On modules M such that
both M and M^* are semi-Gorenstein-projective.

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Abstract: Let A be an artin algebra. An A-module M is semi-Gorenstein-projective provided that $\mathrm{Ext}^i(M,A)=0$ for all $i\geq 1$. If M is Gorenstein-projective, then both M and its A-dual M^* are semi-Gorenstein projective. As we have shown recently, the converse is not true, thus answering a question raised by Avramov and Martsinkovsky. The aim of the present note is to analyse in detail the modules M such that both M and M^* are semi-Gorenstein-projective.

Key words. Gorenstein-projective module, semi-Gorenstein-projective module, finitistic dimension conjecture, Nunke condition.

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1. Introduction.

Let A be an artin algebra. The modules to be considered are usually left A-modules of finite length. Given a module M, let $M^* = \mathrm{Hom}(M, A)$ be its A-dual, and $\phi_M : M \to M^{**}$ the canonical map from M to M^{**}. We will have to deal with complexes $P_\bullet = (P_i, f_i : P_i \to P_{i-1})$ of projective modules. Such a complex is said to be *minimal* provided the image of f_i is contained in the radical of P_{i-1}, for all $i \in \mathbb{Z}$.

A module is said to be *reduced* provided it has no non-zero projective direct summand. The Main Theorem 2.1 asserts that the (isomorphism classes of the) reduced modules M such that both M and M^* are semi-Gorenstein-projective correspond bijectively to the (isomorphism classes of) minimal complexes $P_\bullet$ of projective modules with $H_i(P_\bullet) = 0$ for $i \neq 0, -1$, and such that the A-dual complex $P_\bullet^*$ is acyclic. The essential idea is Lemma 2.2 which shows in which way the canonical map ϕ_M is related to a four term exact sequence of projective right modules with M^* being the image of the middle map. Let us mention that Tr provides a bijection between the reduced semi-Gorenstein-projective modules M with also M^* semi-Gorenstein-projective on the one hand, and the reduced ∞-torsionfree right modules Z with $\Omega^2 Z$ being semi-Gorenstein-projective, on the other hand, see 2.4.

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These results are summarized by exhibiting the complexes $P_\bullet$ and $P_\bullet^*$ with the modules M, M^*, M^{**}, as well as $\operatorname{Tr} M$ and $\operatorname{Tr} M^*$ being inserted, see 2.5.

Section 3 is devoted to two special situations. First, in 3.1, we consider the case that M is a semi-Gorenstein-projective module M with also M^* semi-Gorenstein-projective such that ϕ_M is either an epimorphism or a monomorphism. In 3.2, we deal with semi-Gorenstein-projective modules M such that M^* is projective or even zero. Whereas there do exist modules M such that both M and M^* are semi-Gorenstein-projective, but not Gorenstein-projective (see [RZ1] and [RZ2], and also section 11 of [RZ3]), it is not known whether we may have in addition that ϕ_M is either an epimorphism or a monomorphism. Also, it is not known whether there exists a semi-Gorenstein-projective module which is not projective, such that M^* is projective.

A module M will be said to be a *Nunke* module provided that M is semi-Gorenstein-projective and $M^* = 0$. Note that an indecomposable semi-Gorenstein-projective module M such that ϕ_M is an epimorphism and M^* is projective, is either itself projective or else a Nunke module, see Proposition 3.4. The remaining parts of section 3 are devoted to Nunke modules. There is the old conjecture (one of the classical homological conjectures) that the only Nunke module is the zero module. This conjecture and similar ones are discussed in 3.5. In 3.6 to 3.9 we try to analyze the special case of a simple injective module S which is semi-Gorenstein-projective (thus S is either projective or a Nunke module).

In the final section 4, we consider local algebras, and, in particular, those with radical cube zero (the *short* local algebras). In [RZ1] and [RZ2], we have exhibited short local algebras with modules M such that both M and M^* are semi-Gorenstein-projective, whereas M is not Gorenstein-projective. Here we show: Let A be a local algebra and M a module such that both M and M^* are semi-Gorenstein-projective. If M^* is projective, then also M is projective. If A is short, and ϕ_M is a monomorphism or an epimorphism, then ϕ_M is an isomorphism, thus M is Gorenstein-projective. In this way, we see that for short local algebras, there are no non-trivial examples of modules which satisfy the conditions discussed in section 3.

2. The Main Theorem.

2.1. Main Theorem. *The isomorphism classes of the reduced modules M such that both M and M^* are semi-Gorenstein-projective correspond bijectively to the isomorphism classes of minimal complexes $P_\bullet$ of projective modules with $H_i(P_\bullet) = 0$ for $i \neq 0, -1$, and such that the A-dual complex $P_\bullet^*$ is acyclic, as follows:*

First, let M be a reduced module such that both M, M^* are semi-Gorenstein-projective. Take minimal projective resolutions

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{e} M \to 0 \qquad \textit{and} \qquad 0 \leftarrow M^* \xleftarrow{q} Q_0 \xleftarrow{d_1} Q_1 \xleftarrow{d_2} Q_2 \leftarrow \cdots .$$

For $i < 0$, let $P_i = Q^*_{-i-1}$ and $f_i = d^*_{-i} \colon P_{-i} \to P_{-i-1}$, Finally, let $f_0 = q^*\phi_M e \colon P_0 \to P_{-1}$. In this way, we obtain a minimal complex $P_\bullet$ of projective modules

$$\cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \xrightarrow{f_{-1}} P_{-2} \to \cdots .$$

with $H_i(P_\bullet) = 0$ for $i \neq 0, -1$, and such that the A-dual complex $P_\bullet^*$ is acyclic. By construction, $M = \operatorname{Cok} f_1$ and $M^* = \operatorname{Cok} d_1 = \operatorname{Cok} f_{-1}^*$. Also,

$$H_0(P_\bullet) = \operatorname{Ker} \phi_M \quad \text{and} \quad H_{-1}(P_\bullet) = \operatorname{Cok} \phi_M.$$

Conversely, let $P_\bullet = (P_i, f_i \colon P_i \to P_{i-1})_i$ be a minimal complex of projective modules with $H_i(P_\bullet) = 0$ for $i \neq 0, -1$, and such that the A-dual complex $P_\bullet^*$ is acyclic. Let $M = \operatorname{Cok} f_1$. Then M is reduced and both M and M^* are semi-Gorenstein-projective. Actually, $M^* = \operatorname{Cok} f_{-1}^*$, $M^{**} = \operatorname{Ker} f_{-1}$.

The essential part of the proof is the following general lemma which shows in which way exact sequences of projective modules are related to the canonical maps $\phi_M \colon M \to M^{**}$.

2.2. Lemma. *Let*

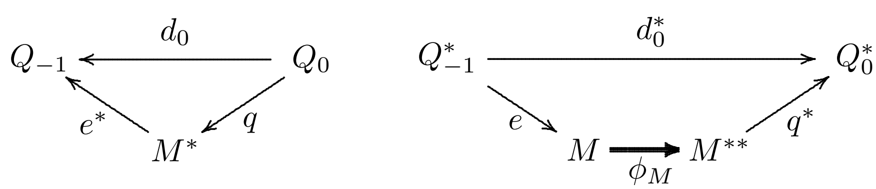
$$Q_{-2} \xleftarrow{d_{-1}} Q_{-1} \xleftarrow{d_0} Q_0 \xleftarrow{d_1} Q_1$$

be a sequence of projective right modules with composition zero. Let $e \colon Q_1^* \to M$ be a cokernel of d_{-1}^* and $c \colon Q_0 \to N$ a cokernel of d_1. The following conditions are equivalent:

(i) *The sequence is exact.*

(ii) *There exists an isomorphism $\zeta \colon N \to M^*$ with $d_0^* = c^*\zeta^*\phi_M e$.*

If the condition (ii) is satisfied, then $q = \zeta c \colon Q_0 \to M^*$ is a cokernel of d_1 and we have the following commutative diagrams:



Before we start with the proof, let us recall: The exact sequence $0 \leftarrow N \xleftarrow{c} Q_0 \xleftarrow{d_1} Q_0$ yields the exact sequence $0 \to N^* \xrightarrow{c^*} Q_0^* \xrightarrow{d_1^*} Q_1^*$, thus $\operatorname{Ker} d_1^* = (\operatorname{Cok} d_1)^*$. Similarly, the exact sequence $Q_{-2}^* \xrightarrow{d_{-1}^*} Q_{-1}^* \xrightarrow{e} M \to 0$ yields the exact sequence $Q_{-2} \xleftarrow{d_{-1}} Q_{-1} \xleftarrow{e^*} M^* \leftarrow 0$, thus $\operatorname{Ker} d_{-1} = (\operatorname{Cok} d_{-1}^*)^*$.

Proof of Lemma. (i) implies (ii). Since the sequence $Q_\bullet$ is exact, the cokernel N of d_1 is equal to the kernel M^* of d_{-1}, thus there is an isomorphism $\zeta \colon N \to M^*$ such that $d_0 = e^*\zeta c$. It follows that $d_0^* = c^*\zeta^* e^{**}$. We have the commutative diagram

$$\begin{array}{ccc} Q_{-1}^* & \xrightarrow{\ e\ } & M \\ {\scriptstyle \phi=1}\downarrow & & \downarrow{\scriptstyle \phi_M} \\ Q_{-1}^* & \xrightarrow{\ e^{**}\ } & M^{**} \end{array}$$

(with $\phi = \phi_{Q_{-1}^*}$ the identity map), thus $e^{**} = \phi_M e$ and therefore $d_0^* = c^*\zeta^* e^{**} = c^*\zeta\phi_M e$.

(ii) implies (i). We assume that $\zeta\colon N \to M^*$ is an isomorphism with $d_0^* = c^*\zeta^*\phi_M e = (\zeta c)^*\phi_M e$. Then $d_0 = d_0^{**} = e^*(\phi_M)^*(\zeta c)^{**}$. There is the commutative diagram

$$\begin{array}{ccc} M^* & \xleftarrow{\ \zeta c\ } & Q_0 \\ {\scriptstyle\phi_{M^*}}\big\downarrow & & \big\downarrow{\scriptstyle\phi=1} \\ M^{***} & \xleftarrow{(\zeta c)^{**}} & Q_0^{**}, \end{array}$$

therefore $(\zeta c)^{**} = \phi_{M^*}\zeta c$, thus $d_0 = e^*(\phi_M)^*(\zeta c)^{**} = e^*(\phi_M)^*\phi_{M^*}\zeta c = e^*\zeta c$. (Here we use that $(\phi_M)^*\phi_{M^*}$ is the identity map of M^*. It implies that ϕ_{M^*} is a splitting monomorphism; but in general, ϕ_{M^*} is not an isomorphism.) This shows that d_0 is the composition of the cokernel map ζc for d_1 with the kernel map e^* for d_{-1}^*. It follows that the sequence $Q_\bullet$ is exact. □

2.3. Proof of Main Theorem. Assume that M is a reduced module such that both M and M^* are semi-Gorenstein-projective. Let

$$\cdots \to P_1 \to P_0 \xrightarrow{e} M \to 0$$

be a minimal projective resolution. Since M is semi-Gorenstein-projective, the A-dual sequence

$$\cdots \leftarrow P_1^* \leftarrow P_0^* \xleftarrow{e^*} M^* \leftarrow 0$$

is exact. Let

$$0 \leftarrow M^* \xleftarrow{q} Q_0 \leftarrow Q_1 \leftarrow \cdots$$

be a minimal projective resolution of the right module M^*. The concatenation is an acyclic minimal complex $Q_\bullet$ of projective right modules (with $Q_{-i} = P_{i-1}^*$ for $i \geq 1$):

$$\cdots \leftarrow P_1^* \leftarrow P_0^* \xleftarrow{e^*q} Q_0 \leftarrow Q_1 \leftarrow \cdots,$$

Let us consider the A-dual $P_\bullet = Q_\bullet^*$

$$(*) \qquad \cdots \to P_1 \to P_0 \xrightarrow{q^*e^{**}} Q_0^* \to Q_1^* \to \cdots.$$

It is the concatenation of the sequence

$$\cdots \to P_1 \to P_0 \xrightarrow{e^{**}} M^{**} \to 0$$

with the exact sequence

$$0 \to M^{**} \xrightarrow{q^*} Q_0^* \to Q_1^* \to \cdots.$$

In particular, the complex $(*)$ is exact at the positions P_i and Q_i^* with $i \geq 1$.

Conversely, let $P_\bullet = (P_i, f_i)_i$ be a minimal complex of projective modules

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \xrightarrow{f_{-1}} P_{-1} \to \cdots$$

such that $H_i(P_\bullet) = 0$ for all $i \neq 0, -1$, and such that the A-dual complex $P_\bullet^*$ is acyclic. Let $e\colon P_0 \to M$ be the cokernel of f_1, thus

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{e} M \to 0$$

is a minimal projective resolution of M. Since the complex $P_\bullet^*$ is acyclic, it follows that M is semi-Gorenstein-projective.

Since M is the cokernel of f_1, we see that M^* is the kernel of f_{-1}^*. Since the complex $P_\bullet^*$ is acyclic, there is the exact sequence

$$0 \leftarrow M^* \leftarrow P_{-1}^* \xleftarrow{f_{-1}^*} P_{-2}^* \xleftarrow{f_{-2}^*} P_{-3}^* \leftarrow \cdots,$$

and this is a projective resolution of M^*. Since the A-dual sequence

$$P_{-1} \xrightarrow{f_{-1}} P_{-2} \xrightarrow{f_{-2}} P_{-3} \to \cdots.$$

is exact, we see that M^* is semi-Gorenstein-projective. □

2.4. The ∞-torsionfree right modules Z with $\Omega^2 Z$ semi-Gorenstein-projective. We recall that a module M is said to be ∞-*torsionfree* provided $\operatorname{Tr} M$ is semi-Gorenstein-projective.

Proposition. *The transpose* Tr *provides a bijection between the reduced modules M such that both M and M^* are semi-Gorenstein-projective and the reduced ∞-torsionfree right modules Z with $\Omega^2 Z$ semi-Gorenstein-projective.*

For the proof, we need the following (well-known) lemma.

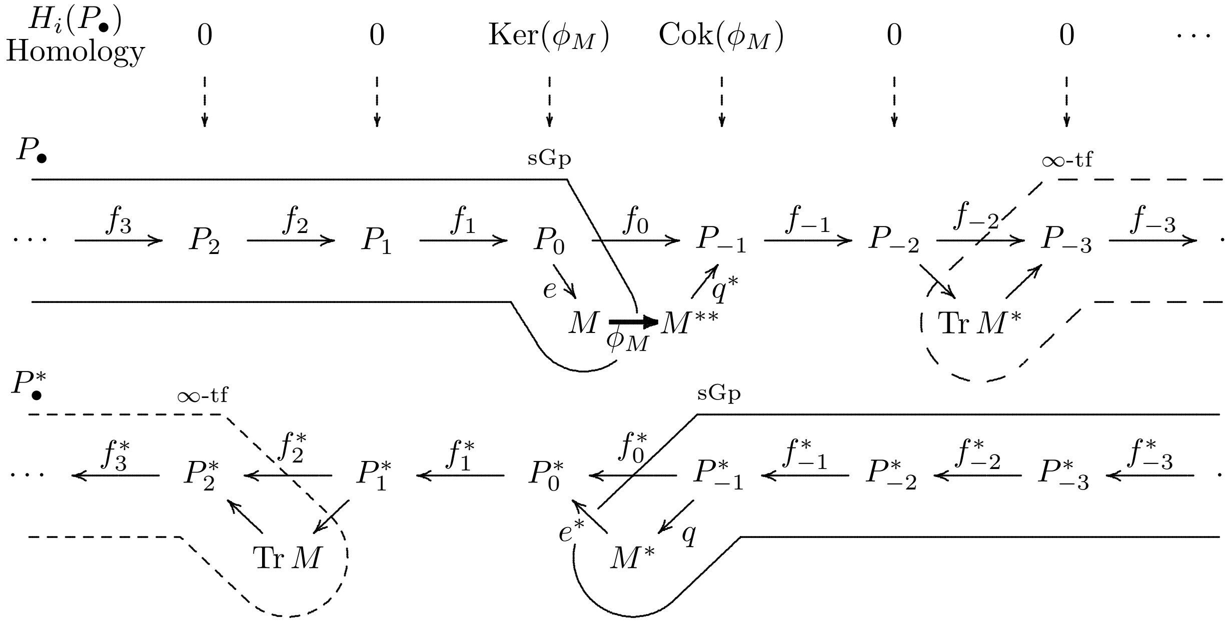
Lemma. *For any module M, we have $\Omega^2 M = (\operatorname{Tr} M)^*$.*

Proof of Lemma. Take a minimal projective presentation $P_1 \xrightarrow{f_1} P_0 \to M \to 0$ Then $\Omega^2 M = \operatorname{Ker} f_1$. By definition of $\operatorname{Tr} M$, there is the exact sequence $P_0^* \xrightarrow{f_1^*} P_1^* \to \operatorname{Tr} M \to 0$. If we apply $^* = \operatorname{Hom}(-, A_A)$, we get the exact sequence $0 \to (\operatorname{Tr} M)^* \to P_1^{**} \xrightarrow{f_1^{**}} P_0^{**}$. But f_1^{**} can be identified with f_1, thus $(\operatorname{Tr} M)^* = \operatorname{Ker} f_1^{**} = \operatorname{Ker} f_1 = \Omega^2 M$. □

Proof of Proposition. Let M be a reduced module such that both M and M^* are semi-Gorenstein-projective and $Z = \operatorname{Tr} M$. Since $M = \operatorname{Tr} Z$ is semi-Gorenstein-projective, Z is ∞-torsionfree. According to the lemma, $\Omega^2 Z = (\operatorname{Tr} Z)^* = (\operatorname{Tr}\operatorname{Tr} M)^* = M^*$, thus $\Omega^2 Z$ is semi-Gorenstein-projective.

Conversely, let Z be a reduced ∞-torsionfree module such that $\Omega^2 Z$ is semi-Gorenstein-projective. Then $M = \operatorname{Tr} Z$ is semi-Gorenste-projective. The Lemma asserts that $M^* = (\operatorname{Tr}\operatorname{Tr} M)^* = (\operatorname{Tr} Z)^* = \Omega^2 Z$. This shows that M^* is semi-Gorenstein-projective. □

2.5 Summary. Let M be a reduced module such that both M and M^* are semi-Gorenstein-projective. The Main Theorem yields a minimal complex $P_\bullet$. Let us display, first, the complex $P_\bullet$ indicating the homology groups above, and second, directly below, the acyclic A-dual complex $P_\bullet^*$. We insert the modules M, M^*, M^{**}, together with the canonical map $\phi_M : M \to M^{**}$ (shown as a bold arrow), as well as the modules $\operatorname{Tr} M$ and $\operatorname{Tr} M^*$. Since the modules M and M^* both are semi-Gorenstein-projective, the modules $\operatorname{Tr} M$ and $\operatorname{Tr} M^*$ are ∞-torsionfree. The complexes $P_\bullet$ and $P_\bullet^*$ provide minimal projective resolutions of the modules M and M^*, respectively (they are encompassed by solid lines, with label sGp added). Similarly, $P_\bullet$ and $P_\bullet^*$ provide minimal projective coresolutions of the modules $\operatorname{Tr} M^*$ and $\operatorname{Tr} M$ which are concatenations of $\mho$-sequences, respectively (these coresolutions are encompassed by dashed lines, with label ∞-tf added).



Be aware that the complexes $P_\bullet$ and $P_\bullet^*$ with the accompagnying modules seem to look quite similar, however there is a decisive difference: whereas the complex $P_\bullet^*$ is acyclic, the complex $P_\bullet$ usually is not acyclic (its homology modules are mentioned above the complex). Let us stress that $P_\bullet$ is acyclic if and only if M is Gorenstein-projective.

2.6. Remarks. **(1)** The Main Theorem illustrates nicely that *an indecomposable module M is Gorenstein-projective if and only if both M and M^* are semi-Gorenstein-projective and M is reflexive* (since the latter means that ϕ_M is an isomorphism), as known from [AB], and stressed for example in [AM].

(2) By construction, the complex $P_\bullet$ (and thus also $P_\bullet^*$) is uniquely determined by the module M. Let us stress that $P_\bullet$ *is usually* **not** *determined by M^*.*

In general, given an acyclic minimal complex $Q_\bullet = (Q_i, d_i : Q_i \to Q_{i-1})$ of projective right modules (such as $Q_\bullet = P_\bullet^*$), say with N_i being the image of d_i, then any module N_i determines uniquely the modules N_j with $j \le i$, since $N_j = \Omega^{j-i} N_i$, but usually not the modules N_j with $j > i$.

If we look at the complexes $P_\bullet$ which are obtained in Main Theorem, then there do exist examples, where $P_\bullet = (P_i, f_i)$ is **not** determined by M^* (this is the image of f_0), as shown in [RZ2]. Namely, let $q \in k$ be an element with infinite multiplicative order and $A = \Lambda(q)$ the algebra defined in [RZ2](1.1). Then the modules M of the form $M(1, -q, c)$ with $c \in k$ are indecomposable, non-projective and semi-Gorenstein-projective and they are pairwise non-isomorphic (thus also the right modules $\operatorname{Tr} M = \operatorname{Tr} M(1,-q,c)$ are pairwise non-isomorphic) — whereas all the right modules $M^* = M(1,-q,c)^*$ are isomorphic, and also semi-Gorenstein-projective, see [RZ2](1.7); they are of the form $M^* = M'(1,-q^{-1},0)$, see [RZ2](9.4). Actually, in this case already all the right modules $\Omega \operatorname{Tr} M = (\Omega M)^*$ are isomorphic, namely of the form $M'(1,-1,0)$, see [RZ2](3.2).

To phrase it differently: [RZ2] provides an infinite family of acyclic minimal complexes $Q(c)_\bullet = (Q(c)_i, d(c)_i)$ indexed by the elements $c \in k$, such that for any $i \in \mathbb{Z}$, the images of the maps $d(c)_i$ are pairwise non-isomorphic if $i \geq 0$, but pairwise isomorphic if $i < 0$.

(3) Even if the modules M and M^* are indecomposable, the module M^{**} may be decomposable, as the example of $M = M(q)$ in [RZ1] shows. Note that if M^* is indecomposable and not projective (this is the case in the example), then also $\operatorname{Tr} M^*$ is indecomposable and not projective, thus in the complex $P_\bullet$ displayed in 2.5, the images of all the maps f_i with $i \neq 0, -1$ can be indecomposable and not projective, whereas M^{**} is decomposable.

(4) Let A be a connected algebra with a non-reflexive module M such that both M and M^* are semi-Gorenstein-projective. Then, of course, A is not left weakly Gorenstein (recall that an algebra A is said to be *left weakly Gorenstein,* provided any semi-Gorenstein-projective module is Gorenstein-projective, see [RZ1]). Is it possible that A is right weakly Gorenstein (this means that any ∞-torsionfree module is Gorenstein-projective)? As we have mentioned in 2.4, the module $\operatorname{Tr} M^*$ is always ∞-torsionfree. Thus, if $\operatorname{Tr} M^*$ is not Gorenstein-projective, then A is not right weakly Gorenstein. But we do not know whether $\operatorname{Tr} M^*$ can be Gorenstein-projective.

In section 3, we discuss the extreme case that M^* is projective (thus $\operatorname{Tr} M^* = 0$). According to the classical homological conjectures, this case should be impossible, see 3.5. But could M^* be Gorenstein-projective?

(5) After completing the paper, the authors became aware of the recent preprint [G] by Gélinas which also deals with the complex $P_\bullet$. There, the central (and decisive) map $q^*\phi_M e$ is called the *Norm map* of the module M, with reference to Buchweitz [B], 5.6.1.

3. Special cases.

3.1. The case where ϕ_M is an epimorphism (or a monomorphism). Let us consider now the special case of a module M with both M, M^* semi-Gorenstein-projective such that ϕ_M is an epimorphism (or a monomorphism). But we stress from the beginning that at present **no non-trivial such example is known** (all known modules M with both M, M^* semi-Gorenstein-projective such that ϕ_M is an epimorphism or a monomorphism, are Gorenstein-projective).

Proposition.

(1) *Let M be a semi-Gorenstein-projective module. Then we have: M^* is semi-Gorenstein-projective and ϕ_M is an epimorphism if and only if $(\Omega M)^*$ is semi-Gorenstein-projective.*

(2) *Let M' be a torsionless semi-Gorenstein-projective module and let $M = \mho M'$. Then M is semi-Gorenstein-projective. And $(M')^*$ is semi-Gorenstein-projective if and only if M^* is also semi-Gorenstein-projective and ϕ_M is an epimorphism.*

Here, we consider in (1) a module M such that both M and M^* are semi-Gorenstein-projective, and in (2), a module M' such that both M' and $(M')^*$ are semi-Gorenstein-projective. In (1) we deal with the case that ϕ_M is an epimorphism. In (2) we deal with the case that $\phi_{M'}$ is an monomorphism (namely, M' is torsionless if and only if $\phi_{M'}$ is a monomorphisms).

Proof of (1). We can assume that M is indecomposable and not projective. Since M is semi-Gorenstein-projective, it follows that the exact sequence $0 \to \Omega M \to P(M) \to M \to 0$ is an $\mho$-sequence, and $0 \to M^* \to P(M)^* \to (\Omega M)^* \to 0$ is exact. Thus, $(\Omega M)^*$ is semi-Gorenstein-projective if and only if M^* is semi-Gorenstein-projective and $\operatorname{Ext}^1((\Omega M)^*, A_A) = 0$. According to Lemma 2.4(b) in [RZ1], we have $\operatorname{Ext}^1((\Omega M)^*, A_A) = 0$ if and only if ϕ_M is an epimorphism.

Proof of (2). We can assume that M' is indecomposable and not projective. Since M' is torsionless and semi-Gorenstein-projective, the module $M = \mho M'$ is semi-Gorenstein-projective. There is an $\mho$-sequence $0 \to M' \to P \to M \to 0$, thus an exact sequence $0 \to M^* \to P^* \to (M')^* \to 0$. Then $(M')^*$ is semi-Gorenstein-projective if and only if M^* is semi-Gorenstein-projective and $\operatorname{Ext}^1((M')^*, A_A) = 0$. According to 2.4(b) of [RZ1], we have $\operatorname{Ext}^1((M')^*, A_A) = 0$ if and only if ϕ_M is an epimorphism. $\square$

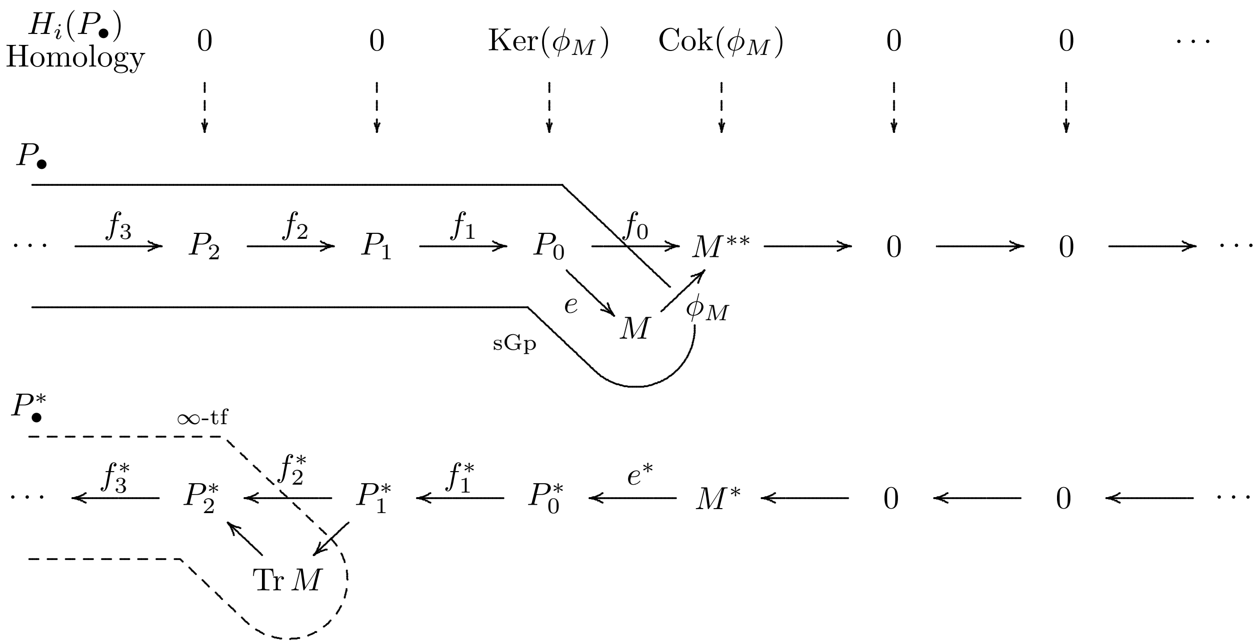
Remark. If we denote by $\mathcal{M}$ the class of reduced modules M such that both M and M^* are semi-Gorenstein-projective and ϕ_M is an epimorphism, and by $\mathcal{M}'$ the class of reduced modules M' such that both M' and $(M')^*$ are semi-Gorenstein-projective and $\phi_{M'}$ is a monomorphism, then Ω and $\mho$ provide inverse bijections between isomorphism classes as follows:

$$\mathcal{M}' \quad \underset{\Omega}{\overset{\mho}{\rightleftarrows}} \quad \mathcal{M}$$

If M belongs to $\mathcal{M}$ and $M' = \Omega M$ (thus $\mho M' = M$), then $\operatorname{Cok} \phi_{M'} \simeq \operatorname{Ker} \phi_M$.

3.2. The semi-Gorenstein-projective modules M with M^* projective. Another special case should be considered, namely the case of a semi-Gorenstein-projective reduced module such that M^* is projective. Also here, let us stress from the beginning that at present **no non-trivial such example is known** (all known semi-Gorenstein-projective modules M with M^* projective are Gorenstein-projective, thus even projective).

Let M be a semi-Gorenstein-projective module with M^* being projective. In addition, we may assume that M is reduced. Since M^* is projective, we take as projective cover $q\colon Q_0 \to M^*$ the identity map $1 = 1_{M^*}$; thus $f_0 = \phi_M \cdot e$. The diagram considered in 2.5 now has the following special form:



Here, $P^*_\bullet$ is acyclic, $\mathrm{Tr}\, M$ is the image of f^*_{-2} and $e\colon M^* = P_0^*$ is an inclusion map. It follows that $\mathrm{Tr}\, M$ has projective dimension at most 2 (and projective dimension at most 1, in case $M^* = 0$).

3.3. Recall that a module M is a Nunke module provided M is semi-Gorenstein-projective and $M^* = 0$ (and the Nunke condition for an algebra A asserts that the zero module is the only Nunke A-module, see 3.5).

Proposition. *The transpose* Tr *provides a bijection between the modules M which are semi-Gorenstein-projective, with M^* being projective, and the ∞-torsionfree right modules Z of projective dimension at most* 2*.*

The transpose Tr *provides a bijection between the Nunke modules M and the ∞-torsionfree right reduced modules Z of projective dimension at most* 1*.* □

3.4. Nunke modules. We consider now Nunke modules. As we will see, this is essentially just the situation where, on the one hand, ϕ_M is an epimorphism (as discussed in 3.1), and, on the other hand, M^* is projective (as discussed in 3.2). Of course, again we have to stress that **no non-trivial example is known** (the only known Nunke module is the zero module) and there is the old conjecture (one of the classical homological conjectures) that no non-zero Nunke module exists (for a further discussion of corresponding conjectures, see 3.5).

Proposition. *Let M be a module. The following conditions are equivalent:*
(i) *M is semi-Gorenstein-projective, ϕ_M is an epimorphism and M^* is projective.*
(ii) *M is the direct sum of a projective module and a Nunke module.*

Proof. We can assume that M is indecomposable.

(i) $\implies$ (ii). Let M be a semi-Gorenstein-projective module, with ϕ_M an epimorphism and M^* projective. Then also M^{**} is projective, thus ϕ_M is a split epimorphism, thus

$M \simeq M' \oplus M^{**}$. Since we assume that M is indecomposable, there are two possibilities: Either $M' = 0$, thus $M \simeq M^{**}$, thus M is projective. Or else $M^{**} = 0$ and thus already $M^* = 0$ (namely, if $M^* \neq 0$, then also $M^{**} \neq 0$, since the module M^* is torsionless), thus M is a Nunke module.

(ii) $\implies$ (i). If M is projective, then M is semi-Gorenstein-projective. Also, M is reflexive, thus ϕ_M is surjective, and with M projective, also M^* is projective.. If M is a Nunke module, then $M^* = 0$ implies that $M^{**} = 0$, thus ϕ_M is surjective and M^*, being the zero module, is projective. □

3.5. Some conjectures.

Proposition. *Let A be an artin algebra. We consider the following conditions:*

(1) *There is a bound b with the following property: If M is a right A-module of finite projective dimension, then the projective dimension of M is at most b.*
(2) *A semi-Gorenstein-projective A-module M with M^* of finite projective dimension is projective.*
(3) *A semi-Gorenstein-projective A-module M with M^* projective is projective.*
(4) *The only Nunke module is the zero module*
(5) *There is no simple module which is a Nunke module.*
(6) *There is no simple injective module which is a Nunke module.*

Then (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6).

Remark. One may conjecture that all these conditions hold true in general.

The condition (1) is called the **finitistic dimension conjecture** for $A^{\rm op}$, and (4) is called the **Nunke condition** for A. Both are classical homological conjectures (that these conditions hold true for all finite-dimensional algebras) and it is well-known that (1) implies (4). See, for example, [H]. The assertion (5) is called the **weak Nunke condition**, it is equivalent to the **generalized Nakayama conjecture** (see [AR], Proposition 1.5). The conditions (1), (4), and (5) are mentioned in [ARS] as conjectures 11, 12, and 9, respectively). The Proposition formulates the intermediate conjectures (2) and (3). Also, we mention the weaker conjecture (6) which will be discussed in 3.6 and 3.7.

Proof of Proposition. (1) $\implies$ (2). Let M be semi-Gorenstein-projective. We may assume that M is indecomposable and not projective. Now $Z = \operatorname{Tr} M$ is ∞-torsionfree, thus there is an exact sequence

$$0 \to Z \to P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots$$

with projective modules P^i, where $i \geq 0$. Let M^i be the image of d^{i-1} for all $i \geq 1$. Since Z is indecomposable and not projective, we see that $\Omega^i M^i = Z$ for all $i \geq 1$.

We apply the Lemma from 2.4 to $Z = \operatorname{Tr} M$ and see that $\Omega^2 Z = (\operatorname{Tr}\operatorname{Tr} M)^* = M^*$, since M is reduced.

By assumption, M^* has finite projective dimension, thus $\Omega^a(M^*) = 0$, for some $a \geq 0$ and therefore $\Omega^{i+2+a} M^i = \Omega^{2+a} Z = \Omega^a M^* = 0$. Thus M^i has finite projective dimension, for all $i \geq 1$. According to (1), we know that M^i has projective dimension at most b, for all $i \geq 1$. Since M^{b+1} has projective dimension at most b, we see that $Z = \Omega^{b+1} M^{b+1} = 0$. But $Z = 0$ imples that $M = \operatorname{Tr} Z = 0$.

(2) $\implies$ (3) is trivial. (3) $\implies$ (4): Let M be semi-Gorenstein-projective and $M^* = 0$. Since M^* is projective, it follows from (3) that M is projective. But a projective module P with $P^* = 0$ is the zero module. The implications (4) $\implies$ (5) $\implies$ (6) are again trivial. □

Remark. As we have mentioned already, the weak Nunke condition 3.5 (5) is equivalent to the following conjecture, now called the **Auslander-Reiten conjecture:** *There is no non-zero semi-Gorenstein-projective* M*, with* $\operatorname{Ext}^i_A(M, M) = 0$ *for all* $i \geq 1$. This was shown by Auslander and Reiten [AR].

The weak Nunke condition asserts that a simple semi-Gorenstein-projective module should be torsionless. One may ask whether a simple semi-Gorenstein-projective module should be even Gorenstein-projective, but at present, nothing is known about simple semi-Gorenstein-projective modules, this is really a pity. There is the special case of a simple injective module. Is it possible that a simple injective module is semi-Gorensetin-projective without being already projective (thus uninteresting)? In 3.5, we have added the corresponding conjecture: *There are no simple injective Nunke modules*, as conjecture (6). One may call this conjecture (6) the **very weak Nunke condition.** In 3.6 and 3.7, we show that the very weak Nunke condition is equivalent to a weak form of the Auslander-Reiten conjecture.

3.6. Simple injective semi-Gorenstein-projective modules. The conjecture 3.5 (6) asserts that a simple injective semi-Gorenstein-projective module S should be projective (thus $\Omega S = 0$). In 3.6, we look at a simple injective semi-Gorenstein-projective module and try to analyse ΩS. Let A be an artin algebra and S a simple injective module with endomorphism ring $\operatorname{End}(S) = D$. Let $M = \Omega S$. Let e be a primitive idempotent of A such that $eS \neq 0$. Let $B = A/AeA$, thus $\mathcal{U}(S) = \operatorname{mod} B$ is a full subcategory of $\operatorname{mod} A$; it consists of all the A-modules which do not have S as a composition factor. Note that M belongs to $\mathcal{U}(S)$.

As we have mentioned, A is supposed to be an artin algebra. We assume that A is a k-algebra, where k is a commutative artinian ring and A is of finite length as a k-module.

Proposition. *Let* S *be a simple injective module with* $\operatorname{End}(S) = D$. *Let* $M = \Omega S$. *Let* e *be a primitive idempotent of* A *such that* $eS \neq 0$ *and* $B = A/AeA$. *The following conditions are equivalent:*

- (i) *The module* S *is semi-Gorenstein-projective.*
- (ii) *The* B*-module* M *is a Nunke* B*-module,* $\operatorname{Ext}^i_B(M, M) = 0$ *for all* $i \geq 1$*, and either* $M = 0$ *or else* $\operatorname{End}(M)$ *is isomorphic to* D *as a* k*-algebra.*

We need some preparations for the proof.

(a) *Projective* B*-module are projective when considered as* A*-modules. Thus* $\mathcal{U}(S)$ *is closed under projective covers.* □

(b) *If* U, U' *are* B*-modules, then* $\operatorname{Ext}^i_A(U, U') = \operatorname{Ext}^i_B(U, U')$ *for all* $i \geq 0$.

Proof. This is clear for $i = 0$. For $i \geq 1$, we start with a projective resolution $P_\bullet$ of the B-module U. According to (a), this is also a projective resolution of U considered as an A-module. We form the complex $\operatorname{Hom}_B(P_\bullet, U') = \operatorname{Hom}_A(P_\bullet, U')$ and consider its homology. □

(c) *If U is a B-module, then* $\operatorname{Ext}^i_A(U, P(S)) = \operatorname{Ext}^i_A(U, M)(= \operatorname{Ext}^i_B(U, M))$ *for all* $i \geq 0$.

Proof. We apply $\operatorname{Hom}_A(U, -)$ to the exact sequence $0 \to M \to P(S) \to S \to 0$. We have $\operatorname{Hom}(U, S) = 0$, since U belongs to $\mathcal{U}(S)$, and we have $\operatorname{Ext}^i(U, S) = 0$ for $i \geq 1$, since S is injective. This shows that $\operatorname{Ext}^i_A(U, M) = \operatorname{Ext}^i_A(U, P(S))$ for all $i \geq 0$. For the second equality, see (b). □

(d) *Let U be a B-module. Then U is semi-Gorenstein-projective as an A-module if and only if U is semi-Gorenstein-projective as a B-module and* $\operatorname{Ext}^i_B(U, M) = 0$ *for all* $i \geq 1$.

Proof. We can assume that A is basic, thus ${}_AA = B \oplus P(S)$. By definition, U is semi-Gorenstein-projective as an A-module if and only if $\operatorname{Ext}^i_A(U, B) = 0$ and $\operatorname{Ext}^i_A(U, P(S)) = 0$ for all $i \geq 1$. According to (b), we have $\operatorname{Ext}^i_B(U, B) = \operatorname{Ext}^i_A(U, B)$ for all $i \geq 1$. But $\operatorname{Ext}^i_A(U, B) = 0$ for all $i \geq 1$ means that U is semi-Gorenstein-projective as a B-module. Also, by (c) we have $\operatorname{Ext}^i_A(U, P(S)) = \operatorname{Ext}^i_B(U, M)$ for all $i \geq 0$. □

(e) *Let $M \neq 0$. The embedding $M \to P(S)$ is a left* $\operatorname{add}({}_AA)$*-approximation if and only if* $\operatorname{End}(M)$ *is isomorphic to D as a k-algebra and* $\operatorname{Hom}(M, {}_BB) = 0$.

Proof. Let $\iota \colon M \to P(S)$ be the inclusion map. We assume again that A is basic, thus ${}_AA = B \oplus P(S)$.

A map $f \colon M \to {}_BB$ factors through ι if and only if $f = 0$ (since $\operatorname{Hom}(P(S), {}_BB) = 0$). Thus all maps $f \colon M \to {}_BB$ factor through ι if and only if $\operatorname{Hom}(M, {}_BB) = 0$.

It remains to look at maps $M \to P(S)$. Any endomorphism of $P(S)$ maps $M = \operatorname{rad} P(S)$ into M, thus there are canonical maps $\pi' \colon \operatorname{End}(P(S)) \to \operatorname{End}(M)$ as well as $\pi \colon \operatorname{End}(P(S)) \to \operatorname{End}(S)$. Since $P(S)$ is a projective cover of S, thus π is surjective. Since S is injective, we have $\operatorname{Hom}(P(S), M) = 0$, π is an injective map. Altogether, we see that π is an isomorphism. Since $M \neq 0$, and S is injective, $\operatorname{Hom}(S, P(S)) = 0$, therefore π' is injective. It follows that $\mu = \pi'\pi^{-1}$ is an embedding of $D = \operatorname{End}(S) \simeq \operatorname{End}(P(S))$ into $\operatorname{End}(M)$. Of course, μ is an surjective if and only if ι is a $\operatorname{add}({}_AA)$-approximation. Note that the embedding μ is an isomorphism if and only if the length of ${}_kD$ is equal to the length of ${}_k\operatorname{End}(M)$, thus if and only if $\operatorname{End}(M)$ is isomorphic to D as a k-algebra. □

Proof of Proposition. (i) $\implies$ (ii). Let S be semi-Gorenstein-projective. If S is projective, then $M = 0$ and the conditions in (ii) are trivially satisfied. Thus, we assume that S is not projective, thus Then ${}_AM = \Omega S$ is an indecomposable semi-Gorenstein-projective A-module. According to (d), M considered as a B-module is semi-Gorenstein-projective and $\operatorname{Ext}^i_B(M, M) = 0$ for all $i \geq 1$. Also, the inclusion $\iota \colon M \to P(S)$ is a left $\operatorname{add}({}_AA)$-approximation (since $\operatorname{Ext}^1(S, {}_AA) = 0$), thus (e) asserts that $\operatorname{End}(M)$ is isomorphic to D as a k-algebra and $\operatorname{Hom}(M, {}_BB) = 0$. In particular, we see that M is a Nunke B-module.

(ii) $\implies$ (i). We can assume that M is non-zero (otherwise S is projective, thus of course semi-Gorenstein-projective). We assume that M is a Nunke B-module, that $\operatorname{End}(M)$ is isomorphic to D as a k-algebra (in particular, M is indecomposable) and that $\operatorname{Ext}^i_B(M, M) = 0$ for all $i \geq 1$. Since M is a semi-Gorenstein-projective B-module and $\operatorname{Ext}^i_B(M, M) = 0$ for all $i \geq 1$, we can apply (d) to $U = M$ and see that M is semi-Gorenstein-projective also as an A-module. Since $\operatorname{End}(M)$ is isomorphic to D as a k-algebra and $\operatorname{Hom}(M, {}_BB) = 0$, we can use (e). It asserts that the embedding $M \to P(S)$ is a left $\operatorname{add}({}_AA)$-approximation. Since M is a Nunke B-module, we have $\operatorname{Hom}(M, {}_BB) = 0$. It follows from $M \neq 0$ that M is not a projective B-module. Of course, M is also not isomorphic to $P(S)$, thus M is not projective as an A-module. Since M is semi-Gorenstein-projective, indecomposable and not projective, it follows that $S = \mho M$ is semi-Gorenstein-projective. □

3.7. Corollary. *There exists an artin algebra A with a simple injective Nunke module if and only if there exists an artin algebra B with an indecomposable semi-Gorenstein-projective module M with* $\operatorname{rad}\operatorname{End}(M) = 0$, *such that* $\operatorname{Hom}(M, {}_BB) = 0$ *and* $\operatorname{Ext}^i_B(M, M) = 0$ *for all* $i \geq 1$.

Proof. Let A be an artin algebra with a simple, injective, semi-Gorenstein-projective module S with $S^* = 0$. Let $\mathcal{U}(S) = \operatorname{mod} B$ for some artin algebra B. Then Proposition 3.6 asserts that the B-module M is semi-Gorenstein-projective, $\operatorname{Hom}(M, {}_BB) = 0$, and $\operatorname{Ext}^i_B(M, M) = 0$ for all $i \geq 1$, and $\operatorname{End}(M)$ is isomorphic to $\operatorname{End}(S)$ as a k-algebra. Of course, if $\operatorname{End}(M)$ is isomorphic to $\operatorname{End}(S)$, then $\operatorname{End}(M)$ is a division ring, thus M is indecomposable and $\operatorname{rad}\operatorname{End}(M) = 0$.

Conversely, let B be an artin algebra with an indecomposable B.module M which is semi-Gorenstein-projective, such that $\operatorname{rad}\operatorname{End}(M) = 0$, $\operatorname{Hom}(M, {}_BB) = 0$, and finally $\operatorname{Ext}^i_B(M, M) = 0$ for all $i \geq 1$. Let $D = \operatorname{End}(M)$. Since M is indecomposable, D is a local artin algebra. Since $\operatorname{rad}\operatorname{End}(M) = 0$, we see that D is a division ring. Let $A = \begin{bmatrix} B & M \\ 0 & D^{\mathrm{op}} \end{bmatrix}$. Let P be the indecomposable projective A-module $P = \begin{bmatrix} M \\ D^{\mathrm{op}} \end{bmatrix}$, let $S = P/\operatorname{rad} P$. Then $\operatorname{End}(S) = D$. Note that S is simple and injective, $\mathcal{U}(S) = \operatorname{mod} B$ and $\Omega S = M$. Proposition 3.6 asserts that S is semi-Gorenstein-projective. Also, S is not projective, since $M \neq 0$. Since S is injective and not projective, we see that $S^* = 0$, thus S is a Nunke module. □

4. (Short) local algebras.

One may wonder whether there do exist non-trivial examples of modules which satisfy the conditions discussed in section 3. Here we want to mention at least one class of algebras, the short local algebras, were no examples of this kind do exist. First, let A be an arbitrary local artin algebra.

4.1. Proposition. *Let A be a local artin algebra. Let M be a semi-Gorenstein-projective module. If M^* is projective, then M is projective.*

Proof. We can assume that M is indecomposable and not projective, thus reduced. According to the Main Theorem, there is a minimal complex $P_\bullet = (P_i, f_i)$ of projective modules such that $P_\bullet^*$ is acyclic, as exhibited in the display 2.5. In particular, M^* is the image of f_0^*. Since f_0 maps into the radical of P_{-1}, f_0^* maps ito the radical of P_0^*, thus M^* is a submodule of the radical of a projective right module. Since A is local, M^* cannot have an indecomposable projective direct summand.

If we assume that M^* is projective, then $M^* = 0$. But for a local algebra A, this implies that $M = 0$, a contradiction. This completes the proof. □

4.2. We recall that a local algebra A is said to be *short* provided that $J^3 = 0$. From now on, let A be a short local artin algebra with $e = |J/J^2|$ and $a = |J^2|$. The pair (e, a) is called the *Hilbert type* of A. For any module M, we denote by $|M|$ its length and set $t(M) = |M/JM|$. If M has Loewy length at most 2, then $\mathbf{dim}\, M = (t(M), |JM|)$ is called the *dimension vector* of M. Note that if $Q_\bullet = (Q_i, d_i \colon Q_i \to Q_{i-1})$ is a minimal complex of projective modules and N_i be the image of d_i, then N_i has Loewy length at most 2, thus its dimension vector is defined.

Proposition. *Let A be a short local artin algebra.*

(a) *Assume that A is not self-injective, and let $Q_\bullet = (Q_i, d_i \colon Q_i \to Q_{i-1})$ be an acyclic minimal complex of projective modules. Let N_i be the image of d_i and assume that at least one of the modules N_i is semi-Gorenstein-projective. Then all modules Q_i have the same rank, say rank t and $\mathbf{dim}\, N_j = (t, at)$ for all $j \in \mathbb{Z}$.*

(b) *Let M be a module such that both M and M^* are semi-Gorenstein-projective. Then* $|\operatorname{Ker}(\phi_M)| = |\operatorname{Cok}(\phi_M)|$. (Thus, if ϕ_M is a monomorphism or an epimorphism, then ϕ_M is an isomorphism, and therefore M is Gorenstein-projective.)

The proof will rely on two results (Theorems 3 and 4) from [RZ3].

Proof of (a). Let $Q_\bullet = (Q_i, d_i \colon Q_i \to Q_{i-1})$ be an acyclic minimal complex of projective modules, with N_i the image of d_i, for all $i \in \mathbb{Z}$. Since $Q_\bullet$ is acyclic and minimal, the canonical maps $Q_i \to N_i$ are projective covers, thus $t(Q_i) = t(N_i)$ for all $i \in \mathbb{Z}$.

According to Theorem 3 of [RZ3], the complex $Q_\bullet$ shows that A is of Hilbert type $(a+1, a)$ with $a \geq 1$, and that either all the modules N_i have the same dimension vector (type I), in particular all the projective modules Q_i have the same rank, or else the rank of the modules Q_i strictly increases for $i \gg 0$ (type II).

Let us now assume that N_0 is semi-Gorenstein-projective. Of course, N_0 is torsionless and not projective and

$$0 \leftarrow N_0 \leftarrow Q_0 \leftarrow Q_1 \leftarrow \cdots$$

is a minimal projective resolution of N_0. We apply Theorem 4 of [RZ3] to the indecomposable direct summands of N_0 and see that $a \geq 2$, and that all the modules $N_i = \Omega^i N_0$ with $i \in \mathbb{N}$ have the same dimension vector $\mathbf{dim}\, N_i = (t, at)$, where $t = t(N_0) = t(Q_0)$. As a consequence, $t(Q_i) = t(N_i) = t$ for all $i \geq 0$. This shows that $Q_\bullet$ cannot be of type II. Thus, $Q_\bullet$ is of type I, and therefore all the projective modules Q_i have the same rank t, and $\mathbf{dim}\, N_i = (t, at)$ for all $i \in \mathbb{Z}$. This completes the proof of (a). $\square$

Proof of (b). If A is self-injective, then all modules are reflexive, thus (b) holds trivially in this case. We therefore may assume that A is not self-injective.

Let M be a module such that both M and M^* are semi-Gorenstein-projective. According to the Main Theorem, there is a minimal complex $P_\bullet = (P_i, f_i)$ of projective modules such that $P_\bullet^*$ is acyclic, as exhibited in the display 2.5. We apply 9.3 to the opposite algebra $A^{\rm op}$, thus to right A-modules, namely to the acyclic complex $P_\bullet^*$ of projective right A-modules. Since the image of f_0^* is the semi-Gorenstein-projective module M^*, we see that all the modules P_i^* have the same rank, say t. Thus also the modules P_i have rank t.

Now ΩM is the image of f_1. Since P_1 is a projective cover of ΩM, we see that top ΩM has length t. Similarly, $\operatorname{Tr} M^*$ is the image of f_{-2} and P_{-2} is a projective cover of $\operatorname{Tr} M^*$, thus $\operatorname{top} \operatorname{Tr} M^*$ has length t. Next, ΩM is an idecomposable torsionless semi-projective module and not projective, thus Theorem 4 of [RZ3] asserts that its dimension vector is (t, at). Similarly, $\operatorname{Tr} M^*$ is an indecomposable ∞-torsionfree module and not projective, thus the same reference shows that the dimension vector of $\operatorname{Tr} M^*$ is also (t, at). We consider the sequence

$$0 \to \Omega M \xrightarrow{u} P_0 \xrightarrow{f_0} P_{-1} \xrightarrow{f_{-1}} P_{-2} \xrightarrow{r} \operatorname{Tr} M^* \to 0,$$

where u is the canonical inclusion and r the canonical projection. This is a complex, and the alternating sum of the length of the modules involved is zero (the modules ΩM and $\operatorname{Tr} M^*$ have length et, whereas the modules P_0, P_{-1}, P_{-2} have length $2et$). It follows that also the alternating sum of the length of the homology modules is 0, but this is $|\operatorname{Ker}(\phi_M)| - |\operatorname{Cok}(\phi_M)|$. $\square$

4.3. Remark. Let $a \geq 2$. In section 11 of [RZ3] we will exhibit a short local algebra A of Hilbert type $(a+1, a)$ which has a Loewy length 2 module M with dimension vector $(1, a)$ such that both M and M^* are semi-Gorenstein-projective, whereas M is not reflexive (actually, we have $|\operatorname{Ker}(\phi_M)| = |\operatorname{Cok}(\phi_M)| = 1$). The construction is a straightforward generalization of the case $a = 2$ algebras as discussed in [RZ1] and [RZ2].

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