Gorenstein-projective and semi-Gorenstein-projective modules. II

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Abstract: Let k be a field and q a non-zero element of k. In Part I, we have exhibited a 6-dimensional k-algebra $\Lambda{=}\Lambda(q)$ and we have shown that if q has infinite multiplicative order, then Λ has a 3-dimensional local module which is semi-Gorenstein-projective, but not torsionless, thus not Gorenstein-projective. This Part II is devoted to a detailed study of all the 3-dimensional local Λ-modules for this particular algebra Λ. If q has infinite multiplicative order, we will encounter a whole family of 3-dimensional local modules which are semi-Gorenstein-projective, but not torsionless.

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1. Introduction.

(1.1) We refer to our previous paper [RZ1] as Part I. As in Part I, let k be a field, and q a non-zero element of k. We consider again the k-algebra $\Lambda = \Lambda(q)$ generated by x, y, z with relations

$$x^2,\ y^2,\ z^2,\ yz,\ xy + qyx,\ xz - zx,\ zy - zx.$$

The algebra Λ is a 6-dimensional local algebra with basis $1, x, y, z, yx, zx$. Its socle is $\operatorname{soc}\Lambda = \operatorname{rad}^2\Lambda = \Lambda yx \oplus \Lambda zx$. If not otherwise stated, all the modules considered will be left Λ-modules.

We follow the terminology used in Part I. In particular, we denote by $\mho M$ the cokernel of a minimal left add(Λ)-approximation of M. In addition, we introduce the following definitions. We say that a module M is *extensionless* if $\operatorname{Ext}^1(M, \Lambda) = 0$. An indecomposable semi-Gorenstein-projective module will be said to be *pivotal* provided it is not torsionless. An indecomposable ∞-torsionfree module will be said to be *pivotal* provided it is not extensionless. Thus, a module M is semi-Gorenstein-projective if and only if $\Omega^t M$ is extensionless for all $t \geq 0$; a torsionless module M is reflexive if and only if $\mho M$ is torsionless (see Part I (2.4)); a module M is ∞-torsionfree if and only if $\mho^t M$ is reflexive for all $t \geq 0$; and M is Gorenstein-projective if and only if M is both semi-Gorenstein-projective and ∞-torsionfree.

(1.2) We are interested in the semi-Gorenstein-projective and the ∞-torsionfree modules and will exhibit those which are 3-dimensional. We recall that a finite length module is said to be *local* provided its top is simple. Thus, a local module is indecomposable; and if R is a left artinian ring, then a left R-module M is local if and only if M is a quotient of an indecomposable projective module. A consequence of our study is the following assertion

Proposition. *Let M be a non-zero module of dimension at most* 3*. If M is semi-Gorenstein-projective, then all the modules $\Omega^t M$ with $t \geq 0$ are* 3*-dimensional and local. If M is ∞-torsionfree, then all the modules $\mho^t M$ with $t \geq 0$ are* 3*-dimensional and local.* In particular, if M is Gorenstein-projective, then all the modules $\Omega^t M$ and $\mho^t M$ with $t \geq 0$ are 3-dimensional and local.

(1.3) The text restricts the attention to the 3-dimensional local modules. We recall that if A is a finite-dimensional algeba, an A-module M is said to have Loewy length at most t provided $\operatorname{rad}^t M = 0$. The starting point of our investigation are two observations. The first one:

Proposition 1. *A module of dimension at most* 3 *has Loewy length at most* 2*.*

The second observation is:

Proposition 2. *An indecomposable* 3*-dimensional torsionless module is local.*

The proof of Proposition 1 will be given in (2.6), the proof of Proposition 2 in (2.7).

(1.4) The 3-dimensional local modules. We identify $(a, b, c) \in k^3 \setminus \{0\}$ with $ax+by+cz$ and denote by $(a:b:c)$ the 1-dimensional subspace of k^3 generated by (a, b, c). The left ideal

$$U(a, b, c) = U(a:b:c) = \Lambda(a, b, c) + \operatorname{soc} \Lambda$$

has dimension 3, and we obtain the left Λ-module

$$M(a, b, c) = M(a:b:c) = {}_\Lambda\Lambda/U(a, b, c).$$

Clearly, *$M(a, b, c)$ is a* 3*-dimensional local module and the modules $M(a, b, c)$, $M(a', b', c')$ are isomorphic if and only if $(a:b:c) = (a':b':c')$.* Let us add that the definition of $M(a, b, c)$ implies that $\Omega M(a, b, c) \simeq U(a, b, c)$, this will be used throughout the text.

Conversely, *any* 3*-dimensional local module is isomorphic to a module of the form $M(a, b, c)$.* In order to see this, one should look at the factor algebra $\overline{\Lambda}$ of Λ modulo $\operatorname{soc} \Lambda = \operatorname{rad}^2 \Lambda$, thus $\overline{\Lambda}$ is the k-algebra generated by x, y, z with relations all monomials of length 2. The Λ-modules of Loewy length at most 2 are just the modules annihilated by all monomials of length 2, thus the $\overline{\Lambda}$-modules. It is clear that the modules $M(a, b, c) = \overline{\Lambda}/(a:b:c)$ are representatives of the 3-dimensional local $\overline{\Lambda}$-modules. According to Proposition 1, all the 3-dimensional Λ-modules are $\overline{\Lambda}$-modules, thus the modules $M(a, b, c)$ are representatives of the 3-dimensional local Λ-modules.

(1.5) The following theorem characterizes the modules of dimension at most 3 which have some relevant properties. We write $o(q)$ for the multiplicative order of q.

Theorem. *An indecomposable module M of dimension at most* 3 *is*
- *torsionless if and only if M is simple or isomorphic to $\Lambda(x-y)$, to Λz, to a module $M(1, b, c)$ with $b \neq -q$, to $M(0, 1, 0)$ or to $M(0, 0, 1)$;*
- *extensionless if and only if M is isomorphic to a module $M(1, b, c)$ with $b \neq -1$;*
- *reflexive if and only if M is isomorphic to a module $M(1, b, c)$ with $b \neq -q^i$ for $i = 1, 2$;*
- *Gorenstein-projective if and only if M is isomorphic to a module $M(1, b, c)$ with $b \neq -q^i$ for $i \in \mathbb{Z}$;*

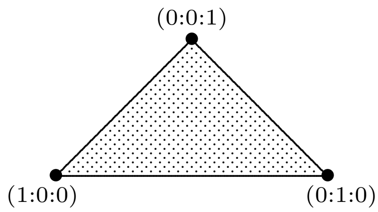
- *semi-Gorenstein-projective if and only if M is isomorphic to a module $M(1,b,c)$ with $b \neq -q^i$ for $i \leq 0$;*
- *∞-torsionfree if and only if M is isomorphic to a module $M(1,b,c)$ with $b \neq -q^i$ for $i \geq 1$;*
- *pivotal semi-Gorenstein-projective if and only if $o(q) = \infty$ and M is isomorphic to a module $M(1,-q,c)$;*
- *pivotal ∞-torsionfree if and only if $o(q) = \infty$ and M is isomorphic to a module $M(1,-1,c)$.*

For the proof of the Theorem, see (7.9).

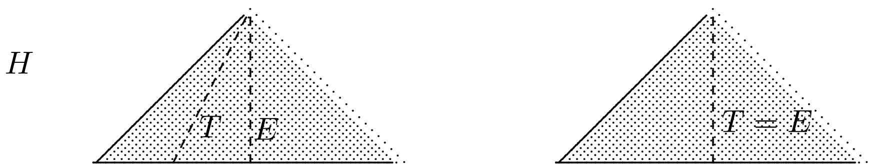
Looking at the Theorem, the reader will be aware that in the context considered here, the relevant modules of dimension at most 3 are of the form $M(1,b,c)$ with $b,c \in k$. Nearly all the modules mentioned in Theorem are of this kind, the only exceptions are four isomorphism classes of torsionless modules, namely the 2-dimensional left ideals $\Lambda(x-y)$ and Λz, as well as the 3-dimensional modules $M(0,1,0)$ and $M(0,0,1)$.

(1.6) As we have seen in (1.4), the set of isomorphism classes of the 3-dimensional local modules can be identified in a natural way with the projective plane $\mathbb{P}^2 = \mathbb{P}(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)$, with the element $(a\!:\!b\!:\!c) \in \mathbb{P}^2$ corresponding to the module $M(a,b,c)$.

We use homogeneous coordinates in order to highlight elements and subsets of $\mathbb{P}^2$ (or the corresponding modules):



Let H be the affine subspace of $\mathbb{P}^2$ given by the points $(1\!:\!b\!:\!c)$ with $b,c \in k$. As we have mentioned already, Theorem (1.5) shows that it is this subset H which is of special interest. We will see in section 7 that H is a union of $\mho$-components, and that the set of 3-dimensional Gorenstein-projective modules is always a (proper) subset of H. A module M in H is torsionless if and only if it does not belong to the line $T = \{(1\!:\!(-q)\!:\!c) \mid c \in k\}$, and is extensionless if and only if it does not belong to the line $E = \{(1\!:\!(-1)\!:\!c) \mid c \in k\}$ (see (6.1) and (5.1), respectively):



(for $q \neq 1$) (for $q = 1$)

In case the multiplicative order $o(q)$ of q is infinite, H is the set of the 3-dimensional modules which are semi-Gorenstein-projective or ∞-torsionfree; the line E consists of the pivotal semi-Gorenstein-projective modules in H; the line T of the pivotal ∞-torsionfree modules in H.

Let us emphasize: *There are* 3*-dimensional pivotal semi-Gorenstein-projective modules if and only if there are* 3*-dimensional pivotal* ∞*-torsionfree modules if and only if the multiplicative order of* q *is infinite.*

Note that a 3-dimensional local module M belongs to H if and only if $\operatorname{soc} M = \operatorname{Ker}(y) = \operatorname{Ker}(z) = yM \oplus zM$, see A.4 in the appendix.

(1.7) The algebra $\Lambda = \Lambda(q)$ with $o(q) = \infty$ was exhibited in Part I in order to present a module M which is not reflexive, such that both M and its Λ-dual M^* are semi-Gorenstein-projective: namely the module $M = M(1, -q, 0)$. Now we see:

Let $o(q) = \infty$ *and assume that* M *is a module of dimension at most* 3*. Then both* M *and* M^* *are semi-Gorenstein-projective, whereas* M *is not reflexive, if and only if* M *is isomorphic to a module of the form* $M(1, -q, c)$ *with* $c \in k$*. In this case* M *is not even torsionless and all the modules* $M(1, -q, c)^*$ *with* $c \in k$ *are isomorphic,* see (9.5). Thus, we encounter a 1-parameter family of pairwise non-isomorphic semi-Gorenstein-projective left modules M such that their Λ-dual modules M^* are isomorphic and semi-Gorenstein-projective.

Also, we see that *for all* $c \neq 0$*, the modules* $M(1, -1, c)$ *are pairwise non-isomorphic* ∞*-torsionfree modules with a fixed module* $\Omega M(1, -1, c) = M(0, 0, 1)$*,* see (4.1). Thus, we encounter non-isomorphic ∞-torsionfree modules with isomorphic first syzygy module (of course, in this situation, the syzygy module cannot be ∞-torsionfree).

(1.8) The modules $M(1, b, 0)$ with $b \in k$ have been studied already in Part I (there, they have been denoted by $M(-b)$). Theorem (1.5) shows that these modules are quite typical for the behavior of the modules $M(1, b, c)$. Namely: *The module* $M(1, b, c)$ *is Gorenstein-projective (or semi-Gorenstein-projective, or* ∞*-torsionfree, or torsionless, or extensionless) if and only if* $M(1, b, 0)$ *has this property.*

(1.9) Outline of the paper. Section 2 provides some preliminary results. Here, the main target is to show that any module of length at most 3 has Loewy length at most 2. In section 3 we collect some formulae which show that certain products of elements in Λ are zero. Sections 4 to 7 deal with the 3-dimensional local left Λ-modules, section 8 with the 3-dimensional local right Λ-modules. Section 9 discusses the Λ-duality. The final section 10 provides an outline of the general frame for this investigation: the study of semi-Gorenstein-projective and ∞-torsionfree modules over local algebras with radical cube zero. There is an appendix which provides a diagrammatic description of the 3-dimensional indecomposable left Λ-modules.

2. Some left ideals and some right ideals of Λ.

(2.1) Lemma. *The left ideal* $\Lambda(a, b, c)$ *is* 2*-dimensional if and only if* $a + b = 0$ *and* $ac = 0$*. We have* $\operatorname{soc} \Lambda(1, -1, 0) = \Lambda yx$ *and* $\operatorname{soc} \Lambda(0, 0, 1) = \Lambda zx$*.*

Proof. An easy calculation shows that $\operatorname{soc} \Lambda(1, -1, 0) = \Lambda yx$ and $\operatorname{soc} \Lambda(0, 0, 1) = \Lambda zx$. Thus, the left ideals $\Lambda(0, 0, 1)$ and $\Lambda(1, -1, 0)$ are 2-dimensional.

Now, let $L = \Lambda(a, b, c)$ be any left ideal. If $a \neq 0$, then $yx \in L$ since $y(a, b, c) = ayx$.

First, assume that $a + b \neq 0$. Then $z(a, b, c) = (a + b)zx$ shows that $zx \in L$. We know already that for $a \neq 0$, also $yx \in L$. If $a = 0$, then $b \neq 0$. Thus $x(a, b, c) = -qbyx + czx$ shows that also in this case $yx \in L$. Thus L cannot be 2-dimensional.

Next, assume that $ac \neq 0$. Since $a \neq 0$, we know that $yx \in L$. Since $c \neq 0$, we use $x(a,b,c) = -qbyx + czx$ in order to see that $zx \in L$. Again, L cannot be 2-dimensional. $\square$

(2.2) *Let L be a 2-dimensional left ideal, different from $\operatorname{soc}\Lambda$. Then either $L \subseteq U(1,-1,0)$ and then $\operatorname{soc} L = \Lambda yx$ and L is isomorphic to $\Lambda(x-y)$ or else $L \subseteq U(0,0,1)$ and then $\operatorname{soc} L = \Lambda zx$ and L is isomorphic to Λz.*

Proof: There is an element $(a,b,c) + w \in L$, with $(a,b,c) \neq 0$ and $w \in \operatorname{soc}\Lambda$. Since $\operatorname{rad}\Lambda((a,b,c)+w) = \operatorname{rad}\Lambda(a,b,c)$, also $L' = \Lambda(a,b,c)$ is 2-dimensional and $L \subseteq L' + \operatorname{soc}\Lambda = U(a,b,c)$. According to (2,1), $(a\!:\!b\!:\!c)$ is equal to $(1\!:\!(-1)\!:\!0)$ or to $(0\!:\!0\!:\!1)$. Of course, L and L' are isomorphic as (left) modules. $\square$

(2.3) Lemma. *There is no 3-dimensional torsionless module with simple socle.*

Proof. Assume that U is a 3-dimensional torsionless module with simple socle. Then U is a submodule of Λ. It is a proper submodule, thus of Loewy length at most 2. Therefore, U is the sum of two 2-dimensional left ideals $L \neq L'$ with $\operatorname{soc} L = \operatorname{soc} L'$. Now we use (2.2). If L, L' have socle equal to Λyx, then $U = L + L' = U(1,-1,0)$. If L, L' have socle equal to Λzx, then also $U = L + L' = U(0,0,1)$. In both cases $\operatorname{soc}\Lambda \subseteq U$, a contradiction. $\square$

(2.4) *Any 3-dimensional left ideal contains* $\operatorname{soc}\Lambda$.

(2.5) *The 3-dimensional left ideals are the subspaces $U(a,b,c)$. They have the following structure: $U(1,-1,0) = \Lambda(1,-1,0) \oplus \Lambda zx$; $U(0,0,1) = \Lambda(0,0,1) \oplus \Lambda yx$; and if $a+b \neq 0$ or $ac \neq 0$, then $U(a,b,c) = \Lambda(a,b,c)$ is a local module (in particular, indecomposable).*

Proof. The left ideals $U(a,b,c)$ are 3-dimensional. Conversely, let U be a 3-dimensional left ideal of Λ. Since $\operatorname{soc}\Lambda$ is contained in U, there is an element $(a,b,c) \neq 0$ with $(a,b,c) \in U$, thus $U = U(a,b,c)$.

If $a + b = 0$ and $ac = 0$, then $(a\!:\!b\!:\!c)$ is equal to $(1\!:\!(-1)\!:\!0)$ or to $(0\!:\!0\!:\!1)$. By (2.1), we have $U(1,-1,0) = \Lambda(1,-1,0) \oplus \Lambda zx$ and $U(0,0,1) = \Lambda(0,0,1) \oplus \Lambda yx$. If $a+b \neq 0$ or $ac \neq 0$, then $U(a,b,c) = \Lambda(a,b,c)$ is a local module, thus indecomposable. $\square$?

(2.6) Proposition. *Any module of dimension at most 3 has Loewy length at most 2.*

Proof. Let M be a module of dimension at most 3. If M is not local, then clearly M has Loewy length at most 2. If $\dim M \leq 2$, then again M has Loewy length at most 2. Thus, we can assume that M is 3-dimensional and local and therefore a factor module of Λ, say $M = \Lambda/U$. According to (2.4), $\operatorname{soc}\Lambda \subseteq U$, thus M is annihilated by $\operatorname{soc}\Lambda$, and therefore M has Loewy length at most 2. $\square$

(2.7) Lemma. *Any indecomposable torsionless module M of dimension at most 3 is local and isomorphic to a left ideal of Λ. If $\dim M = 3$, then M is of the form $U(a,b,c)$.*

Proof. Let M be indecomposable and torsionless. If $\dim M \leq 2$, then M is of course local and isomorphic to a left ideal. Thus we can assume that $\dim M = 3$.

Since M is torsionless, there is a set of non-zero maps $u_i\colon M \to {}_\Lambda\Lambda$ (say with index set I) such that $\bigcap_{i\in I} K_i = 0$, where K_i is the kernel of u_i.

If $K_i = 0$ for some i, then already u_i is an embedding (thus M is isomorphic to a left ideal). In particular, if the socle of M is simple, then we must have $K_i = 0$ for some i.

Thus, we can assume that the socle of M is not simple. Therefore M has to be a local module and we have a surjective map $\pi\colon {}_\Lambda\Lambda \to M$.

It remains to look at the case where $\dim K_i = 1$ or 2 for all i. Since the only 2-dimensional submodule of M is its radical, we have $\bigcap_{i\in I'} K_i = 0$, where I' is the set of indices i with $\dim K_i = 1$. But then $K_i \cap K_j = 0$ for some $i \neq j$ in I'. This shows that we can assume that $I = \{1,2\}$ and that K_1, K_2 are different 1-dimensional submodules of M.

Now u_i provides an isomorphism from M/K_i onto a (2-dimensional) left ideal of Λ. Since M/K_i is indecomposable, (2.2) shows that M/K_i is isomorphic to $\Lambda(1,-1,0)$ or to $\Lambda(0,0,1)$. Let $K_i' = \operatorname{Ker}(u_i\pi)$ for $i = 1,2$.

If $M/K_i \simeq \Lambda(1,-1,0)$, then K_i' is equal to $\Lambda(x+qy)+\Lambda z$, since $\Lambda(0,0,1)$ is annihilated by $x+qy$ and by z. Similarly, if $M/K_i \simeq \Lambda(0,0,1)$, then K_i' is equal to $\Lambda(x+qy)+\Lambda z$. Thus one of M/K_i has to be isomorphic to $\Lambda(1,-1,0)$, the other one to $\Lambda(0,0,1)$ and $\operatorname{Ker}(\pi) = K_1' \cap K_2' = U(0,0,1)$. It follows that $M \simeq {}_\Lambda\Lambda/\operatorname{Ker}(\pi) = {}_\Lambda\Lambda/U(0,0,1)$. But ${}_\Lambda\Lambda/U(0,0,1)$ is isomorphic to the left ideal $\Lambda(1,-1,1) = U(1,-1,1)$.

We have shown that M is isomorphic to a left ideal, thus of the form $U(a,b,c)$, see (2.5). Since we assume that M is indecomposable, (2.5) asserts that M is local. □

We need to know also the right ideals $(a,b,c)\Lambda$. Note that $U(a,b,c)$ is always a twosided ideal and it will be pertinent to denote $U(a,b,c)$ by $U'(a,b,c)$, if we consider it as a right ideal (thus as a right module).

(2.8) The right ideals $(a,b,c)\Lambda$. *If $a \neq 0$ or $bc \neq 0$, then $(a,b,c)\Lambda = U(a,b,c)$ is 3-dimensional. The right ideals $(0,1,0)\Lambda$ and $(0,0,1)\Lambda$ are 2-dimensional with* $\operatorname{soc}(0,1,0)\Lambda = yx\Lambda$ *and* $\operatorname{soc}(0,0,1)\Lambda = zx\Lambda$.

Proof: Let $V = (a,b,c)\Lambda$. First, let $a \neq 0$. Then zx belongs to V, since $(a,b,c)z = azx$. Also $yx \in V$, since $(a,b,c)y = -qayx + czx$. Second, assume that $a = 0$ and $bc \neq 0$. Then $(0,b,c)y = czx$ shows that $zx \in V$, and $(0,b,c)x = byx + czx$ shows that also $yx \in V$. □

(2.9) *If a 3-dimensional indecomposable right module N is torsionless, then it is isomorphic to a right ideal, thus to $U'(a,b,c)$ for some $(a,b,c) \neq 0$.*

Proof. Let N be a 3-dimensional indecomposable torsionless right module. As in (2.7) one shows that N is isomorphic to a right ideal, using (2.8) instead of (2.2). It remains to see that all 3-dimensional right ideals are of the form $U'(a,b,c)$. Here, one has to copy the proof of (2.5).

3. The transformations ω and ω'.

If $(a\!:\!b\!:\!c)$ is different from $(1\!:\!(-1)\!:\!0)$ and $(0\!:\!0\!:\!1)$, then (2.5) shows that $U(a,b,c)$ is a 3-dimensional local module, thus of the form $M(a'\!:\!b'\!:\!c')$. In order to describe in which way $(a'\!:\!b'\!:\!c')$ depends on $(a\!:\!b\!:\!c)$, we will need the transformations ω and ω'. We start with some equalities in Λ.

(3.1) Formulae. *Let $a,b,c \in k$. Then*

$$(1)\qquad \big(a\,x + qb\,y - \tfrac{a}{a+b}c\,z\big)\big(a\,x + b\,y + c\,z\big) = 0 \quad \textit{if} \quad a+b \neq 0$$

$$(2)\qquad z(ax - ay + cz) = 0$$

$$(3)\qquad \big(a\,x + b\,y + c\,z\big)\big(a\,x + q^{-1}b\,y - \tfrac{a+q^{-1}b}{a}c\,z\big) = 0 \quad \textit{if} \quad a \neq 0$$

$$(4)\qquad (by + cz)z = 0$$

Proof of the equality (1):

$$\begin{aligned}
&\big(a\,x{+}qb\,y - \tfrac{a}{a+b}c\,z\big)\big(a\,x + b\,y + c\,z\big)\\
&\quad= ab\,xy + ac\,xz + qab\,yx - \tfrac{a}{a+b}ac\,zx - \tfrac{a}{a+b}bc\,zy\\
&\quad= ab\,(xy + q\,yx) + \big(1 - \tfrac{a}{a+b} - \tfrac{b}{a+b}\big)ac\,zx \;=\; 0.
\end{aligned}$$

The proof of the remaining equalities is similar. □

(3.2) In case $a+b \neq 0$, let $\omega(a,b,c) = (a, qb, -\frac{a}{a+b}c)$. In case $a' \neq 0$, let $\omega'(a',b',c') = (a', q^{-1}b', -\frac{a'+q^{-1}b'}{a'}c')$.

Proposition. *The transformation ω provides a bijection from the set $\{(a,b,c) \in k^3 \mid a(a+b) \neq 0\}$ onto the set $\{(a',b',c') \in k^3 \mid a'(a'+q^{-1}b') \neq 0\}$, with inverse ω'.*

Proof. Let $a(a+b) \neq 0$. Then $(a',b',c') = \omega(a,b,c)$ is defined and $a' = a \neq 0$, and $a' + q^{-1}b' = a + q^{-1}qb = a + b \neq 0$. Thus ω maps $\{(a,b,c) \in k^3 \mid a(a+b) \neq 0\}$ into $\{(a,b,c) \in k^3 \mid a(a+q^{-1}b) \neq 0$. Similarly, ω' maps $\{(a',b',c') \in k^3 \mid a'(a'+q^{-1}b') \neq 0$ into $\{(a,b,c) \in k^3 \mid a(a+b) \neq 0$. It is easy to check that $\omega'\omega(a,b,c) = (a,b,c)$ for $a(a+b) \neq 0$ and that $\omega\omega'(a',b',c') = (a',b',c')$ for $a'(a'+q^{-1}b') \neq 0$. □

4. The isomorphism class of $U(a,b,c) \simeq \Omega M(a,b,c)$.

(4.1) Proposition. *Let $(a,b,c) \neq 0$. Then*

$$\Omega M(a,b,c) \simeq \begin{cases} M(\omega(a,b,c)) & \text{if} \quad a \neq 0,\ a+b \neq 0, & (1)\\ M(0,0,1) & \text{if} \quad a \neq 0,\ a+b = 0,\ c \neq 0, & (2)\\ \Lambda(x-y) \oplus \Lambda zx & \text{if} \quad a \neq 0,\ a+b = 0,\ c = 0, & (3)\\ M(0,1,0) & \text{if} \quad a = 0,\ b \neq 0, & (4)\\ \Lambda z \oplus \Lambda yx & \text{if} \quad a = 0,\ b = 0. & (5) \end{cases}$$

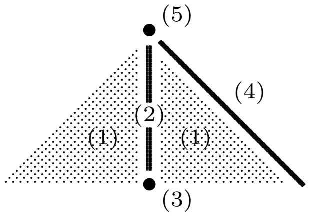
Proof: If $a = 0$ and $b = 0$, then $U(a,b,c) = U(0,0,1)$. If $a+b = 0$ and $c = 0$, then $U(a,b,c) = U(1,-1,0)$. According to (2.3), $U(0,0,1) = \Lambda z \oplus \Lambda yx$ and $U(1,-1,0) = \Lambda(x-y) \oplus \Lambda zx$, This shows (5) and (3). In this way, we have considered all triples (a,b,c) with $a+b = 0$ and $ac = 0$.

Thus, let $a+b \neq 0$ or $ac \neq 0$. By (2.5), $U(a,b,c) = \Lambda(a,b,c)$ is local and we look at the surjective map $\phi\colon {}_\Lambda\Lambda \to U(a,b,c)$ which sends 1 to (a,b,c).

Let $a+b \neq 0$. According to formula (1) of (3.1), $\Lambda(a,b,c)$ is annihilated by $\omega(a,b,c)$, thus $M(\omega(a,b,c))) = {}_\Lambda\Lambda/\Lambda(\omega(a,b,c))$ maps onto $\Lambda(a,b,c)$. Since the modules $M(\omega(a,b,c))$ and $\Lambda(a,b,c)$ both have dimension 3, we see that $U(a,b,c) = \Lambda(a,b,c)$ is isomorphic to $M(\omega(a,b,c))$. This yields (1) and (4) (namely, if $a = 0$, and $b \neq 0$, we have $\omega(0,b,c) = (0,qb,0)$).

Finally, we show (2). For $c \neq 0$, the module $U(1,-1,c)$ is isomorphic to $M(0,0,1)$. Now we use in the same way formula (2) of (3.1). □

The following picture outlines the position of the partition of $\mathbb{P}^2$ which is used in the Proposition.



(4.2) Corollary. *The syzygy functor Ω provides a bijection from the set of isomorphism classes of modules $M(a,b,c)$ with $a(a+b) \neq 0$ onto the set of isomorphism classes of modules $M(a',b',c')$ with $a'(a'+q^{-1}b') \neq 0$ and we have $\Omega M(a,b,c) = M(\omega(a,b,c))$ for $a(a+b) \neq 0$.*

Proof. This follows directly from Propositions (3.2) and (4.1). □

5. The extensionless modules $M(a,b,c)$.

(5.1) Proposition. *The module $M(a,b,c)$ is extensionless if and only if $a(a+b) \neq 0$.*

For the proof, we need some preparations.

(5.2) Lemma. *The following conditions are equivalent:*

(i) *The module $M(a,b,c)$ is extensionless.*

(ii) *The inclusion map $\iota\colon U(a,b,c) \to {}_\Lambda\Lambda$ is a left* add(Λ)*-approximation.*

(iii) *$U(a,b,c) = \Lambda(a,b,c)$ and the inclusion map $\iota\colon \Lambda(a,b,c) \to {}_\Lambda\Lambda$ is a left* add(Λ)*-approximation.*

(iv) *The subspace $U(a,b,c)$ is indecomposable both as a left module and as a right module, and the image of every homomorphism ${}_\Lambda U(a,b,c) \to {}_\Lambda\Lambda$ is contained in $U(a,b,c)$.*

Proof. The equivalence of (i) and (ii) follows from Part I, Lemma 2.1.

(ii) $\implies$ (iii): We assume (ii). If $U(a,b,c) = U_1 \oplus U_2$ with U_1, U_2 both non-zero, then a minimal left add(Λ)-approximation $U(a,b,c) \to \Lambda^t$ is the direct sum of minimal left add(Λ)-approximations $U_1 \to \Lambda^{t_1}$ and $U_2 \to \Lambda^{t_2}$, thus $t = t_1 + t_2 \geq 2$. This shows that $U(a,b,c)$ is indecomposable. According to (2.5), this means that $U(a,b,c) = \Lambda(a,b,c)$.

(iii) $\implies$ (iv). Since $\Lambda(a,b,c)$ is a local module, it is indecomposable. Thus $U(a,b,c) = \Lambda(a,b,c)$ implies that $U(a,b,c)$ considered as a left module is indecomposable. Given any homomorphism $\phi\colon U(a,b,c) \to {}_\Lambda\Lambda$, (iii) provides $\lambda \in \Lambda$ with $\phi(a,b,c) = (a,b,c)\lambda \in (a,b,c)\Lambda \subseteq U(a,b,c)$. Now assume that $(a,b,c)\Lambda$ is a proper subset of $U(a,b,c)$. Let $w \in \operatorname{soc}\Lambda$. Since Λw is simple, there is a homomorphism $\phi\colon \Lambda(a,b,c) \to \Lambda$ with $\phi(a,b,c) = w$ and (iii) asserts that $w = \phi(a,b,c) = (a,b,c)\lambda$ for some $\lambda \in \Lambda$. This shows that $\operatorname{soc}\Lambda \subseteq (a,b,c)\Lambda$ and therefore $U(a,b,c) = (a,b,c)\Lambda$. In particular, $U(a,b,c)$ is indecomposable also as a right Λ-module.

(iv) $\implies$ (ii). Let $\phi\colon U(a,b,c) \to {}_\Lambda\Lambda$ be a homomorphism. Since $U(a,b,c)$ is indecomposable as a left module, we have $U(a,b,c) = \Lambda(a,b,c)$. Since $U(a,b,c)$ is indecomposable as a right module, we have $U(a,b,c) = (a,b,c)\Lambda$. According to (iv), $\phi(a,b,c) \in U(a,b,c) = (a,b,c)\Lambda$, thus $\phi(a,b,c) = (a,b,c)\lambda = r_\lambda\iota(a,b,c)$ for some $\lambda \in \Lambda$, where $r_\lambda\colon {}_\Lambda\Lambda \to {}_\Lambda\Lambda$ is the right multiplication by λ. Since the left module $U(a,b,c) = \Lambda(a,b,c)$ is generated by (a,b,c), the equality $\phi(a,b,c) = r_\lambda\iota(a,b,c)$ implies that $\phi = r_\lambda\iota$. □

(5.3) Lemma. *Let R be a ring and X a left R-module. If $\phi\colon {}_RR \to X$ is an R-module homomorphism and $w \in R$ annihilates X, then $Rw \subseteq \operatorname{Ker}\phi$.*

Corollary. *Let L be a left ideal of R and X an R-module annihilated by $w_1, \ldots, w_t \in R$. The image of any map $R/L \to X$ is a factor module of $R/(L + Rw_1 + \cdots Rw_t)$.*

Proof. Let $\phi\colon R/L \to X$ be a homomorphism. Let $\pi\colon R \to R/L$ be the canonical projection. By construction, L is contained in $\operatorname{Ker}(\phi\pi)$. By the lemma, also the left ideals Rw_i are contained in $\operatorname{Ker}(\phi\pi)$. Thus $L + Rw_1 + \cdots + Rw_t \subseteq \operatorname{Ker}(\phi\pi)$. □

(5.4) Proof of Proposition (5.1). According to (5.2), $M(a, b, c)$ is extensionless if and only if condition (iv) is satisfied. We look at all the elements $(a\,{:}\,b\,{:}\,c) \in \mathbb{P}^2$, using the partition of $\mathbb{P}^2$ into the subsets (1) to (5) as in (4.1).

The cases (3) and (5): Both $U(1, -1, 0)$ and $U(0, 0, 1)$ are decomposable as left modules, see (2.5). Case (4): According to (4.1), $U(0, 1, c) \simeq M(0, 1, 0)$. Obviously, $M(0, 1, 0)$ has Λz as a factor module, thus there is a homomorphism $U(0, 1, c) \to {}_\Lambda\Lambda$ with image Λz and $\Lambda z \not\subseteq U(0, 1, c)$. The case (2) is similar: (4.1) shows that $U(1, -1, c) \simeq M(0, 0, 1)$, and $M(0, 0, 1)$ maps onto Λz; thus there is a homomorphism $U(1, -1, c) \to {}_\Lambda\Lambda$ with image Λz and $\Lambda z \not\subseteq U(1, -1, c)$. This shows that none of the modules $M(a, b, c)$ with $a(a + b) = 0$ is extensionless.

It remains to consider the case (1). Thus, assume that $a(a + b) \neq 0$. Let $(1, b', c') = \omega(1, b, c)$, thus $b' = qb$. We want to show that the conditions (iv) of (5.2) are satisfied. According to (2.5) and (2.8), $U(a, b, c)$ is indecomposable both as a left module and as a right module, It remains to show that the image of every homomorphism ${}_\Lambda U(a, b, c) \to {}_\Lambda\Lambda$ is contained in $U(a, b, c)$.

(a) *The only left ideal isomorphic to $U(1, b, c)$ is $U(1, b, c)$ itself.* Proof. The 3-dimensional left ideals are of the form $U(a'', b'', c'')$, for some $(a'', b'', c'') \neq 0$, see (2.5). Assume that $U(1, b, c) \simeq U(a'', b'', c'')$. We have $U(a'', b'', c'') \simeq \Omega M(a'', b'', c'')$ and by (4.1) we must be in case (1), namely $a'' \neq 0$ and $a'' + b'' \neq 0$. In particular, we may assume that $a'' = 1$ and (4.1)(1) asserts that $\Omega M(1, b'', c'') = M(\omega(1, b'', c''))$. The isomorphy $M(\omega(1, b, c)) \simeq M(\omega(1, b'', c''))$ implies that the triples $\omega(1, b, c)$ and $\omega(1, b'', c'')$ yield the same element in $\mathbb{P}^2$, and since the first coordinate of both triples is equal to 1, we have $\omega(1, b, c) = \omega(1, b'', c'')$. Since $1 + b \neq 0$ and $1 + b'' \neq 0$, we use (3.2) in oder to conclude that $(1, b, c) = (1, b'', c'')$.

(b) *The left ideal Λz is not a factor module of $U(1, b, c)$.* The proof uses Corollary (5.3) for the left ideal $L = U(1, b', c')$ and the module $X = \Lambda z$ which is annihilated by y and z. Namely, on the one hand, we have $U(1, b, c) \simeq \Omega M(1, b, c) \simeq M(\omega(1, b, c)) = M(1, b', c') = \Lambda/U(1, b', c') = \Lambda/L$. On the other hand, $\operatorname{rad}\Lambda = \Lambda(x + b'y + c'z) + \Lambda y + \Lambda z \subseteq U(1, b', c') + \Lambda y + \Lambda z \subseteq \operatorname{rad}\Lambda$ shows that $L + \Lambda y + \Lambda z = \operatorname{rad}\Lambda$. Therefore, (5.3) asserts that the image of any homomorphism $U(1, b, c) \to \Lambda z$ is a factor module of $\Lambda/\operatorname{rad}\Lambda$, thus simple or zero.

(c) *The left ideal $\Lambda(x - y)$ is not a factor module of $U(1, b, c)$.* Again, we use Corollary (5.3) for $L = U(1, b', c')$ and now for $X = \Lambda(x - y)$. Note that $\Lambda(x - y)$ is annihilated by $x - qy$ and z. We recall from (b) that $U(1, b, c) \simeq \Lambda/L$. And we have $\operatorname{rad}\Lambda = \Lambda(x + b'y + c'z) + \Lambda(x - qy) + \Lambda z$, since $b' = qb \neq -q$. Therefore, we also have $U(1, b', c') + \Lambda(x - qy) + \Lambda z = \operatorname{rad}\Lambda$, and (5.3) asserts that the image of any homomorphism $U(1, b, c) \to \Lambda z$ is simple or zero.

Any homomorphism $\phi\colon U(1,b,c) \to {}_\Lambda\Lambda$ *maps into* $U(1,b,c)$. Proof. According to (b) and (c), the image I of ϕ is not of dimension 2. If the image I is of dimension 3, then (a) shows that I is equal to $U(1,b,c)$. Of course, if I is of dimension at most 1, then $I \subseteq \operatorname{soc}\Lambda \subseteq U(1,b,c)$. $\square$

(5.5) Corollary. *If* $M(a,b,c)$ *is extensionless, then* $\Omega M(a,b,c) \simeq M(\omega(a,b,c))$.

Proof. This follows directly from (5.1) and the case (1) of (4.1). $\square$.

6. The torsionless modules $M(a,b,c)$.

(6.1) Proposition. *The module* $M(a,b,c)$ *is torsionless if and only if either* $a(a+q^{-1}b) \neq 0$ *or else* $a = 0$ *and* $bc = 0$ (so that $(a\!:\!b\!:\!c)$ is equal to $(0\!:\!1\!:\!0)$ or to $(0\!:\!0\!:\!1)$).

In order to prove (6.1), we consider the possible cases separately. First, we consider the modules $M(a,b,c)$ with $a \neq 0$. In section 5 we have seen that $M(1,b,c)$ is extensionless if and only if $b \neq -1$, and then $\Omega M(1,b,c) \simeq M(\omega(1,b,c))$. There is the following corresponding assertion concerning the torsionless modules (see also (7.1)).

(6.2) *The module* $M(1,b,c)$ *is torsionless if and only if* $b \neq -q$, *and in this case* $\mho M(1,b,c) \simeq M(\omega'(1,b,c))$.

Proof. Let $b \neq -q$. Then $\omega'(1,b,c) = (1,q^{-1}b,c')$ for some c'. According to (5.1) and (5.5), $M(1,q^{-1}b,c')$ is extensionless and $\Omega M(1,q^{-1}b,c') \simeq M(1,b,c)$, since $\omega(1,q^{-1}b,c') = \omega\omega'(1,b,c) = (1,b,c)$. This shows that $M(1,b,c)$ is torsionless and that $\mho M(1,b,c) \simeq M(\omega'(1,b,c))$.

Conversely, we consider $M(1,-q,c)$ and assume, for the contrary, that $M(1,-q,c)$ is torsionless. According to (2.7), this means that $M(1,-q,c)$ is isomorphic to a left ideal $U(a',b',c') = \Omega M(a',b',c')$. According to (4.1), we must be in the case $a'+b' \neq 0$ and $a' \neq 0$. We can assume that $a' = 1$, thus $1+b' \neq 0$. We have $\Omega M(1,b',c') \simeq M(\omega(1,b',c')) = M(1,qb',c'')$ for some c''. Since $M(1,-q,c) \simeq \Omega M(1,b',c') \simeq M(1,qb',c'')$, we see that $(1,-q,c) = (1,qb',c'')$, thus $b' = -1$. But this is a contradiction to $1+b' \neq 0$. $\square$

(6.3) *For* $M = M(0,1,0)$ *and* $M(0,0,1)$, *there is no monomorphism* $M \to {}_\Lambda\Lambda$ *which is an* $\operatorname{add}(\Lambda)$*-approximation.*

Proof. Let M be equal to $M(0,1,0)$ or to $M(0,0,1)$. Assume that there is a monomorphism $u\colon M \to {}_\Lambda\Lambda$ which is an $\operatorname{add}(\Lambda)$-approximation. The image $u(M)$ is a 3-dimensional left ideal, thus of the form $U(a,b,c)$ for some $(a,b,c) \neq 0$, see (2.7). The implication (ii) $\Longrightarrow$ (iv) in (5.2) asserts that any homomorphism $U(a,b,c) \to {}_\Lambda\Lambda$ maps into $U(a,b,c)$.

Obviously, both modules $M(0,1,0)$ and $M(0,0,1)$ have a factor module isomorphic to Λz, thus there is a surjective homomorphism $U(a,b,c) \to \Lambda z$, and therefore $\Lambda z \subseteq U(a,b,c)$. But Λz is an indecomposable module of length 2, and $U(a,b,c) \simeq M$ is a local module of length 3 with socle of length 2. A local module of length 3 with socle of length 2 has no indecomposable submodule of length 2, thus we obtain a contradiction. $\square$

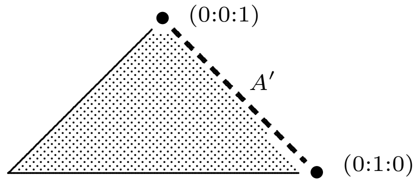
(6.4) Proposition. *The modules* $M(0,b,c)$ *with* $bc \neq 0$ *are not torsionless.*

Proof. Let $M = M(0,b,c)$ with $bc \neq 0$ and assume that M is torsionless. According to (2.7), this means that $M \simeq U(a',b',c') \simeq \Omega M(a',b',c')$ for some triple (a',b',c'), and (2.5) asserts that $a'+b' \neq 0$ or $a'c' \neq 0$. Now we use (4.1) and have to distinguish the three cases (1), (2) and (4). Case (1) means that $a' + b' \neq 0$ and $a' \neq 0$, then $\Omega M(a', b', c') \simeq M(\omega(a', b', c'))$ and the first component of $\omega(a', b', c')$ is a', thus non-zero. But then $M(\omega(a', b', c'))$ cannot be isomorphic to $M(0, b, c)$. Case (4) means that $a' = 0$ and $b' \neq 0$. Then $\Omega M(a', b', c') \simeq M(0, 1, 0)$, thus not isomorphic to $M(0, b, c)$ with $bc \neq 0$. Finally, there is the case (2) with $a' + b' = 0$ and $a'c' \neq 0$. Then $\Omega M(a', b', c') \simeq M(0, 0, 1)$, again not isomorphic to $M(0, b, c)$ with $bc \neq 0$. In all cases, we get a contradiction. □

(6.5) Proposition. *If M is equal to $M(0, 1, 0)$ or $M(0, 0, 1)$, then M is torsionless and the module $\mho M$ has Loewy length 3. Since $\mho M$ is indecomposable and non-projective, it is not torsionless.*

Proof. The modules M of the form $M(0, 1, 0)$ and $M(0, 0, 1)$ are torsionless, since (4.1), (4) and (2) assert that $M(0, 1, 0) \simeq \Omega M(0, 1, 0)$ and that $M(0, 0, 1) \simeq \Omega M(1, -1, 1)$. According to (5.2), in both cases there is no inclusion map $M \to \Lambda$ which is an add(Λ)-approximation. Thus, a minimal left add(Λ)-approximation of M is an injective map $M \to \Lambda^t$ with $t \geq 2$. This shows that $\mho M$ has dimension $6t - 3$ and its top has dimension t. According to Part I (3.2), $\mho M$ is indecomposable and not projective. The Loewy length of $\mho M$ has to be 3. [Namely, an indecomposable module with Loewy length at most 2 and top of dimension $t \geq 2$ has dimension at most $4t - 1$, since it is a proper factor module of $\overline{\Lambda}^t$. But $6t - 3 \leq 4t - 1$ implies $t \leq 1$, a contradiction.] An indecomposable non-projective module of Loewy length 3 cannot be torsionless. □

(6.6) We finish this section by reformulating the results concerning the modules of the form $M(0, b, c)$ in terms of $\mho$-components. Here, we will exhibit the structure of all the $\mho$-components containing modules of the form $M(0, b, c)$. We have to distinguish between the modules $M(0, 1, 0)$ and $M(0, 0, 1)$ and the modules $M(0, b, c)$ with $bc \neq 0$, thus lying on the dashed line $A' = \{(0\!:\!b\!:\!c) \mid bc \neq 0\}$:



The modules in A' are singletons (that is, components of type $\mathbb{A}_1$) in the $\mho$-quiver. And, there are the following two $\mho$-components of the form $\mathbb{A}_2$:

$$\underset{M(0,0,1)}{\blacklozenge} \dashleftarrow \underset{\mho M(0,0,1)}{\blacksquare} \qquad\qquad \underset{M(0,1,0)}{\blacklozenge} \dashleftarrow \underset{\mho M(0,1,0)}{\blacksquare}$$

(If M is an indecomposable module, then we represent $[M]$ in the $\mho$-quiver usually just by a circle ∘. We use a bullet • in case we know that M is torsionless and extensionless, a black square ■ in case we know that M is extensionless, but not torsionless; and a black lozenge ♦ in case we know that M is torsionless, but not extensionless.)

7. The modules $M(1, b, c)$ and proof of Theorem (1.5).

We consider now the affine subspace H of $\mathbb{P}^2$ given by the points $(1\!:\!b\!:\!c)$ with $b, c \in k$ and the corresponding modules $M(1, b, c)$. We recall that $o(q)$ denotes the multiplicative order of q.

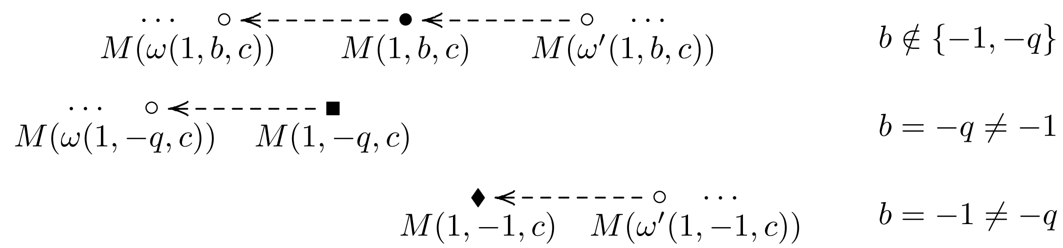
(7.1) We have seen in (4.2) that Ω provides a bijection from the set of modules $M(1,b,c)$ with $b \neq -1$ onto the set of modules $M(1,b',c')$ with $b' \neq -q$. The sections 5 and 6 strengthen this bijection as follows:

If $b \neq -1$, then the exact sequence

$$0 \to M(1,b',c') \to {}_\Lambda\Lambda \to M(1,b,c) \to 0$$

with $(1,b',c') = \omega(1,b,c)$ is an $\mho$-sequences (here, $(1,b',c')$ is an arbitrary triple with $b' \neq -q$, and $(1,b,c) = \omega'(1,b',c')$). We obtain in this way all the $\mho$-sequences involving modules of the form $M(1,b,c)$.

(7.2) Reformulation. The neighborhood of $M(1,b,c)$ in the $\mho$-quiver looks like:



and $M(1,b,c)$ is a singleton in the $\mho$-quiver if $q = 1$ and $b = -1$.

(7.3) *The module $M(1,b,c)$ is semi-Gorenstein-projective if and only if $b \neq -q^t$ for all $t \leq 0$. The module $M(1,b,c)$ is ∞-torsionfree if and only if $b \neq -q^t$ for all $t \geq 1$.*

Proof: $M(1,b,c)$ is semi-Gorenstein-projective if and only if $\omega^s(1,b,c) \notin E$ for all $s \geq 0$. Since $\omega^s(1,b,c) = (1, q^s b, c_s)$ for some $c_s \in k$, we see that $M(1,b,c)$ is semi-Gorenstein-projective if and only if $1 + q^s \neq 0$ for all $s \geq 0$, thus if and only if $q^{-s} \neq -b$ for all $s \geq 0$. Write $t = -s$.

Similarly, $M(1,b,c)$ is ∞-torsionfree if and only if $\omega^{-s}(1,b,c) \notin T$ for all $s \geq 0$, thus if and only if $1 + q^{-1}q^{-s}b \neq 0$ for all $s \geq 0$, if and only if $-b \neq q^{s+1}$ for all $s \geq 0$. Write $t = s+1$. □

Corollary. *The module $M(1,b,c)$ is Gorenstein-projective if and only if $b \neq -q^t$ for all $t \in \mathbb{Z}$.*

(7.4) *Any module $M(1,0,c)$ with $c \in k$ is Gorenstein-projective with Ω-period* 1 *or* 2.

Proof. According to (6.2), the modules $M(1,0,c)$ are extensionless and torsionless. Since $\omega(1,0,c) = (1,0,-c)$, we see that $M(1,0,0)$ has Ω-period 1, and $M(1,0,c)$ with $c \neq 0$ has Ω-period 2 in case the characteristic of k is different from 2, otherwise its Ω-period is also 1. □

(7.5) Proposition. *If $o(q) = \infty$, then any module of the form $M(1,b,c)$ is semi-Gorenstein-projective or ∞-torsionfree* (whereas the modules of the form $M(0,b,c)$ are never semi-Gorenstein-projective nor ∞-torsionfree).

Proof. The first assertion follows immediately from (7.3), the additional assertion in the bracket is a consequence of (5.1), (6.4) and (6.5). □

(7.6) Proposition. *If $M(1,b,c)$ belongs to an $\mho$-component of the form $\mathbb{A}_n$, then $o(q)=n$.*

Proof. We consider an $\mho$-component of type $\mathbb{A}_n$, say containing a module M which is not torsionless. Since M belongs to T, we have $M=M(1,-q,c)$ and the component consists of the modules $M,\ \Omega M,\ \dots,\ \Omega^{n-1}M$. In particular, $\omega^{n-1}(1,-q,c)$ belongs to E. Now $\Omega^{n-1}M = M(\omega^{n-1}(1,-q,c)) = M(1,-q^n,c')$ for some c'. Since $\Omega^{n-1}M$ is not extensionless, $(1,-q^n,c')$ belongs to E, thus $-q^n=-1$. This shows that $q^n=1$. Finally, for $1\le t<n$, we have $q^t\neq 1$, since otherwise $\omega^{t-1}(1,-q,c)$ would belong to E. $\square$

Corollary. *If $o(q)=\infty$, then all the $\mho$-components in H are cycles or of type $\mathbb{Z}$, or $-\mathbb{N}$, or $\mathbb{N}$. Thus, any module in H is semi-Gorenstein-projective or ∞-torsionfree.*

For $o(q)=\infty$, there are the following $\mho$-components of the form $-\mathbb{N}$ and $\mathbb{N}$:

$$\begin{array}{ccccccc} \cdots & \bullet & \leftarrow & \bullet & \leftarrow & \blacksquare \\ & M(1,-q^3,c_3) & & M(1,-q^2,c_2) & & M(1,-q,c_1) \end{array}$$

$$\begin{array}{ccccccc} \blacklozenge & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \cdots \\ M(1,-1,d_0) & & M(1,-q^{-1},d_1) & & M(1,-q^{-2},d_2) & & \end{array}$$

with arbitrary elements $c_0,d_1\in k$ and $c_{t+1}=-\frac{1}{1-q^t}c_t$ for $t\ge 1$, whereas $d_{t+1}=-(1-q^{-t})d_t$ for $t\ge 0$. Of course, $(1,-q,c_1)\in T$ and $(1,-1,d_0)\in E$, thus the module $M(1,-q,c_1)$ is pivotal semi-Gorenstein-projective, whereas $M(1,-1,d_0)$ is pivotal ∞-torsionfree.

(7.7) The case that q has finite multiplicative order. *Now let $o(q)=n<\infty$. Then the modules $M(1,-q^t,c)$ with $0\le t<n$ and $c\in k$ belong to $\mho$-components of the form $\mathbb{A}_n$.* These $\mho$-components look as follows:

$$\begin{array}{ccccccccc} \blacklozenge & \leftarrow & \bullet & \leftarrow & \cdots & \bullet & \leftarrow & \blacksquare \\ M(1,-1,c_n) & & M(1,-q^{n-1},c_{n-1}) & & & M(1,-q^2,c_2) & & M(1,-q,c_1) \end{array}$$

with an arbitrary element $c_1\in k$ and $c_{t+1}=-\frac{1}{1-q^t}c_t$ for $1\le t<n$ (of course, $(1,-1,c_n)\in E$ and $(1,-q,c_1)\in T$).

Corollary (7.3) asserts that the remaining modules $M(1,b,c)$ (those with $-b\notin q^{\mathbb{Z}}$) are Gorenstein-projective.

(7.9) Proof of Theorem (1.5).

Torsionless modules: According to (2.7), an indecomposable torsionless module is isomorphic to a left ideal. Of course, k is torsionless. According to (2.2), a 2-dimensional indecomposable left ideal is isomorphic to $\Lambda(x-y)$ or Λz. According to (2.3), a 3-dimensional indecomposable torsionless module has to be local, thus it is of the form $M(a,b,c)$, and (6.1) says that $a(a+q^{-1}b)\neq 0$ or else $M(a,b,c)$ is equal to $M(0,1,0)$ or to $M(0,0,1)$.

Extensionless modules: We show: *An indecomposable module M of dimension at most 3 with simple socle is not extensionless.*

Of course, $\mathrm{Ext}^1(k,\Lambda)\neq 0$, since otherwise we would have $\mathrm{Ext}^1(X,\Lambda)=0$ for all modules X.

Let I be an indecomposable module of length 2. A projective cover of I as an $\overline{\Lambda}$-module provides an exact sequence $0 \to k^2 \to \overline{\Lambda} \to I \to 0$. We apply $\operatorname{Hom}_{\overline{\Lambda}}(-, J)$, where $J = \operatorname{rad}\Lambda$. We obtain the exact sequence

$$0 \to \operatorname{Hom}_{\overline{\Lambda}}(I, J) \to \operatorname{Hom}_{\overline{\Lambda}}(\overline{\Lambda}, J) \to \operatorname{Hom}_{\overline{\Lambda}}(k^2, J) \to \operatorname{Ext}^1_{\overline{\Lambda}}(I, J) \to 0.$$

Now, $\dim \operatorname{Hom}_{\overline{\Lambda}}(I, J) \geq \dim \operatorname{Hom}_{\overline{\Lambda}}(k, J) = 2$, $\dim \operatorname{Hom}_{\overline{\Lambda}}(\overline{\Lambda}, J) = \dim J = 5$, and finally $\dim \operatorname{Hom}_{\overline{\Lambda}}(k^2, J) = 4$, thus $\dim \operatorname{Ext}^1_{\overline{\Lambda}}(I, J) \geq 1$. This shows that there exists a non-split exact sequence $\epsilon\colon 0 \to J \xrightarrow{u} E \to I \to 0$ with some $\overline{\Lambda}$-module E. The inclusion map $\iota\colon J \to \Lambda$ yields an induced exact sequence $\epsilon'\colon 0 \to \Lambda \to E' \to I \to 0$. Assume that ϵ' splits. Then we obtain a map $v\colon E \to \Lambda$ such that $vu = \iota$. Now E is an $\overline{\Lambda}$-module, thus of Loewy length at most 2. Therefore $v\colon E \to \Lambda$ maps into $\operatorname{rad}\Lambda = J$, thus $v = \iota v'$ for some $v'\colon E \to J$. But $\iota v' u = vu = \iota$ implies that $v'u$ is the identity map of E, thus ϵ splits, a contradiction. The exact sequence ϵ' shows that $\operatorname{Ext}^1_\Lambda(I, \Lambda) \neq 0$. Thus I is not extensionless.

A similar proof shows that $\operatorname{Ext}^1(V, \Lambda) \neq 0$ for any 3-dimensional module V with simple socle. Again, we use that V is an $\overline{\Lambda}$-module (see (1.3) Proposition 1), thus we start with an exact sequence $0 \to k^5 \to \overline{\Lambda}^2 \to V \to 0$.

This completes the proof that an indecomposable module M of dimension at most 3 with simple socle is not extensionless. The remaining indecomposable modules of dimension at most 3 are the modules of the form $M(1, b, c)$. According to (5.1) $M(1, b, c)$ is extensionless if and only if $b \neq -1$.

Reflexive modules: We recall from Part I that a module M is reflexive if and only if both M and $\mho M$ are torsionless. We show: *A module M with simple socle is not reflexive.* Assume that M has simple socle and is torsionless. Since M has simple socle, there is an embedding $M \to {}_\Lambda\Lambda$, say with cokernel Q. The elements yx and zx cannot both belong to $u(M)$, since the socle of $u(M)$ is simple. If $yx \notin u(M)$, then $yxQ \neq 0$, otherwise $zxQ \neq 0$. Let $f\colon M \to {}_\Lambda\Lambda^t$ be a minimal left $\operatorname{add}(\Lambda)$-approximation; its cokernel is $\mho M$. There is $u'\colon {}_\Lambda\Lambda^t \to \Lambda$ with $u'f = u$. The map u' has to be surjective, since otherwise u' would vanish on the socle of ${}_\Lambda\Lambda^t$. This implies that the map $\mho M \to Q$ induced by u' is also surjective. Since $\mho M$ is indecomposable, non-projective and not annihilated by $\operatorname{rad}^2\Lambda$, $\mho M$ cannot be torsionless.

Let us assume that M is reflexive and $\dim M \leq 3$. It follows that M has to be a torsionless module with $\dim M = 3$. Since also $\mho M$ has to be torsionless, (6.5) shows that the cases $M(0, 1, 0)$ and $M(0, 0, 1)$ are not possible, thus M is of the form $M(1, b, c)$ with $b \neq -q$. Using (6.2) and (6.1), we see that we also must have $b \neq -q^2$. Conversely, the same references show that all the modules $M(1, b, c)$ with $b \neq -q^i$ for $i = 1, 2$ are reflexive.

Semi-Gorenstein-projective and ∞-torsionfree modules. The semi-Gorenstein-projective modules are extensionless, the ∞-torsionfree modules are reflexive. The previous considerations therefore show that we only have to consider the modules of the form $M(1, b, c)$. (7.3) provides the conditions on b so that $M(1, b, c)$ is semi-Gorenstein-projective, ∞-torsionfree, or Gorenstein-projective.

If $M(1, b, c)$ is pivotal semi-Gorenstein-projective, then $M(1, b, c)$ is not torsionless, thus $b = -q$. If $M(1, -q, c)$ is semi-Gorenstein-projective, then $-q \neq -q^{-s}$ for all $s \geq 0$, thus $q^{s+1} \neq 1$ for all $s \geq 0$. This means that $o(q) = \infty$. Of course, there is also the converse: if $o(q) = \infty$, then $M(1, -q, c)$ is pivotal semi-Gorenstein-projective.

A similar argument shows that $M(1, b, c)$ is pivotal ∞-torsionfree if and only if $o(q) = \infty$ and $b = -1$. □

Remark. It seems worthwhile to note that *the set of modules $M(1, b, c)$ with $b, c \in k$ is a union of $\mho$-components.*

8. Right modules.

Recall that we write $U'(a, b, c)$ instead of $U(a, b, c)$, if we consider $U(a, b, c)$ as a right ideal. Let $M'(a, b, c) = \Lambda_\Lambda / U'(a, b, c)$, this is a right module (of course, the sets $M(a, b, c)$ and $M'(a, b, c)$ are the same, but we use the notation $M'(a, b, c)$ if we want to stress that we deal with a right module).

(8.1) Proposition. *Let $(a, b, c) \neq 0$. Then*

$$\Omega M'(a,b,c) \simeq \begin{cases} M'(\omega'(a,b,c)) & \text{if} \quad a \neq 0, & (1)\\ M'(0,0,1) & \text{if} \quad a = 0,\ bc \neq 0, & (2)\\ y\Lambda \oplus zx\Lambda & \text{if} \quad a = 0,\ c = 0, & (3)\\ z\Lambda \oplus yx\Lambda & \text{if} \quad a = 0,\ b = 0. & (4)\end{cases}$$

Proof. We have $\Omega M'(a, b, c) = U'(a, b, c)_\Lambda$. According to (2.8), $U'(a, b, c)_\Lambda = (a, b, c)\Lambda$ if $a \neq 0$ or $bc \neq 0$, and $U'(0, 1, 0) = y\Lambda \oplus zx\Lambda$, $U'(0, 0, 1) = z\Lambda \oplus yx\Lambda$.

Consider the map $\pi\colon \Lambda_\Lambda \to U'(a, b, c)$ defined by $\pi(1) = (a, b, c)$. We assume that $a \neq 0$ or $bc \neq 0$, thus π is surjective. If $a \neq 0$, the formula (3.1) (3) asserts that $\omega'(a, b, c)$ is in the kernel of π, thus π yields an epimorphism $M'(\omega'(a, b, c)) = \Lambda_\Lambda/\omega'(a, b, c)\Lambda \to U'(a, b, c)$. Since this is a map between 3-dimensional modules, it has to be an isomorphism.

If $a = 0$ and $bc \neq 0$, we use formula (3.1) (4) in order to get similarly an isomorphism $M'(0, 0, 1) = \Lambda_\Lambda/(0, 0, 1)\Lambda \to U'(0, b, c)$. □

(8.2) *If a 3-dimensional indecomposable right module N is torsionless and no embedding $N \to \Lambda_\Lambda$ is a left* $\operatorname{add}(\Lambda_\Lambda)$*-approximation, then $\mho N$ has Loewy length 3 and is not torsionless.*

Proof. Let $\phi\colon N \to \Lambda_\Lambda^t$ be a minimal left $\operatorname{add}(\Lambda_\Lambda)$-approximation of N. Since N is torsionless, ϕ is a monomorphism. By assumption, we must have $t \geq 2$. It follows that the cokernel $\mho N$ of ϕ is an indecomposable right Λ-module of length $6t-3$ with top of length t. But an indecomposable right Λ-module of Loewy length at most 2 with top of length $t \geq 2$ is a right $\overline{\Lambda}$-module of length at most $4t-1$. Thus $6t-3 \leq 4t-1$, therefore $2t \leq 2$, thus $t \leq 1$, a contradiction. This shows that $\mho N$ has Loewy length equal to 3. Of course, $\mho N$ is not projective. Since an indecomposable non-projective torsionless right Λ-module has Loewy length at most 2, we see that $\mho N$ cannot be torsionless. □

(8.3) The right modules $M'(0, b, c)$. *The only right module of the form $M'(0, b, c)$ which is torsionless is $M'(0, 0, 1)$. The right module $\mho M'(0, 0, 1)$ has Loewy length 3 and thus it is not torsionless. No right module of the form $M'(0, b, c)$ is extensionless.*

Proof. Let $N = M'(0, b, c)$.

(a) If N is torsionless, then $b = 0$ (thus $(0:b:c) = (0:0:1)$). Namely, According to (2.9), $M'(0,b,c)$ arises as a right ideal and (8.1) shows that this happens only for $b = 0$.

(b) *No embedding* $M'(0,0,1) \to \Lambda_\Lambda$ *is a left* $\mathrm{add}(\Lambda_\Lambda)$*-approximation.* Proof. Let $\phi\colon M'(0,0,1) \to \Lambda_\Lambda$ be an embedding. According to (2.9), the image of ϕ is of the form $U'(0,b,c)$ with $bc \neq 0$. Now $M'(0,0,1)$ has a factor module isomorphic to $(0,0,1)\Lambda$, thus there is $f\colon M'(0,0,1) \to \Lambda_\Lambda$ with image $(0,0,1)\Lambda$. If ϕ is a left $\mathrm{add}(\Lambda_\Lambda)$-approximation, then there exists $f' : \Lambda_\Lambda \to \Lambda_\Lambda$ with $f = f'\phi$. The homomorphism f' is the left multiplication by some element λ in Λ. If λ belongs to $\mathrm{rad}\,\Lambda$, then the image of $f'\phi$ is contained in $\mathrm{rad}^2\,\Lambda = \mathrm{soc}\,\Lambda$. If λ is invertible, then the image of $f'\phi$ is 3-dimensional. In both cases, we get a contradiction, since the image of f is $(0,0,1)\Lambda$, thus 2-dimensional and not contained in $\mathrm{soc}\,\Lambda$.

(c) It follows from (8.2) that $\mho M'(0,0,1)$ has Loewy length 3 and is not torsionless.

(d) A right module of the form $M'(0,b,c)$ is never extensionless: either $\Omega M'(0,b,c)$ is decomposable, or else $\Omega M'(0,b,c) = M'(0,0,1)$ and according to (b), no embedding $M'(0,0,1) \to \Lambda_\Lambda$ is a left $\mathrm{add}(\Lambda_\Lambda)$-approximation. □

Reformulation. *The right modules* $M'(0,1,c)$ *are singletons in the* $\mho$*-quiver. The right module* $M'(0,0,1)$ *belongs to an* $\mho$*-component of the form* $\mathbb{A}_2$*:*

$$\underset{M'(0,0,1)}{\blacklozenge} \longleftarrow\!-\,-\,-\,-\,-\,-\,- \underset{\mho M'(0,0,1)}{\blacksquare}$$

(8.4) The right modules $M'(1,b,c)$ with $c \neq 0$.

Proposition. *Let* $c \neq 0$*. The right module* $M'(1,b,c)$ *is torsionless if and only if* $b \neq -1$*, and then* $\mho M'(1,b,c) = M'(\omega(1,b,c))$*. Let* $c' \neq 0$*. The right module* $M'(1,b',c')$ *is extensionless if and only if* $b' \neq -q$*, and then* $\Omega M'(1,b',c') = M'(\omega'(1,b',c'))$*.*

Remark. If $b \neq -1$ and $c \neq 0$, then $\omega(1,b,c) = (1,b',c')$ with $b' \neq -q$ and some $c' \neq 0$. If $b' \neq -q$, then $\omega'(1,b',c') = (1,b,c)$ with $b \neq -1$ and some $c \neq 0$. Thus, the proposition provides $\mho$-sequences

$$0 \to M'(1,b,c) \to \Lambda_\Lambda \to M'(1,b',c') \to 0$$

with $b \neq -1$ and $b' \neq -q$ (and both c, c' being non-zero). Any triple $(1,b,c)$ with $b \neq -1$ and $c \neq 0$ occurs on the left and given $(1,b,c)$, then we have $(1,b',c') = \omega(1,b,c)$ on the right. Any triple $(1,b',c')$ with $b' \neq -q$ and $c' \neq 0$ occurs on the right and given $(1,b',c')$, then we have $(1,b,c) = \omega'(1,b',c')$ on the left.

Proof of Proposition. We follow closely the proof of (5.1) and (6.1). We always assume that $c \neq 0$. As in (5.2) one sees that $M'(1,b,c)$ is extensionless if and only if the image of every homomorphism $U'(1,b,c) \to \Lambda_\Lambda$ is contained in $U'(1,b,c)$.

(a) *The module* $M'(1,-q,c)$ *is not extensionless.* Proof. According to (8.1), we have $U'(1,-q,c') \simeq \Omega M'(1,-q,c') \simeq M'(\omega'(1,-q,c')) = M'(1,-1,0)$ for all $c' \in k$. Thus, there is a homomorphism $U'(1,-q,0) \to \Lambda_\Lambda$ with image $U'(1,-q,0)$ and this image $U'(1,-q,0)$ is not contained in $U'(1,-q,c)$.

(b) *If* $b \neq -q$*, then the module* $M'(1,b,c)$ *is extensionless.* For the proof, we need three assertions (b1), (b2) (b3). Note that (8.1) asserts that $U'(1,b,c) \simeq \Omega M'(1,b,c) \simeq M'(\omega'(1,b,c)) = M'(1,q^{-1}b,c')$, where $\omega'(1,b,c) = (1,q^{-1}b,c')$.

(b1) *The only right ideal isomorphic to $U'(1,b,c)$ is $U'(1,b,c)$ itself.* Proof. Let V be a right ideal of Λ_Λ which is isomorphic to $U'(1,b,c)$, say $V = U'(a'',b'',c'')$ for some triple (a'',b'',c''). By (8.1), we have $U(a'',b'',c'') \simeq \Omega M'(a'',b'',c'') = M'(a'',q^{-1}b'',d)$, where $\omega'(a'',b'',c'') = (a'',q^{-1}b'',d)$ for some d. We must have $a'' \neq 0$, since $M(a'',q^{-1}b'',d) \simeq U'(1,b,c) \simeq M'(1,q^{-1}b,c')$. Thus, we may assume that $a''=1$ and then $M'(1,q^{-1}b'',d) \simeq M'(1,q^{-1}b,c')$ implies that $(1,q^{-1}b'',d) = (1,q^{-1}b,c')$. In particular, we have $b''=b \neq -q$. The equality $\omega'(1,b'',c'') = \omega'(1,b,c)$ yields $(1,b'',c'') = (1,b,c)$, see Proposition (3.2). Therefore $V = U(1,b'',c'') = U(1,b,c)$.

(b2) *The right ideal $z\Lambda$ is not a factor module of $U'(1,b,c)$.* Proof. The right ideal $z\Lambda$ is annihilated by $x-y$ and z, thus Corollary (5.3) asserts that the image I of any homomorphism $M'(1,b',c') \to z\Lambda$ is a factor module of $\Lambda/((1,b,c)\Lambda + (x-y)\Lambda + z\Lambda)$. Now $(x+by+cz)\Lambda + (x-y)\Lambda + z\Lambda = \operatorname{rad}\Lambda$, since $b \neq -1$, thus I is simple or zero.

(b3) *The right ideal $y\Lambda$ is not a factor module of $U'(1,b,c)$.* Proof. The right ideal $y\Lambda$ is annihilated by y and z, thus Corollary (5.3) asserts that the image I of any homomorphism $M'(1,b',c') \to y\Lambda$ is a factor module of $\Lambda/((1,b,c)\Lambda + y\Lambda + z\Lambda)$. Now $(x+by+cz)\Lambda + y\Lambda + z\Lambda = \operatorname{rad}\Lambda$, since $b \neq -1$, thus I is simple or zero.

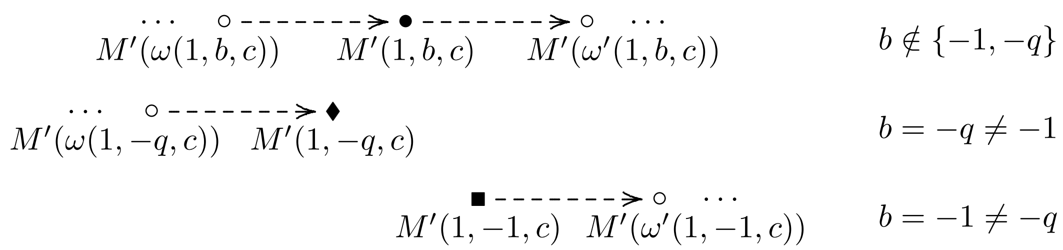
The assertions (b1), (b2) and (b3) show: if ϕ is any homomorphism $U'(1,b,c) \to \Lambda_\Lambda$ and its image I is of dimension at least 2, then I is contained in $U'(1,b,c)$. Of course, if I is 1-dimensional, then I is contained in $\operatorname{soc}\Lambda_\Lambda$ and $\operatorname{soc}\Lambda_\Lambda \subseteq U'(1,b,c)$. Thus, we have obtained a proof of (b). In addition, (8.1) asserts that $\Omega M'(1,b,c) \simeq M'(\omega'(1,b,c))$.

(c) *If $b \neq -1$, then $M'(1,b,c)$ is torsionless and $\mho M'(1,b,c) = M'(\omega(1,b,c))$.* Proof. Let $\omega(1,b,c) = (1,b',c')$. Then $b' = qb \neq -q$, and $\omega'(1,b',c') = \omega'\omega(1,b,c) = (1,b,c)$ by Proposition (3.2). According to (8.1), we have $\Omega M'(1,b',c') \simeq M'(\omega'(1,b',c')) = M'(1,b,c)$. This shows that $M'(1,b,c)$ is torsionless.According to (b), the module $M'(\omega(1,b,c))$ is extensionless, thus $\mho M'(1,b,c) = M'(1,b',c') = M'(\omega(1,b,c))$.

(d) *The modules $M'(1,-1,c)$ are not torsionless.* Proof. Assume, for the contrary, that $M'(1,-1,c)$ is torsionless, thus isomorphic to $U'(a',b',c')$ for some (a',b',c'). According to (8.1), we must have $a' \neq 0$, thus we can assume that $a'=1$, and $(1,-1,c) = \omega'(1,b',c') = (1,q^{-1}b',-(1+q^{-1}b')c')$. It follows that $b'=-q$ and therefore $c = -(1+q^{-1}b')c' = 0$, a contradiction.

This completes the proof of (8.4). □

Reformulation. *The neighborhood of $M'(1,b,c)$ with $c \neq 0$ in the $\mho$-quiver looks as follows:*



and $M'(1,b,c)$ is a singleton in the $\mho$-quiver if $q=1$ and $b=-1$.

Note that we want to use a fixed index set $\mathbb{P}^2$ both for the (left) modules $M(a\!:\!b\!:\!c)$ and the right modules $M'(a\!:\!b\!:\!c)$, Since we have drawn the dashed arrows in the $\mho$-quiver of the left Λ-modules from right to left, we now have drawn the dashed arrows in the $\mho$-quiver of the right Λ-modules from left to right.

As in section 7, we see that the $\mho$-components of the modules $M'(1,b,c)$ with $c \neq 0$ are cycles, or of type $\mathbb{Z}, \mathbb{N}$ or $-\mathbb{N}$ in case $o(q) = \infty$, and cycles or of type $\mathbb{Z}$ or $\mathbb{A}_n$ in case $o(q) = n < \infty$.

For $o(q) = \infty$, the right modules $M'(1,-1,c)$ with $c \neq 0$ are pivotal semi-Gorenstein-projective, and the right modules $M'(1,-q,c)$ with $c \neq 0$ are pivotal ∞-torsionfree.

(8.5) The right modules $M'(1,b,0)$.

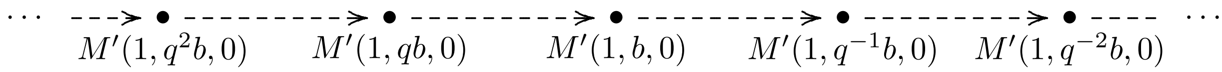
The right modules $M'(1,b,0)$ have been considered already in Part I: these are just the right ideals $m_\alpha\Lambda$, where $m_\alpha = x - \alpha y$. Namely, we have

$$M'(1,b,0) = (x + qby)\Lambda = m_{-qb}\Lambda$$

for all $b \in k$. (Proof: We have $M'(1,b,0) = \Lambda_\Lambda/U'(1,b,0) = \Lambda_\Lambda/(x+by)\Lambda \simeq (x+qby)\Lambda$, where we use that $(x+qby)(x+by) = 0$ and that both right ideals $(x+by)\Lambda$ and $(x+qby)\Lambda$ are 3-dimensional, see (2.8).)

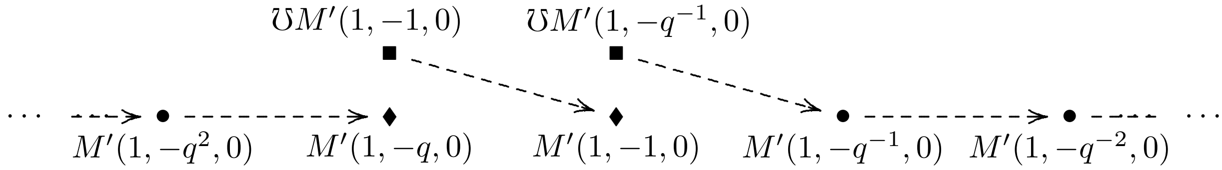
Let us recall the results presented in Part I using the present notation:

If $b \notin -q^{\mathbb{Z}}$, then $M'(1,b,0)$ is Gorenstein-projective and its $\mho$-component looks as follows:



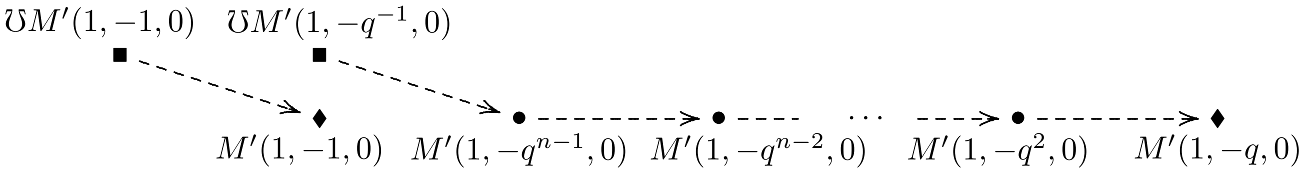
In particular, if $o(q) = n$, then these $\mho$-components are cycles with n vertices, whereas for $o(q) = \infty$, one obtains $\mho$-components of type $\mathbb{Z}$.

For $o(q) = \infty$, there are three remaining $\mho$-components:

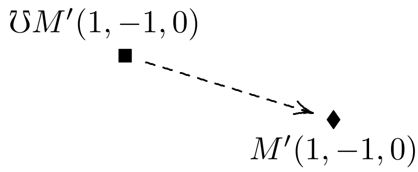


These $\mho$-components are of type $\mathbb{N}, \mathbb{A}_2$ and $-\mathbb{N}$, respectively.

For $2 \le n = o(q) < \infty$, there are two remaining $\mho$-components, one is of type $\mathbb{A}_2$, the other of type $\mathbb{A}_n$:



In case $q = 1$, there is only one additional $\mho$-component (of type $\mathbb{A}_2$), namely



(8.6) Similar to Theorem (1.5), here is the summary which characterizes the right modules of dimension at most 3 with relevant properties.

Theorem. *An indecomposable right module N of dimension at most 3 is*
- *torsionless if and only if N is simple or isomorphic to $y\Lambda$, to $z\Lambda$, to a module $M'(1,b,c)$ with $b \neq -1$, to $M'(1,-1,0)$ or to $M'(0,0,1)$.*
- *extensionless if and only if N is isomorphic to a module $M'(1,b,c)$ with $b \neq -q$;*
- *reflexive if and only if M is isomorphic to a module $M'(1,b,c)$ with $b \neq -q^i$ for $i = -1, 0$;*
- *Gorenstein-projective if and only if N is isomorphic to a module $M'(1,b,c)$ with $b \neq -q^i$ for $i \in \mathbb{Z}$;*
- *semi-Gorenstein-projective if and only if N is isomorphic to a module $M'(1,b,c)$ with $b \neq -q^i$ for $i \geq 0$ or to a module $M'(1,-1,c)$ with $c \neq 0$;*
- *∞-torsionfree if and only if N is isomorphic to a module $M'(1,b,c)$ with $b \neq -q^i$ for $i \leq 0$;*
- *pivotal semi-Gorenstein-projective if and only if $o(q) = \infty$ and N is isomorphic to a module $M'(1,-1,c)$ with $c \neq 0$;*
- *pivotal ∞-torsionfree if and only if $o(q) = \infty$ and N is isomorphic to a module $M'(1,-q,c)$.*

Whereas the set of modules $M(1,b,c)$ with $b,c \in k$ is a union of $\mho$-components, the right modules behave differently: as we have seen already in Part I, 7.2, the $\mho$-component containing the right module $M(1,-1,0)$ consists of $M(1,-1,0)$ and the 9-dimensional right module $\mho M(1,-1,0)$.

9. The Λ-dual of $M(1,b,c)$ and $M'(1,b,c)$.

We need the following (of course well-known) Lemma.

(9.1) Lemma. *Let R be a ring and $w \in R$. If any left-module homomorphism $Rw \to {}_RR$ maps w into wR, then* $\operatorname{Hom}(Rw, {}_RR) \simeq wR$ *as right R-modules.*

Proof. Let $u\colon Rw \to {}_RR$ be the inclusion map. We have $\operatorname{Hom}(Rw, {}_RR) = uR$, since for any homomorphism $f\colon Rw \to {}_RR$, there is $\lambda \in R$ with $f(w) = w\lambda$, thus $f = u\lambda$. Now $I = \{r \in R \mid wr = 0\}$ is a right ideal and $R_R/I \simeq wR$ as right modules (an isomorphism is given by the map $R_R \to wR$ defined by $1 \mapsto w$). Since $I = \{r \in R \mid ur = 0\}$, we have in the same way $R_R/I \simeq uR$, and therefore $wR \simeq R_R/I \simeq uR = \operatorname{Hom}(Rw, {}_RR)$. $\square$

(9.2) Lemma. *If $(1,b,c)$ is different from $(1,-1,0)$, then $M'(1,b,c) \simeq \operatorname{Tr} M(1,b,c)$ and $M(1,b,c) \simeq \operatorname{Tr} M'(1,b,c)$.*

Proof. We have $U'(1,b,c) = (1,b,c)\Lambda$, and since $(1,b,c) \neq (1,-1,0)$, we also have $U(1,b,c) = \Lambda(1,b,c)$. By definition, $M(1,b,c) = {}_\Lambda\Lambda/U(1,b,c)$, thus $M(1,b,c)$ is the cokernel of the right multiplication $r_{(1,b,c)}\colon {}_\Lambda\Lambda \to {}_\Lambda\Lambda$ and $\operatorname{Tr} M(1,b,c)$ is the cokernel of the left multiplication $l_{(1,b,c)}\colon \Lambda_\Lambda \to \Lambda_\Lambda$, thus isomorphic to $\Lambda_\Lambda/(1,b,c)\Lambda = \Lambda_\Lambda/U'(1,b,c)$. $\square$

(9.3) Proposition. *If $b \notin \{-q, -q^2\}$, then $M(1,b,c)$ is reflexive and*

$$M(1,b,c)^* = M'((\omega')^2(1,b,c)).$$

If $b \notin \{-1, -q^{-1}\}$, then $M'(1,b,c)$ is reflexive and $M'(1,b,c)^* = M(\omega^2(1,b,c))$.

Proof. According to (7.1), we have the following two $\mho$-sequences:

$$0 \to M(1,b,c) \to {}_\Lambda\Lambda \to M(\omega'(1,b,c)) \to 0,$$
$$0 \to M(\omega'(1,b,c)) \to {}_\Lambda\Lambda \to M((\omega')^2(1,b,c)) \to 0$$

(the first one, since $\omega'(1,b,c) = (1,b',c')$ with $b' = q^{-1}b \neq -1$; the second one, since $(\omega')^2(1,b,c) = (1,b'',c'')$ with $b'' = q^{-2}b \neq -1$) This implies that $M(1,b,c)$ is reflexive and that $X = \mho^2 M(1,b,c) = M((\omega')^2(1,b,c))$ is a module with $\operatorname{Ext}^i(X,\Lambda) = 0$ for $i = 1,2$. According to Part I, Lemma 2.5, we have $\operatorname{Tr} X = (\Omega^2 X)^*$. On the one hand, $\Omega^2 X = \mho M(1,b,c) = M(1,b,c)$. On the other hand, (9.2) shows that $\operatorname{Tr} X = \operatorname{Tr} M((\omega')^2(1,b,c)) = M'((\omega')^2(1,b,c))$, since $(\omega')^2(1,b,c) = (1,q^{-2}b,c'')$ for some c'' and $q^{-2}b \neq -1$. This yields the first assertion. The second can be shown in the same way, or just by applying the Λ-duality to $M(1,b,c)^* = M'((\omega')^2(1,b,c))$. $\square$

(9.4) Proposition. *For all $b, c \in k$,*

$$M(1,b,c)^* = M'((\omega')^2(1,b,c)).$$

In particular, for all $b, c \in k$, the right module $M(1,b,c)^*$ is again 3-dimensional and local.

Whereas $(\omega')^2$ is a bijection from $\{(1,b,c) \mid b \notin \{-q,-q^2\}\}$ onto $\{(1,b,c) \mid b \notin \{-1,-q^{-1}\}\}$, we should stress that $(\omega')^2(1,-q,c) = (1,-q^{-1},0)$ and that $(\omega')^2(1,-q^2,c) = (1,-1,0)$ for all $c \in k$. Thus, (9.3) combines the first assertion of (9.2) with the corresponding assertion for the remaining cases, namely:

$$M(1,-q,c)^* = M'(1,-q^{-1},0) \quad \textit{and} \quad M(1,-q^2,c)^* = M'(1,-1,0),$$

for all $c \in k$.

Proof of Proposition. According to (9.2), we only have to consider the cases where $b = -q$ or $b = -q^2$.

Case 1. Let $b = -q$. As we have seen in (6.2), the module $M(1,-q,c)$ is not torsionless. Now obviously, there is a surjective homomorphism $M(1,-q,c) \to \Lambda(1,-1,0)$ with kernel $zM(1,-q,c)$. It follows that $zM(1,-q,c)$ is contained in the kernel of every homomorphism $M(1,-q,c) \to {}_\Lambda\Lambda$ and therefore $M(1,-q,c)^* = (\Lambda(1,-1,0))^*$. Now, $(\Lambda(1,-1,0))^* \simeq (1,-1,0)\Lambda = U'(1,-1,0)$, as shown in Part I, 6.5. On the other hand, according to (8.1), we have $U'(1,-1,0) = \Omega M'(1,-1,0) = M'(\omega'(1,-1,0))$ and $\omega'(1,-1,0) = (1,-q^{-1},0)$.

Case 2: $b = -q^2$ and $o(q) = 2$. The assumption $o(q) = 2$ means that $q = -1 \neq 1$, in particular, the characteristic of k is different from 2, and we have $b = -1$. Since $q = -1$ and the characteristic of k is different from 2, (4.1) asserts that

$$\Lambda(1,1,-2c) = U(1,1,-2c) = \Omega M(1,1,-2c) = M(\omega(1,1,-2c)) = M(1,-1,c).$$

On the other hand, we have

$$(1,1,-2c)\Lambda = U'(1,1,-2c) = \Omega M'(1,1,-2c) = M'(\omega'(1,1,-2c)) = M'(1,-1,0).$$

We claim that any homomorphism $\Lambda(1,1,-2c) \to {}_\Lambda\Lambda$ maps $(1,1,-2c)$ into $(1,1,-2c)\Lambda$. Namely, let $\phi\colon \Lambda(1,1,-2c) \to {}_\Lambda\Lambda$ be a homomorphism. Now $\Lambda(1,1,-2c)$ is 3-dimensional, thus equal to $U(1,1,-2c)$, and ${}_\Lambda\Lambda/U(1,1,-2c) \simeq M(1,1,-2c)$. According to (5.1), the module $M(1,1,-2c)$ is extensionless, since $1+1 \neq 0$. The implication (i) to (iv) in (5.2) shows that $\phi(1,1,-2c) \in (1,1,-2c)\Lambda$.

Since any homomorphism $\Lambda(1,1,-2c) \to {}_\Lambda\Lambda$ maps $(1,1,-2c)$ into $(1,1,-2c)\Lambda$, Lemma (9.0) implies that the right modules $(\Lambda(1,1,-2c))^*$ and $(1,1,-2c)\Lambda$ are isomorphic, thus $M(1,-1,c)^* \simeq M'(1,-1,0)$.

Case 3. $b = -q^2$ and $o(q) \geq 3$. There is the $\mho$-sequence

$$\epsilon\colon \qquad 0 \to M(1,-q^3,c') \to {}_\Lambda\Lambda \to M(1,-q^2,c) \to 0$$

for some c' (here we use that $q^2 \neq 1$). The Λ-dual of ϵ is the exact sequence

$$0 \to M(1,-q^2,c)^* \to \Lambda_\Lambda \to M(1,-q^3,c')^* \to 0.$$

Since $q^2 \neq 1$, proposition (9.3) asserts that $M(1,-q^3,c')^* = M'(1,-q,c'')$ for some c''. Altogether we see that

$$M(1,-q^2,c)^* \simeq \Omega(M(1,-q^3,c')^*) = \Omega M'(1,-q,c'') \simeq M'(1,-1,0),$$

where the final isomorphism is due to (8.1). □

(9.5) The algebra $\Lambda = \Lambda(q)$ with $o(q) = \infty$ was exhibited in Part I in order to present a module M which is not torsionless, such that M and M^* both are semi-Gorenstein-projective: namely the module $M = M(1,-q,0)$ with $M^* = M'(1,-q,0)$. Now we see: *all the modules $M(1,-q,c)$ with $c \in k$ are modules which are semi-Gorenstein-projective and not torsionless, and that the Λ-duals $M(1,-q,c)^* \simeq M'(1,-q^{-1},0)$ are semi-Gorenstein-projective.* We should stress that this concerns a 1-parameter family $M(1,-q,c)$ (with $c \in k$) of semi-Gorenstein-projective left modules, and the single semi-Gorenstein-projective right module $M(1,-q^{-1},0)$.

(9.6) Proposition. *Let $b, c \in k$.*

$$M'(1,b,c)^* = \begin{cases} M(\omega^2(1.b.c)) & \text{if} \quad b \notin \{-1,-q^{-1}\}, \\ U(0,0,1) & \text{if} \quad b=-1,\ c \neq 0, \\ U(1,-q,0)+U(0,0,1) & \text{if} \quad b=-1,\ c=0, \\ M(0,0,1) & \text{if} \quad b=-q^{-1},\ c\neq 0,\ q \neq 1, \\ U(1,-1,0) & \text{if} \quad b=-q^{-1},\ c=0,\ q\neq 1. \end{cases}$$

Whereas we saw in (9.4) that all the right modules $M(1,b,c)^*$ are 3-dimensional and local, not all the modules $M'(1,b,c)^*$ are 3-dimensional and local: the module $M'(1,-1,0)^* = U(1,-q,0)+U(0,0,1)$ has dimension 4, whereas the modules $M'(1,-1,c)^* = U(0,0,1)$ for $c \neq 0$ and, in case $q \neq 1$, the module $M'(1,-q^{-1},0)^* = U(1,-1,0)$ are decomposable.

Proof. According to (9.3), we only have to deal with the cases with $b \in \{-1,-q^{-1}\}$. If $c = 0$, then we can refer to Part I. For $b=-1$, the end of 7.1 in Part I shows that

$M'(1,-1,0)^* \simeq M(1,-q^2,0)^{**} \simeq U(1,-q,0)+U(0,0,1)$. For $b=-q^{-1}\neq -1$, the end of 6.7 in Part I asserts that $M'(1,-q^{-1},0)^* \simeq (M(1,-q,0)^{**} \simeq \Omega M(1,-1,0) \simeq U(1,-1,0)$.

Now, we assume that $c \neq 0$. As in the proof of (9.4), we consider again 3 cases.

Case 1. $b=-1$. The module $M'(1,-1,c)$ with $c\neq 0$ is not torsionless, see (8.4). Since the factor module $M'(1,-1,c)/M'(1,-1,c)z$ is isomorphic to $(0,0,1)\Lambda$, it follows that $M'(1,-1,c)^* \simeq ((0,0,1)\Lambda)^*$ and an easy calculation yields $((0,0,1)\Lambda)^* \simeq U(0,0,1)$. Namely, the inclusion map $u\colon z\Lambda \to \Lambda_\Lambda$ satisfies $yu=0$ and $zu=0$, thus a basis of $(z\Lambda)^*$ is given by u, xu and the map $f\colon z\Lambda \to \Lambda_\Lambda$ with $f(z)=yx$, so that $(z\Lambda)^* \simeq {}_\Lambda\Lambda/(\Lambda y+\Lambda z)\oplus k \simeq U(0,0,1)$.

Case 2. $b=-q^{-1}$ and $o(q)=2$. Thus, the characteristic of k is different from 2, $q=-1$ and $b=1$. The module $M'(1,1,c)$ is torsionless: namely, by (8.1) we have $M'(1,1,c) \simeq \Omega M'(1,-1,-\frac{c}{2})$, since $\omega'(1,-1,-\frac{c}{2})=(1,1,c)$. Now, $\Omega M'(1,-1,-\frac{c}{2}) \simeq U'(1,-1,\frac{c}{2})=(1,-1,\frac{c}{2})\Lambda$. Since $q\neq 1$, the right module $M'(1,-1,-\frac{c}{2})$ is extensionless by (8.4), thus we can use (5.2) and (9.1) in order to see that $((1,-1,\frac{c}{2})\Lambda)^* \simeq \Lambda(1,-1,\frac{c}{2})$. By (4.1) (2), we have $\Lambda(1,-1,\frac{c}{2})=U(1,-1,\frac{c}{2}) \simeq \Omega M((1,-1,-\frac{c}{2})) \simeq M(0,0,1)$.

Case 3. $b=-q^{-1}$ and $o(q)\geq 3$. There is the $\mho$-sequence

$$0 \to M'(1,-q^{-2},c') \to \Lambda_\Lambda \to M'(1,-q^{-1},c) \to 0$$

for $c'=\lambda c$ with $\lambda \neq 0$ (here we use that $q^2\neq 1$). The Λ-dual is the exact sequence

$$0 \to M'(1,-q^{-1},c)^* \to {}_\Lambda\Lambda \to M'(1,-q^{-2},c')^* \to 0.$$

We assume that $q\neq 1$ and $q\neq 2$. Then by Proposition (9.2), we have $M'(1,-q^{-2},c')^* = M(1,-1,c'')$ for some multiple $c''=\lambda' c'$ with $\lambda'\neq 0$. It follows that $M'(1,-q^{-1},c)^* = \Omega M(1,-1,c'')$ and $c''=0$ if and only if $c=0$. By (4.1), we have $\Omega M(1,-1,c'')=M(0,0,1)$ in case $c\neq 0$, and $\Omega M(1,-1,0)=U(1,-1,0)$ in case $c=0$. □

(9.7) Corollary. *Let N be a right Λ-module of dimension at most 3 which is semi-Gorenstein-projective, but not Gorenstein-projective. Then N^* is not semi-Gorenstein-projective.*

Proof. According to (8.6), N is isomorphic to a right module of the form $M'(1,-q^i,c)$ with $i\leq -1$ and $c\in k$ or of the form $M'(1,-1,c)$ with $c\neq 0$. We apply (9.6). If $i\leq -2$, then $N^*=M'(1,-q^i,c)^*=M(1,-q^{i+2},c')$ for some c', and according to (1.5), N^* is not semi-Gorenstein-projective, since $i+2\leq 0$. If $i=-1$, then N^* is isomorphic to $M(0,0,1)$ or to $U(1,-1,0)$. If $N=M'(1,-1,c)$ with $c\neq 0$, then N^* is isomorphic to $U(0,0,1)$. But by (1.5), $M(0,0,1)$, $U(1,-1,0)$ and $U(0,0,1)$ are not semi-Gorenstein-projective. □

10. The general context.

Our detailed study of the algebra $\Lambda(q)$ in Part I and Part II should be seen in the frame of looking at Gorenstein-projective (or, more general, semi-Gorenstein-projective and ∞-torsionfree modules) over local algebras with radical cube zero.

(10.1) Let A be a finite-dimensional local k-algebra with radical J such that $A/J=k$. Such an algebra is said to be *short* provided $J^3=0$. In commutative ring theory, the short local algebras have attracted a lot of interest, since some conjectures have been disproved by looking at modules over short algebras, see [AIŞ] for a corresponding account. We have to thank D. Jorgensen for his advice concerning the present knowledge in the commutative case.

Let us assume now that A is short, but not necessarily commutative. Let $e = \dim J/J^2$ and $a = \dim J^2$ (thus $0 \leq a \leq e^2$). Here is a report about the relevant general results: If there exists an indecomposable module which is semi-Gorenstein-projective or ∞-torsionfree, but not projective, then either A is self-injective, so that $a \leq 1$ (and $e = 1$ in case $a = 0$), or else $a = e - 1 \geq 2$. Always, both modules ${}_AJ$ and J_A have to be indecomposable.

Of course, if A is self-injective, then all modules are Gorenstein-projective, thus the interesting case is the case $a = e - 1 \geq 2$. Our algebra $\Lambda(q)$ is of this kind (with $a = 2$), as is the Jorgensen-Şega algebra [JŞ] (with $a = 3$).

Not only the shape of the algebras is very restricted, also the modules themselves are very special: Let A be a short local algebra which is not self-injective. Let M be indecomposable and not projective. If M is semi-Gorenstein-projective and torsionless, or if M is ∞-torsionfree (in particular, if M is Gorenstein-projective), then $\operatorname{soc} M = \operatorname{rad} M$ and $\dim \operatorname{soc} M = a \cdot \dim \operatorname{top} M$ (by definition, $\operatorname{top} M = M/\operatorname{soc} M$). Also, if M is semi-Gorenstein-projective and torsionless, then $\dim \Omega^i M = \dim M$ for all $i \in \mathbb{N}$, whereas if M is ∞-torsionfree, then $\dim \mho^i M = \dim M$ for all $i \in \mathbb{N}$.

These assertions have been shown by Christensen and Veliche in the case that A is commutative and M is Gorenstein-projective, see [CV], and the proof can be modified in order to work in general, see [RZ2]. There is an essential difference between the commutative and the non-commutative algebras: If A is commutative, then all local modules which are semi-Gorenstein-projective or ∞-torsionfree are Gorenstein-projective, whereas this is not true for A non-commutative.

Thus, for our algebra $\Lambda(q)$, the non-projective indecomposable modules which are semi-Gorenstein-projective and torsionless, or which are ∞-torsionfree, are of dimension $3t$ with socle of dimension $2t$, where $t = \dim \operatorname{top} M$. For $t = 1$, we deal with local modules with 2-dimensional socle: these are precisely the modules studied in the present paper.

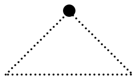
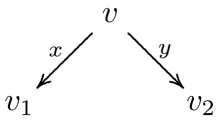
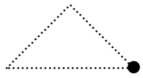
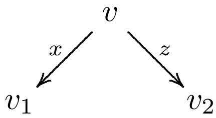
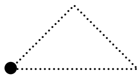
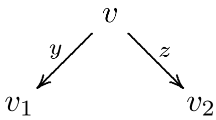
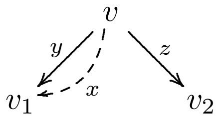
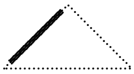
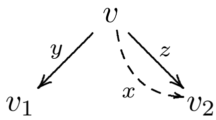
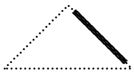
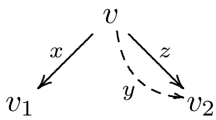
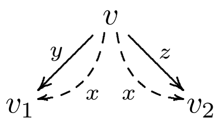
(10.2) As we have mentioned, a 3-dimensional local $\Lambda(q)$-module M belongs to H if and only if $\operatorname{soc} M = \operatorname{Ker}(y) = \operatorname{Ker}(z) = yM \oplus zM$. Thus, it seems to be of interest to study the full subcategory $\mathcal{H}$ of all the $\Lambda(q)$-modules M with $\operatorname{soc} M = \operatorname{Ker}(y) = \operatorname{Ker}(z) = yM \oplus zM$.

It will be shown in [RZ3] that all reflexive modules which are semi-Gorenstein-projective or ∞-torsionfree belong to $\mathcal{H}$. On the other hand, we will exhibit a representation equivalence between $\mathcal{H}$ and the category of finite-dimensional $k\langle x_1, x_2\rangle$-modules, where $k\langle x_1, x_2\rangle$ is the free algebra in two variables x_1, x_2.

Appendix. A diagrammatic description of the modules $M(a\!:\!b\!:\!c)$.

(A.1) If M is a left Λ-module annihilated by $\operatorname{rad}^2 \Lambda$, then it is a left $\overline{\Lambda}$-module. Since $\overline{\Lambda}$ is a commutative k-algebra, also $D(M) = \operatorname{Hom}(M, k)$ is a left $\overline{\Lambda}$-module, thus a left Λ-module. As mentioned in (1.6), we identify the set of isomorphism classes of the 3-dimensional local modules with the projective plane $\mathbb{P}^2 = \mathbb{P}(\operatorname{rad} \Lambda/\operatorname{rad}^2 \Lambda)$.

Proposition. *Let M be an indecomposable 3-dimensional left Λ-module. Then M or $D(M)$ is isomorphic to one of the following pairwise non-isomorphic $\overline{\Lambda}$-modules $M(a,b,c)$:*

Case	Modules	Position in $\mathbb{P}^2$	Diagram	Characterization
(1)	$M(0,0,1)$			$zM=0$
(2)	$M(0,1,0)$			$yM=0$
(3)	$M(1,0,0)$			$xM=0$
(4)	$M(1,b,0)$ $b\in k^*$		 with $xv=-bv_1$	$xM=yM$
(5)	$M(1,0,c)$ $c\in k^*$		 with $xv=-cv_2$	$xM=zM$
(6)	$M(0,1,c)$ $c\in k^*$		 with $yv=-cv_2$	$yM=zM$
(7)	$M(1,b,c)$ $b,c\in k^*$		 with $xv=-bv_1-cv_2$	xM, yM, zM non-zero and pairwise different

The diagrams describe the modules $M=M(a,b,c)$ as follows: The elements v,v_1,v_2 form a basis of M. Both elements v_1,v_2 are annihilated by x,y,z. If there is drawn a solid arrow $v \longrightarrow v_i$ with $i\in\{1,2\}$ and with label $\alpha\in\{x,y,z\}$, then $\alpha v=v_i$. If there is a dashed arrow $v \dashrightarrow v_i$ with label α, then $\alpha v=c_1v_1+c_2v_2$ with $c_i\neq 0$ (and we provide the coefficients c_1,c_2 below the diagram). Finally, $zv=0$ in case (1), $yv=0$ in case (2), $xv=0$ in case (3).

The last column provides a characterization of the corresponding modules $M(a,b,c)$: For example, a local 3-dimensional Λ-module M is a case-(1)-module provided $zM = 0$, and so on.

(A.2) Remark. *If M is an indecomposable 3-dimensional Λ-module, then its annihilator is equal to $U(a,b,c)$ for some $(a,b,c) \neq 0$ and M considered as a $\Lambda/U(a,b,c)$-module is either the unique indecomposable projective $\Lambda/U(a,b,c)$-module* (and then a local module, thus isomorphic to $M(a,b,c)$) *or the unique indecomposable injective $\Lambda/U(a,b,c)$-module* (and then a module with simple socle, thus isomorphic to $D(M(a,b,c))$).

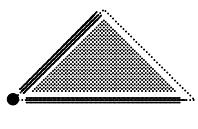
(A.3) Proof of the Proposition and the Remark. First, let us assume that M is local. According to (2.6) and (1.4), we know that $M \simeq M(a\!:\!b\!:\!c)$ for some $(a\!:\!b\!:\!c) \in \mathbb{P}^2$ and that these modules are pairwise non-isomorphic. As representatives of the elements of $\mathbb{P}^2$, we choose (as usual) the triples (c_1, c_2, c_3) with $c_i = 1$ for some i and $c_j = 0$ for $j < i$. Clearly, there are the seven cases (1) to (7) as listed above. It remains to choose in every case a basis $\mathcal{B}(a,b,c) = \{v, v_1, v_2\}$ of $M(a,b,c)$. Recall that $M(a,b,c) = \overline{\Lambda}/(a\!:\!b\!:\!c)$ is a factor module of Λ and $\overline{\Lambda}$ has the basis $\{1, x, y, z\}$. We choose as elements of $\mathcal{B}(a,b,c)$ the residue class $v = \overline{1}$ as well as two of the three residue classes $\overline{x}, \overline{y}, \overline{z}$, namely $v_1 = \overline{x}$ if $a = 0$ and $v_1 = \overline{y}$ otherwise, and then $v_2 = \overline{y}$ in case $(a,b,c) = (0,0,1)$ and $v_2 = \overline{z}$ otherwise. (We should remark that the vertices and the arrows of the diagram are those of the coefficient quiver $\Gamma(M(a,b,c), \mathcal{B}(a,b,c))$ as considered in [R], and the solid arrows focus the attention to a spanning tree.)

Second, assume that M is not local. Since M is an indecomposable module of length 3 and Loewy length 2, it follows that M has simple socle, thus $D(M)$ is local and therefore of the form (1) to (7).

Finally, M and $D(M)$ have the same annihilator, this is a 3-dimensional ideal, thus of the form $U(a,b,c)$. The 3-dimensional local algebra $\Lambda/U(a,b,c)$ has a unique 3-dimensional local module, this is the indecomposable projective $\Lambda/U(a,b,c)$-module, and dually, it has a unique 3-dimensional module with simple socle, this is the unique indecomposable injective $\Lambda/U(a,b,c)$-module. This completes the proof. $\square$

(A.4) As we have mentioned in (1.6), of special interest is the affine subspace H of $\mathbb{P}^2$ given by the points $(1\!:\!b\!:\!c)$ with $b, c \in k$. *A 3-dimensional local module M belongs to H if and only if* $\operatorname{soc} M = \operatorname{Ker}(y) = \operatorname{Ker}(z) = yM \oplus zM$.

H



Namely, H is the union of the sets (3), (4), (5) and (7).

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