the \( \bar{\Omega} \)-quiver if \( q = 1 \) and \( b = -1 \).

Note that we want to use a fixed index set \( \mathbb{P}^2 \) both for the (left) modules \( M(a : b : c) \) and the right modules \( M'(a : b : c) \), since we have drawn the dashed arrows in the \( \bar{\Omega} \)-quiver.
of the left \( \Lambda \)-modules from right to left, we now have drawn the dashed arrows in the \( \mathcal{U} \)-quiver of the right \( \Lambda \)-modules from left to right.

As in section 7, we see that the \( \mathcal{U} \)-components of the modules \( M'(1, b, c) \) with \( c \neq 0 \) are cycles, or of type \( \mathbb{Z}, \mathbb{N} \) or \( -\mathbb{N} \) in case \( o(q) = \infty \), and cycles or of type \( \mathbb{Z} \) or \( \mathbb{A}_n \) in case \( o(q) = n < \infty \).

For \( o(q) = \infty \), the right modules \( M'(1, -1, c) \) with \( c \neq 0 \) are pivotal semi-Gorenstein-projective, and the right modules \( M'(1, -q, c) \) with \( c \neq 0 \) are pivotal \( \infty \)-torsionfree.

(8.5) The right modules \( M'(1, b, 0) \).

The right modules \( M'(1, b, 0) \) have been considered already in Part I: these are just the right ideals \( m_\alpha \Lambda \), where \( m_\alpha = x - \alpha y \). Namely, we have

\[
M'(1, b, 0) = (x + qby)\Lambda = m_{-qb}\Lambda
\]

for all \( b \in k \). (Proof: We have \( M'(1, b, 0) = \Lambda_\Lambda/\mathcal{U}'(1, b, 0) = \Lambda_\Lambda/(x + by)\Lambda \cong (x + qby)\Lambda \), where we use that \( (x + qby)(x + by) = 0 \) and that both right ideals \( (x + by)\Lambda \) and \( (x + qby)\Lambda \) are 3-dimensional, see (2.8).)

Let us recall the results presented in Part I using the present notation:

If \( b \notin -q\mathbb{Z} \), then \( M'(1, b, 0) \) is Gorenstein-projective and its \( \mathcal{U} \)-component looks as follows:

\[
\cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots
\]

\[
M'(1, q^2b, 0) \quad M'(1, q^b, 0) \quad M'(1, b, 0) \quad M'(1, -q, b, 0) \quad M'(1, -q^{-1}, 0)
\]

In particular, if \( o(q) = n \), then these \( \mathcal{U} \)-components are cycles with \( n \) vertices, whereas for \( o(q) = \infty \), one obtains \( \mathcal{U} \)-components of type \( \mathbb{Z} \).

For \( o(q) = \infty \), there are three remaining \( \mathcal{U} \)-components:

\[
\cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots
\]

\[
M'(1, -q^2, 0) \quad M'(1, -q, 0) \quad M'(1, -1, 0) \quad M'(1, -q^{-1}, 0) \quad M'(1, -q^{-2}, 0)
\]

These \( \mathcal{U} \)-components are of type \( \mathbb{A}_2 \) and \(-\mathbb{N} \), respectively.

For \( 2 \leq n = o(q) < \infty \), there are two remaining \( \mathcal{U} \)-components, one is of type \( \mathbb{A}_2 \), the other of type \( \mathbb{A}_n \):

\[
\cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \cdots
\]

\[
\mathcal{U}M'(1, -1, 0) \quad \mathcal{U}M'(1, -q^{-1}, 0)
\]

\[
M'(1, -1, 0) \quad M'(1, -q^{n-1}, 0) \quad M'(1, -q^{n-2}, 0) \quad M'(1, -q^{n-3}, 0) \quad M'(1, -q, 0)
\]

In case \( q = 1 \), there is only one additional \( \mathcal{U} \)-component (of type \( \mathbb{A}_2 \)), namely

\[
\mathcal{U}M'(1, -1, 0)
\]

\[
M'(1, -1, 0)
\]
(8.6) Similar to Theorem (1.5), here is the summary which characterizes the right modules of dimension at most 3 with relevant properties.

**Theorem.** An indecomposable right module $N$ of dimension at most 3 is

- torsionless if and only if $N$ is simple or isomorphic to $y\Lambda$, to $z\Lambda$, to a module $M'(1, b, c)$ with $b \neq -1$, to $M'(1, -1, 0)$ or to $M'(0, 0, 1)$.
- extensionless if and only if $N$ is isomorphic to a module $M'(1, b, c)$ with $b \neq -q$;
- reflexive if and only if $M$ is isomorphic to a module $M'(1, b, c)$ with $b \neq -q^i$ for $i = -1, 0$;
- Gorenstein-projective if and only if $N$ is isomorphic to a module $M'(1, b, c)$ with $b \neq -q^i$ for $i \in \mathbb{Z}$;
- semi-Gorenstein-projective if and only if $N$ is isomorphic to a module $M'(1, b, c)$ with $b \neq -q^i$ for $i \geq 0$ or to a module $M'(1, -1, c)$ with $c \neq 0$;
- $\infty$-torsionfree if and only if $N$ is isomorphic to a module $M'(1, b, c)$ with $b \neq -q^i$ for $i \leq 0$;
- pivotal semi-Gorenstein-projective if and only if $o(q) = \infty$ and $N$ is isomorphic to a module $M'(1, -1, c)$ with $c \neq 0$;
- pivotal $\infty$-torsionfree if and only if $o(q) = \infty$ and $N$ is isomorphic to a module $M'(1, -q, c)$.

Whereas the set of modules $M(1, b, c)$ with $b, c \in k$ is a union of $\tilde{O}$-components, the right modules behave differently: as we have seen already in Part I, 7.2, the $M$ modules of dimension at most 3 with relevant properties.

9. The $\Lambda$-dual of $M(1, b, c)$ and $M'(1, b, c)$.

We need the following (of course well-known) Lemma.

**Lemma.** Let $R$ be a ring and $w \in R$. If any left-module homomorphism $Rw \to R$ maps $w$ into $wR$, then $\text{Hom}(Rw, R) \simeq wR$ as right $R$-modules.

Proof. Let $u: Rw \to R$ be the inclusion map. We have $\text{Hom}(Rw, R) = uR$, since for any homomorphism $f: Rw \to R$, there is $\lambda \in R$ with $f(w) = w\lambda$, thus $f = u\lambda$. Now $I = \{r \in R \mid wr = 0\}$ is a right ideal and $R/I \simeq wR$ as right modules (an isomorphism is given by the map $R \to Rw$ defined by $1 \mapsto w$). Since $I = \{r \in R \mid wr = 0\}$, we have in the same way $R/I \simeq uR$, and therefore $wR \simeq R/I \simeq uR = \text{Hom}(Rw, R)$. \(\square\)

**Lemma.** If $(1, b, c)$ is different from $(1, -1, 0)$, then $M'(1, b, c) \simeq \text{Tr} M(1, b, c)$ and $M(1, b, c) \simeq \text{Tr} M'(1, b, c)$.

Proof. We have $U'(1, b, c) = (1, b, c)\Lambda$, and since $(1, b, c) \neq (1, -1, 0)$, we also have $U(1, b, c) = \Lambda(1, b, c)$. By definition, $M(1, b, c) = \Lambda(1, b, c)$, thus $M(1, b, c)$ is the cokernel of the right multiplication $r_{(1, b, c)}: \Lambda \to \Lambda$ and $\text{Tr} M(1, b, c)$ is the cokernel of the left multiplication $l_{(1, b, c)}: \Lambda \to \Lambda$, thus isomorphic to $\Lambda / (1, b, c)\Lambda = \Lambda / U'(1, b, c)$. \(\square\)

**Proposition.** If $b \notin \{-q, -q^2\}$, then $M(1, b, c)$ is reflexive and $M(1, b, c)^* = M'((\omega')^2(1, b, c))$. 19
If \( b \notin \{-1, -q^{-1}\} \), then \( M'(1, b, c) \) is reflexive and \( M'(1, b, c)^* = M(\omega^2(1, b, c)) \).

Proof. According to (7.1), we have the following two \( \mathcal{U} \)-sequences:

\[
0 \rightarrow M(1, b, c) \rightarrow \Lambda \mathcal{A} \rightarrow M(\omega'(1, b, c)) \rightarrow 0,
\]
\[
0 \rightarrow M(\omega'(1, b, c)) \rightarrow \Lambda \mathcal{A} \rightarrow M((\omega')^2(1, b, c)) \rightarrow 0
\]

(the first one, since \( \omega'(1, b, c) = (1, b', c') \) with \( b' = q^{-1}b \neq -1 \); the second one, since \( (\omega')^2(1, b, c) = (1, b'', c'') \) with \( b'' = q^{-2}b \neq -1 \)). This implies that \( M(1, b, c) \) is reflexive and that \( X = \mathcal{U}^2M(1, b, c) = M((\omega')^2(1, b, c)) \) is a module with \( \text{Ext}^i(X, \Lambda) = 0 \) for \( i = 1, 2 \). According to Part I, Lemma 2.5, we have \( \text{Tr}X = (\Omega^2X)^* \). On the one hand, \( \Omega^2X = \mathcal{U}M(1, b, c) = M(1, b, c) \). On the other hand, (9.2) shows that \( \text{Tr}X = \text{Tr}M((\omega')^2(1, b, c)) = M'((\omega')^2(1, b, c)) \), since \( (\omega')^2(1, b, c) = (1, q^{-2}b, c'') \) for some \( c'' \) and \( q^{-2}b \neq -1 \). This yields the first assertion. The second can be shown in the same way, or just by applying the \( \Lambda \)-duality to \( M(1, b, c)^* = M'((\omega')^2(1, b, c)) \).

\((9.4)\) Proposition. For all \( b, c \in k \),

\[
M(1, b, c)^* = M'((\omega')^2(1, b, c)).
\]

In particular, for all \( b, c \in k \), the right module \( M(1, b, c)^* \) is again 3-dimensional and local.

Whereas \( (\omega')^2 \) is a bijection from \( \{(1, b, c) \mid b \notin \{-q, -q^2\}\} \) onto \( \{(1, b, c) \mid b \notin \{-1, -q^{-1}\}\} \), we should stress that \( \omega^2(1, -q, c) = (1, -q^{-1}, 0) \) and that \( \omega^2(1, -q^2, c) = (1, -1, 0) \) for all \( c \in k \). Thus, (9.3) combines the first assertion of (9.2) with the corresponding assertion for the remaining cases, namely:

\[
M(1, -q, c)^* = M'(1, -q^{-1}, 0) \quad \text{and} \quad M(1, -q^2, c)^* = M'(1, -1, 0),
\]

for all \( c \in k \).

Proof of Proposition. According to (9.2), we only have to consider the cases where \( b = -q \) or \( b = -q^2 \).

Case 1. Let \( b = -q \). As we have seen in (6.2), the module \( M(1, -q, c) \) is not torsionless. Now obviously, there is a surjective homomorphism \( M(1, -q, c) \rightarrow \Lambda(1, -1, 0) \) with kernel \( zM(1, -q, c) \). It follows that \( zM(1, -q, c) \) is contained in the kernel of every homomorphism \( M(1, -q, c) \rightarrow \Lambda \mathcal{A} \) and therefore \( M(1, -q, c)^* = (\Lambda(1, -1, 0))^* \). Now, \( (\Lambda(1, -1, 0))^* \simeq (1, -1, 0)\Lambda = U'(1, -1, 0) \), as shown in Part I, 6.5. On the other hand, according to (8.1), we have \( \Lambda^c(1, -1, 0) = \Omega M'(1, -1, 0) = M'((\omega')^2(1, -1, 0)) \) and \( \omega'(1, -1, 0) = (1, -q^{-1}, 0) \).

Case 2: \( b = -q^2 \) and \( o(q) = 2 \). The assumption \( o(q) = 2 \) means that \( q = -1 \neq 1 \), in particular, the characteristic of \( k \) is different from 2, and we have \( b = -1 \). Since \( q = -1 \) and the characteristic of \( k \) is different from 2, (4.1) asserts that

\[
\Lambda(1, 1, -2c) = U(1, 1, -2c) = \Omega M(1, 1, -2c) = M(\omega(1, 1, -2c)) = M(1, -1, c).
\]

On the other hand, we have

\[
(1, 1, -2c)\Lambda = U'(1, 1, -2c) = \Omega M'(1, 1, -2c) = M'((\omega')(1, 1, -2c)) = M'(1, -1, 0).
\]
We claim that any homomorphism $\Lambda(1, 1, -2c) \to \Lambda\Lambda$ maps $(1, 1, -2c)$ into $(1, 1, -2c)\Lambda$. Namely, let $\phi: \Lambda(1, 1, -2c) \to \Lambda\Lambda$ be a homomorphism. Now $\Lambda(1, 1, -2c)$ is $3$-dimensional, thus equal to $U(1, 1, -2c)$, and $\Lambda\Lambda/U(1, 1, -2c) \simeq M(1, 1, -2c)$. According to (5.1), the module $M(1, 1, -2c)$ is extensionless, since $1 + 1 \neq 0$. The implication (i) to (iv) in (5.2) shows that $\phi(1, 1, -2c) \in (1, 1, -2c)\Lambda$.

Since any homomorphism $\Lambda(1, 1, -2c) \to \Lambda\Lambda$ maps $(1, 1, -2c)$ into $(1, 1, -2c)\Lambda$, Lemma (9.0) implies that the right modules $(\Lambda(1, 1, -2c))^*$ and $(1, 1, -2c)\Lambda$ are isomorphic, thus $M(1, -1, c)^* \simeq M'(1, -1, 0)$.

Case 3. $b = -q^2$ and $o(q) \geq 3$. There is the $\partial$-sequence

$$
\epsilon: \quad 0 \to M(1, -q^3, c') \to \Lambda\Lambda \to M(1, -q^2, c) \to 0
$$

for some $c'$ (here we use that $q^2 \neq 1$). The $\Lambda$-dual of $\epsilon$ is the exact sequence

$$
0 \to M(1, -q^2, c)^* \to \Lambda\Lambda \to M(1, -q^3, c')^* \to 0.
$$

Since $q^2 \neq 1$, proposition (9.3) asserts that $M(1, -q^3, c')^* = M'(1, -q, c'')$ for some $c''$. Altogether we see that

$$
M(1, -q^2, c)^* \simeq \Omega(M(1, -q^3, c')^*) = \Omega M'(1, -q, c'') \simeq M'(1, -1, 0),
$$

where the final isomorphism is due to (8.1). □

(9.5) The algebra $\Lambda = \Lambda(q)$ with $o(q) = \infty$ was exhibited in Part I in order to present a module $M$ which is not torsionless, such that $M$ and $M^*$ both are semi-Gorenstein-projective: namely the module $M = M(1, -q, 0)$ with $M^* = M'(1, -q, 0)$. Now we see: all the modules $M(1, -q, c)$ with $c \in k$ are modules which are semi-Gorenstein-projective and not torsionless, and that the $\Lambda$-duals $M(1, -q, c)^* \simeq M'(1, -q^{-1}, 0)$ are semi-Gorenstein-projective. We should stress that this concerns a $1$-parameter family $M(1, -q, c)$ (with $c \in k$) of semi-Gorenstein-projective left modules, and the single semi-Gorenstein-projective right module $M(1, -q^{-1}, 0)$.

(9.6) Proposition. Let $b, c \in k$.

$$
M'(1, b, c)^* = \begin{cases} 
M(\omega^2(1, b, c)) & \text{if } b \notin \{-1, -q^{-1}\}, \\
U(0, 0, 1) & \text{if } b = -1, c \neq 0, \\
U(1, -q, 0) + U(0, 0, 1) & \text{if } b = -1, c = 0, \\
M(0, 0, 1) & \text{if } b = -q^{-1}, c \neq 0, q \neq 1, \\
U(1, -1, 0) & \text{if } b = -q^{-1}, c = 0, q \neq 1.
\end{cases}
$$

 Whereas we saw in (9.4) that all the right modules $M(1, b, c)^*$ are $3$-dimensional and local, not all the modules $M'(1, b, c)^*$ are $3$-dimensional and local: the module $M'(1, -1, 0)^* = U(1, -q, 0) + U(0, 0, 1)$ has dimension $4$, whereas the modules $M'(1, -1, c)^* = U(0, 0, 1)$ for $c \neq 0$ and, in case $q \neq 1$, the module $M'(1, -q^{-1}, 0)^* = U(1, -1, 0)$ are decomposable.

Proof. According to (9.3), we only have to deal with the cases with $b \in \{-1, -q^{-1}\}$. If $c = 0$, then we can refer to Part I. For $b = -1$, the end of 7.1 in Part I shows that

\[21\]
Such an algebra is said to be projective.

Now, we assume that $c \neq 0$. As in the proof of (9.4), we consider again 3 cases.

Case 1. $b = -1$. The module $M'(1, -1, c)$ with $c \neq 0$ is not torsionless, see (8.4). Since the factor module $M'(1, -1, c)/M'(1, -1, c)z$ is isomorphic to $(0, 0, 1)\Lambda$, it follows that $M'(1, -1, c)^* \simeq ((0, 0, 1)\Lambda)^*$ and an easy calculation yields $((0, 0, 1)\Lambda)^* \simeq U(0, 0, 1)$. Namely, the inclusion map $u: z\Lambda \to \Lambda\Lambda$ satisfies $yu = 0$ and $zu = 0$, thus a basis of $(z\Lambda)^*$ is given by $u, xu$ and the map $f: z\Lambda \to \Lambda\Lambda$ with $f(z) = yx$, so that $(z\Lambda)^* \simeq \Lambda\Lambda/((\Lambda y + \Lambda z) \oplus k) \simeq U(0, 0, 1)$.

Case 2. $b = -q^{-1}$ and $o(q) = 2$. Thus, the characteristic of $k$ is different from 2, $q = -1$ and $b = 1$. The module $M'(1, 1, c)$ is torsionless: namely, by (8.1) we have $M'(1, 1, c) \simeq \Omega M'(1, -1, -\frac{c}{i})$, since $\omega'(1, -1, -\frac{c}{i}) = (1, 1, c)$. Now, $\Omega M'(1, -1, -\frac{c}{i}) \simeq U'(1, -1, -\frac{c}{2}) = (1, -1, \frac{c}{i})\Lambda$. Since $q \neq 1$, the right module $M'(1, -1, -\frac{c}{i})$ is extensionless by (8.4), thus we can use (5.2) and (9.1) in order to see that $((1, -1, -\frac{c}{i})\Lambda)^* \simeq \Lambda(1, -1, -\frac{c}{i})$. By (4.1) (2), we have $\Lambda(1, -1, -\frac{c}{i}) = U(1, -1, -\frac{c}{i}) \simeq \Omega M((1, -1, -\frac{c}{i})) \simeq M(0, 0, 1)$.

Case 3. $b = -q^{-1}$ and $o(q) \geq 3$. There is the $\mathcal{U}$-sequence

$$0 \to M'(1, -q^{-2}, c') \to \Lambda\Lambda \to M'(1, -q^{-1}, c) \to 0$$

for $c' = \lambda c$ with $\lambda \neq 0$ (here we use that $q^2 \neq 1$). The $\Lambda$-dual is the exact sequence

$$0 \to M'(1, -q^{-1}, c)^* \to \Lambda\Lambda \to M'(1, -q^{-2}, c')^* \to 0.$$ 

We assume that $q \neq 1$ and $q \neq 2$. Then by Proposition (9.2), we have $M'(1, -q^{-2}, c')^* = M(1, -1, c'')$ for some multiple $c'' = \lambda' c'$ with $\lambda' \neq 0$. It follows that $M'(1, -q^{-1}, c)^* = \Omega M(1, -1, c'')$ and $c'' = 0$ if and only if $c = 0$. By (4.1), we have $\Omega M(1, -1, c'') = M(0, 0, 1)$ in case $c \neq 0$, and $\Omega M(1, -1, 0) = U(1, -1, 0)$ in case $c = 0$.

(9.7) Corollary. Let $N$ be a right $\Lambda$-module of dimension at most 3 which is semi-Gorenstein-projective, but not Gorenstein-projective. Then $N^*$ is not semi-Gorenstein-projective.

Proof. According to (8.6), $N$ is isomorphic to a right module of the form $M'(1, -q^i, c)$ with $i \leq -1$ and $c \in k$ or of the form $M'(1, -1, c)\Lambda$ with $c \neq 0$. We apply (9.6). If $i \leq -2$, then $N^* = M'(1, -q^i, c)^* = M(1, -q^{i+2}, c')$ for some $c'$, and according to (1.5), $N^*$ is not semi-Gorenstein-projective, since $i + 2 \leq 0$. If $i = -1$, then $N^*$ is isomorphic to $M(0, 0, 1)$ or to $U(1, -1, 0)$. If $N = M'(1, -1, c)$ with $c \neq 0$, then $N^*$ is isomorphic to $U(0, 0, 1)$. But by (1.5), $M(0, 0, 1)$, $U(1, -1, 0)$ and $U(0, 0, 1)$ are not semi-Gorenstein-projective.

10. The general context.

Our detailed study of the algebra $\Lambda(q)$ in Part I and Part II should be seen in the frame of looking at Gorenstein-projective (or, more general, semi-Gorenstein-projective and $\infty$-torsionfree modules) over local algebras with radical cube zero.

(10.1) Let $A$ be a finite-dimensional local $k$-algebra with radical $J$ such that $A/J = k$. Such an algebra is said to be short provided $J^3 = 0$. In commutative ring theory, the short local algebras have attracted a lot of interest, since some conjectures have been disproved.
by looking at modules over short algebras, see [AIS] for a corresponding account. We have
to thank D. Jorgensen for his advice concerning the present knowledge in the commutative
case.

Let us assume now that \( A \) is short, but not necessarily commutative. Let \( e = \dim J/J^2 \)
and \( a = \dim J^2 \) (thus \( 0 \leq a \leq e^2 \)). Here is a report about the relevant general
results: If there exists an indecomposable module which is semi-Gorenstein-projective or
\( \infty \)-torsionfree, but not projective, then either \( A \) is self-injective, so that \( a \leq 1 \)
(and \( e = 1 \) in case \( a = 0 \)), or else \( a = e - 1 \geq 2 \). Always, both modules \( AJ \) and \( J_A \)
have to be indecomposable.

Of course, if \( A \) is self-injective, then all modules are Gorenstein-projective, thus the
interesting case is the case \( a = e - 1 \geq 2 \). Our algebra \( \Lambda(q) \) is of this kind (with \( a = 2 \)),
as is the Jorgensen-Šega algebra [JS] (with \( a = 3 \)).

Not only the shape of the algebras is very restricted, also the modules themselves
are very special: Let \( A \) be a short local algebra which is not self-injective. Let \( M \) be
indecomposable and not projective. If \( M \) is semi-Gorenstein-projective and torsionless, or
\( M \) is \( \infty \)-torsionfree (in particular, if \( M \) is Gorenstein-projective), then \( \text{soc } M = \text{rad } M \)
and \( \dim \text{soc } M = a \cdot \dim \text{top } M \) (by definition, \( \text{top } M = M/\text{soc } M \)). Also, if \( M \) is semi-
Gorenstein-projective and torsionless, then \( \dim \Omega^i M = \dim M \) for all \( i \in \mathbb{N} \), whereas if \( M \)
is \( \infty \)-torsionfree, then \( \dim \Omega^i M = \dim M \) for all \( i \in \mathbb{N} \).

These assertions have been shown by Christensen and Veliche in the case that \( A \) is
commutative and \( M \) is Gorenstein-projective, see [CV], and the proof can be modified in
order to work in general, see [RZ2]. There is an essential difference between the commuta-
tive and the non-commutative algebras: If \( A \) is commutative, then all local modules which
are semi-Gorenstein-projective or \( \infty \)-torsionfree are Gorenstein-projective, whereas this is
not true for \( A \) non-commutative.

Thus, for our algebra \( \Lambda(q) \), the non-projective indecomposable modules which are
semi-Gorenstein-projective and torsionless, or which are \( \infty \)-torsionfree, are of dimension
\( 3t \) with socle of dimension \( 2t \), where \( t = \dim \text{top } M \). For \( t = 1 \), we deal with local modules
with 2-dimensional socle: these are precisely the modules studied in the present paper.

(10.2) As we have mentioned, a 3-dimensional local \( \Lambda(q) \)-module \( M \) belongs to \( H \) if
and only if \( \text{soc } M = \text{Ker}(y) = \text{Ker}(z) = yM \oplus zM \). Thus, it seems to be of interest to
study the full subcategory \( \mathcal{H} \) of all the \( \Lambda(q) \)-modules \( M \) with \( \text{soc } M = \text{Ker}(y) = \text{Ker}(z) = yM \oplus zM \)

It will be shown in [RZ3] that all reflexive modules which are semi-Gorenstein-projective
or \( \infty \)-torsionfree belong to \( \mathcal{H} \). On the other hand, we will exhibit a representation equiva-
lence between \( \mathcal{H} \) and the category of finite-dimensional \( k\langle x_1, x_2 \rangle \)-modules, where \( k\langle x_1, x_2 \rangle \)
is the free algebra in two variables \( x_1, x_2 \).

Appendix. A diagrammatic description of the modules \( M(a:b:c) \).

(A.1) If \( M \) is a left \( \Lambda \)-module annihilated by \( \text{rad}^2 \Lambda \), then it is a left \( \Lambda \)-module. Since \( \Lambda \)
is a commutative \( k \)-algebra, also \( D(M) = \text{Hom}(M, k) \) is a left \( \Lambda \)-module, thus a left \( \Lambda \)-module. As mentioned in (1.6), we identify the set of isomorphism classes of the 3-
dimensional local modules with the projective plane \( \mathbb{P}^2 = \mathbb{P}(\text{rad } \Lambda/\text{rad}^2 \Lambda) \).
Proposition. Let $M$ be an indecomposable 3-dimensional left $\Lambda$-module. Then $M$ or $D(M)$ is isomorphic to one of the following pairwise non-isomorphic $\overline{K}$-modules $M(a, b, c)$:

<table>
<thead>
<tr>
<th>Case</th>
<th>Modules</th>
<th>Position in $\mathbb{P}^2$</th>
<th>Diagram</th>
<th>Characterization</th>
</tr>
</thead>
</table>
| (1)  | $M(0, 0, 1)$ | \[
\begin{array}{c}
  \bullet \\
  \bullet
\end{array}
\] | $v_1 \xrightarrow{x} v \xrightarrow{y} v_2$ | $zM = 0$ |
| (2)  | $M(0, 1, 0)$ | \[
\begin{array}{c}
  \bullet \\
  \bullet
\end{array}
\] | $v_1 \xrightarrow{x} v \xrightarrow{z} v_2$ | $yM = 0$ |
| (3)  | $M(1, 0, 0)$ | \[
\begin{array}{c}
  \bullet \\
  \bullet
\end{array}
\] | $v_1 \xrightarrow{y} v \xrightarrow{z} v_2$ | $xM = 0$ |
| (4)  | $M(1, b, 0)$ | $b \in k^*$ | $v_1 \xrightarrow{y} v \xrightarrow{z} v_2$ | $xM = yM$ with $xv = -bv_1$ |
| (5)  | $M(1, 0, c)$ | $c \in k^*$ | $v_1 \xrightarrow{y} v \xrightarrow{z} v_2$ | $xM = zM$ with $xv = -cv_2$ |
| (6)  | $M(0, 1, c)$ | $c \in k^*$ | $v_1 \xrightarrow{y} v \xrightarrow{z} v_2$ | $yM = zM$ with $yv = -cv_2$ |
| (7)  | $M(1, b, c)$ | $b, c \in k^*$ | $v_1 \xrightarrow{y} v \xrightarrow{z} v_2$ | $xM, yM, zM$ non-zero and pairwise different with $xv = -bv_1 - cv_2$ |

The diagrams describe the modules $M = M(a, b, c)$ as follows: The elements $v, v_1, v_2$ form a basis of $M$. Both elements $v_1, v_2$ are annihilated by $x, y, z$. If there is drawn a solid arrow $v \rightarrow v_i$ with $i \in \{1, 2\}$ and with label $\alpha \in \{x, y, z\}$, then $\alpha v = v_i$. If there is a dashed arrow $v \dashrightarrow v_i$ with label $\alpha$, then $\alpha v = c_1 v_1 + c_2 v_2$ with $c_i \neq 0$ (and we provide the coefficients $c_1, c_2$ below the diagram). Finally, $zv = 0$ in case (1), $yv = 0$ in case (2), $xv = 0$ in case (3). 

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The last column provides a characterization of the corresponding modules $M(a, b, c)$: For example, a local 3-dimensional $A$-module $M$ is a case-(1)-module provided $z M = 0$, and so on.

(A.2) Remark. If $M$ is an indecomposable 3-dimensional $A$-module, then its annihilator is equal to $U(a, b, c)$ for some $(a, b, c) \neq 0$ and $M$ considered as a $\Lambda/U(a, b, c)$-module is either the unique indecomposable projective $\Lambda/U(a, b, c)$-module (and then a local module, thus isomorphic to $M(a, b, c)$) or the unique indecomposable injective $\Lambda/U(a, b, c)$-module (and then a module with simple socle, thus isomorphic to $D(M(a, b, c))$).

(A.3) Proof of the Proposition and the Remark. First, let us assume that $M$ is local. According to (2.6) and (1.4), we know that $M \simeq M(a : b : c)$ for some $(a : b : c) \in \mathbb{P}^2$ and that these modules are pairwise non-isomorphic. As representatives of the elements of $\mathbb{P}^2$, we choose (as usual) the triples $(c_1, c_2, c_3)$ with $c_i = 1$ for some $i$ and $c_j = 0$ for $j < i$. Clearly, there are the seven cases (1) to (7) as listed above. It remains to choose in every case a basis $B(a, b, c) = \{v, v_1, v_2\}$ of $M(a, b, c)$. Recall that $M(a, b, c) = \Lambda/(a : b : c)$ is a factor module of $\Lambda$ and $\Lambda$ has the basis $\{1, x, y, z\}$. We choose as elements of $B(a, b, c)$ the residue class $v = 1$ as well as two of the three residue classes $\bar{x}, \bar{y}, \bar{z}$, namely $v_1 = \bar{x}$ if $a = 0$ and $v_1 = \bar{y}$ otherwise, and then $v_2 = \bar{y}$ in case $(a, b, c) = (0, 0, 1)$ and $v_2 = \bar{z}$ otherwise. (We should remark that the vertices and the arrows of the diagram are those of the coefficient quiver $\Gamma(M(a, b, c), B(a, b, c))$ as considered in [R], and the solid arrows focus the attention to a spanning tree.)

Second, assume that $M$ is not local. Since $M$ is an indecomposable module of length 3 and Loewy length 2, it follows that $M$ has simple socle, thus $D(M)$ is local and therefore of the form (1) to (7).

Finally, $M$ and $D(M)$ have the same annihilator, this is a 3-dimensional ideal, thus of the form $U(a, b, c)$. The 3-dimensional local algebra $\Lambda/U(a, b, c)$ has a unique 3-dimensional local module, this is the indecomposable projective $\Lambda/U(a, b, c)$-module, and dually, it has a unique 3-dimensional module with simple socle, this is the unique indecomposable injective $\Lambda/U(a, b, c)$-module. This completes the proof. \[\square\]

(A.4) As we have mentioned in (1.6), of special interest is the affine subspace $H$ of $\mathbb{P}^2$ given by the points $(1 : b : c)$ with $b, c \in k$. A 3-dimensional local module $M$ belongs to $H$ if and only if $\text{soc } M = \text{Ker}(y) = \text{Ker}(z) = y M \oplus z M$.

![Diagram](image)

Namely, $H$ is the union of the sets (3), (4), (5) and (7).
References.


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