The eigenvector variety of a matrix pencil.

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Abstract. Let k be a field and n, a, b natural numbers. A matrix pencil P is given by n matrices of the same size with coefficients in k, say by $(b \times a)$-matrices, or, equivalently, by n linear transformations $\alpha_i \colon k^a \to k^b$ with $i = 1, \dots, n$. We say that P is *reduced* provided the intersection of the kernels of the linear transformations α_i is zero. If P is a reduced matrix pencil, a vector $v \in k^a$ will be called an *eigenvector* of P provided the subspace $\langle \alpha_1(v), \dots, \alpha_n(v) \rangle$ of k^b generated by the elements $\alpha_1(v), \dots, \alpha_n(v)$ is 1-dimensional. Eigenvectors are called *equivalent* provided they are scalar multiples of each other. The set $\epsilon(P)$ of equivalence classes of eigenvectors of P is a Zariski closed subset of the projective space $\mathbb{P}(k^a)$, thus a projective variety. We call it the *eigenvector variety* of P. The aim of this note is to show that any projective variety arises as an eigenvector variety of some reduced matrix pencil.

1. Introduction.

Let k be a field and n, a, b natural numbers. A *matrix pencil* P is given by n matrices of the same size with coefficients in k, say $(b \times a)$-matrices or, equivalently, by n linear transformations $\alpha_i \colon k^a \to k^b$ with $i = 1, \dots, n$, thus we write $P = (a, b; \alpha_1, \dots, \alpha_n)$ and we call (a, b) the *dimension vector* of P. We say that P is *reduced* provided the intersection of the kernels of the linear transformations α_i is zero.

Let us assume that $P = (a, b; \alpha_1, \dots, \alpha_n)$ is a reduced matrix pencil. A vector $v \in k^a$ will be called an *eigenvector* of P provided the subspace of k^b generated by the elements $\alpha_1(v), \dots, \alpha_n(v)$ is 1-dimensional. Of course, if v is an eigenvector of P, then any non-zero multiple of v is also an eigenvector; eigenvectors are called *equivalent* provided they are multiples of each other. The set $\epsilon(P)$ of equivalence classes of the eigenvectors of P is a Zariski closed subset of the projective space $\mathbb{P}(k^a)$, thus a projective variety. We call it the *eigenvector variety* of P. If $n = 2$ and α_1 is the identity $(a \times a)$-matrix, then the eigenvectors of the matrix pencil $(a, a; \alpha_1, \alpha_2)$ are just the eigenvectors of α_2 as usually considered.

The aim of this note is to show the following.

Theorem 1. *Let k be algebraically closed. Any projective variety arises as the eigenvector variety $\epsilon(P)$ of a reduced matrix pencil P.*

We may require, in addition, that the reduced matrix pencil P is a matrix pencil of square matrices, see 4.2.

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In order to prove Theorem 1, we will reformulate the setting in terms of rings and modules, namely as dealing with bristle varieties of Kronecker modules. But before we do this, let us mention that the paper includes an appendix which provides further information on the existence of eigenvectors of reduced matrix pencils.

Here is the reformulation of Theorem 1 in terms of quivers and their representations (see, for example, [9]), thus in terms of rings and modules. Let $K(n)$ be the n-Kronecker quiver; it consists of two vertices, denoted by 1 and 2, and n arrows $\alpha_i\colon 1 \to 2$, with $1 \le i \le n$.

$$1 \quad \begin{array}{c} \xrightarrow{\alpha_1} \\ \vdots \\ \xrightarrow[\alpha_n]{} \end{array} \quad 2$$

The n-Kronecker modules are the representations of $K(n)$ over k. Kronecker modules are written in the form $M = (M_1, M_2; \alpha_1, \dots, \alpha_n)$ where M_1, M_2 are vector spaces and $\alpha_i\colon M_1 \to M_2$ are linear transformations, for $1 \le i \le n$; the dimension vector of M is the pair $\mathbf{dim}\, M = (\dim M_1, \dim M_2)$. If $(a, b; \alpha_1, \dots, \alpha_n)$ is matrix pencil, we call $M(P) = (k^a, k^b; \alpha_1, \dots, \alpha_n)$ the corresponding Kronecker module. Conversely, given a Kronecker module $M = (M_1, M_2; \alpha_1, \dots, \alpha_n)$, we may choose bases $\mathcal{B}_1, \mathcal{B}_2$ of M_1, M_2, respectively, so that we can write the linear transformations α_i as matrices. We obtain in this way a corresponding matrix pencil $P(M; \mathcal{B}_1, \mathcal{B}_2)$.

The n-Kronecker algebra $\Lambda(n) = kK(n)$ is the path algebra of $K(n)$, thus the $\Lambda(n)$-modules are just n-Kronecker modules. There are precisely two (isomorphism classes of) simple n-Kronecker modules, $S(1)$ and $S(2)$ (in general, if x is a vertex of an acyclic quiver, we denote by $S(x)$ the corresponding simple module; it is defined by $S(x)_x = k$ and $S(x)_y = 0$ for all vertices $y \neq x$). The module $S(1)$ is injective, the module $S(2)$ is projective. A $\Lambda(n)$-module M is called *reduced* provided $S(1)$ is not a submodule of M, thus iff the corresponding matrix pencils are reduced.

In general, given a ring R, a *bristle* is by definition an indecomposable R-module of length 2. An n-Kronecker module is a bristle if and only if it is indecomposable with dimension vector $(1, 1)$. Such a bristle is of the form $B(\lambda) = (k, k; \lambda_1, \dots, \lambda_n)$, where $(\lambda_1, \dots, \lambda_n)$ is a non-zero element of k^n.

Given a representation M of a quiver and $\mathbf{d}$ a dimension vector, one denotes by $\mathbb{G}_{\mathbf{d}}(M)$ the set of submodules of M with dimension vector $\mathbf{d}$ and calls it a *quiver Grassmannian;* this is a projective variety.

Let M be a reduced Kronecker module. Since we assume that M is reduced, any submodule of M with dimension vector $(1, 1)$ is indecomposable, thus a bristle. We write $\beta(M) = \mathbb{G}_{(1,1)}(M)$ for the set of bristle submodules of M and call it the *bristle variety* of M. Of course, $\beta(M)$ is nothing else than the eigenvector variety of the matrix pencils corresponding to M:

$$\beta(M) = \epsilon(P(M, \mathcal{B}_1, \mathcal{B}_2))$$

for bases $\mathcal{B}_1, \mathcal{B}_2$ of M_1, M_2, respectively.

Theorem 1 may be formulated as follows:

Theorem 1$'$. *Let k be algebraically closed. Any projective variety arises as the bristle variety $\beta(M)$ of a reduced Kronecker module M.*

This formulation shows that here we are in the realm of a highly contested topic: the realization of any projective variety as the moduli space of serial modules, as studied by Huisgen-Zimmermann and several coauthors [7,1,2], as the moduli space of thin modules, studied by Hille [5], and as a quiver Grassmannian, studied by Reineke [8]; see [10] and the discussions in various internet blogs quoted there. The main result of the present note was first presented at the conference in honor of Jerzy Weyman's 60th birthday, April 2015, at the University of Connecticut and then on various other occasions. Our construction is obviously based on the previous accounts, but the observation that it is sufficient to consider length 2 submodules seems to have been new at that time. For further discussions, we want to refer to Hille [6].

We say that a module is *bristled* provided it is generated by modules of length 2. The paper [11] has been devoted to the study of bristled Kronecker modules. There, it has been shown that for $n \geq 3$, there is an abundance of bristled Kronecker modules (see the appendix of the present paper). Of course, also the main result of the present note deals with bristled Kronecker modules. A module M will be called *fully bristled* provided it is bristled and any simple submodule of M is contained in a bristle submodule of M.

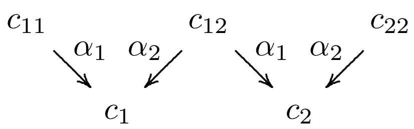
There is an interesting fully bristled n-Kronecker module C which will be exhibited in the next section, we call it the canonical bristled n-Kronecker module. The proof of Theorem $1'$ will be established by looking at submodules of C.

2. The canonical bristled module C.

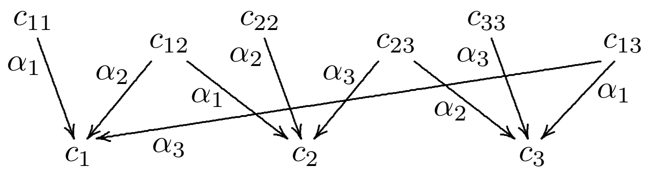
The aim of this section is to construct a special n-Kronecker module C which we will call the canonical bristled module. Here we allow that k is an arbitrary field.

We define $C = (C_1, C_2; \alpha_1, \dots, \alpha_n)$ as follows: its dimension vector is $\mathbf{dim}\, C = (\binom{n+1}{2}, n)$, the space C_2 has the basis elements c_i, with $1 \leq i \leq n$, the space C_1 has the basis elements $c_{ij} = c_{ji}$, where $1 \leq i \leq j \leq n$, and $\alpha_i(c_{ij}) = c_j$, whereas $\alpha_r(c_{ij}) = 0$ in case $r \notin \{i,j\}$. As we will see, C is always an indecomposable bristled module; we call it the *canonical bristled module.*

For $n = 1$, the module C is the only indecomposable module which is not simple; it is just the unique bristle. For $n = 2$, the canonical bristled module is the unique indecomposable 2-Kronecker module with dimension vector $(3,2)$ (it is the Auslander-Reiten translate of the simple injective module $S(1)$). Here are illustrations for $n = 2$ and $n = 3$:



$n = 2$



$n = 3$

(2.1) *For every non-zero element $c = \sum_i \lambda_i c_i$ of C_2, there is a unique bristle B with $c \in B$, it is generated by $d(c) = \sum_{i \leq j} \lambda_i \lambda_j c_{ij} \in C_1$ and has type $(\lambda_1 : \lambda_2 : \cdots : \lambda_n)$. It follows that C is a fully bristled module, that* $\dim \operatorname{Hom}(B, C) = 1$ *for any bristle B and that* $\operatorname{End}(C) = k$.

Proof. Given a linear combination $d = \sum_{i\leq j} \lambda_{ij}c_{ij}$, we have $\alpha_r(d) = \sum_i \lambda_{ir}c_i$ where we write $\lambda_{ij} = \lambda_{ji}$ for all i, j. It follows that we have $\alpha_r(d) = 0$ if and only if $\lambda_{ir} = 0$ for $1 \leq i \leq r$. As a consequence, $\alpha_r(d) = 0$ for all $1 \leq r \leq n$ implies that $d = 0$. This shows that C has no direct summand of the form $S(1)$.

Next, consider a non-zero element $c = \sum_i \lambda_i c_i$ of C_2 and let $d = d(\lambda) = \sum_{i\leq j} \lambda_i\lambda_j c_{ij} \in C_1$. We see that $\alpha_r(d) = \sum_i \lambda_i\lambda_r c_i = \lambda_r c$ is a multiple of c, and for at least one index r, the multiple $\lambda_r c$ is non-zero. Thus, the submodule generated by d is the vector space with basis c, d. It is a bristle and $\alpha_r(d) = \lambda_r \cdot c$ shows that this bristle is of type $(\lambda_1 : \lambda_2 : \cdots : \lambda_n)$.

On the other hand, consider a submodule U of C which is a bristle and assume that $c = \sum_i \lambda_i c_i$ generates its socle. Assume that $d = \sum_{i\leq j} \lambda_{ij}c_{ij}$ generates U. Since U is a bristle, $\alpha_j(d)$ is a multiple of c, for any j, say $\alpha_j(d) = \mu_j \cdot c$ for some μ_j. This means that

$$\sum_i \lambda_{ij}c_i = \alpha_j(d) = \mu_j \cdot c = \sum_i \mu_j\lambda_i c_i,$$

and therefore $\lambda_{ij} = \lambda_i\mu_j$ for all i, j. Since $\lambda_{ij} = \lambda_{ji}$, we have $\lambda_i\mu_j = \lambda_j\mu_i$ for all i, j. Now $0 \neq c = \sum_i \lambda_i c_i$, thus $\lambda_t \neq 0$ for some t. The equality $\lambda_t\mu_j = \lambda_j\mu_t$ implies that

$$\mu_j = \frac{\mu_t}{\lambda_t}\lambda_j,$$

for all j. As a consequence,

$$\lambda_{ij} = \lambda_i\mu_j = \frac{\mu_t}{\lambda_t}\lambda_i\lambda_j,$$

thus d is a multiple of $d(\lambda)$. This shows that the only submodule U which is a bristle with socle generated by $c = \sum_i \lambda_i c_i$ is the bristle generated by $d(c)$. For later reference, let us note that we have shown in this way that C has no indecomposable submodule with dimension vector $(2, 1)$.

Let $B_i = B(e_i)$ and $B_{ij} = B(e_i + e_j)$ where $e_1, \dots, e_n$ is the canonical basis of k^n and $i \neq j$. The bristle B_i is embedded as the submodule generated by $d(c_i) = c_{ii}$ and the bristle B_{ij} with $i \neq j$ is embedded as the submodule generated by $d(c_i + c_j) = c_{ii} + c_{ij} + c_{jj}$. These are $\binom{n+1}{2}$ submodules and their sum is equal to C. In particular, we see that C is a bristled module. Since every element of the socle of C is obviously contained in a bristle, C is a fully bristled module. On the other hand, we also know that any bristle B occurs as a unique submodule of C, thus $\dim \mathrm{Hom}(B, C) = 1$.

It remains to be seen that $\mathrm{End}(C) = k$. The assertion is clear for $n = 1$, thus we assume that $n \geq 2$. First, let us show that any non-zero endomorphism f of C is an automorphism. Namely, assume that f is an endomorphism with non-zero kernel. Let U be a proper non-zero subspace of C_2 which belongs to the kernel of f. Let $c \in C_2 \setminus U$. We claim that C/U has an indecomposable submodule V with dimension vector $(2, 1)$ whose socle is generated by $c + U$. Namely, if $0 \neq u \in C$, then there are bristles B and B' with socle kc and $k(c + u)$, respectively, and the sum of the images of B and B' in C/U is the requested submodule. Now f induces a homomorphism $C/U \to C$. Since $\mathrm{Hom}(V, B) = 0$, it follows that also c belongs to the kernel of f. Using induction, we see that f vanishes on C_2 and therefore $f = 0$.

Consider now an automorphism f of C. Since $\dim \mathrm{Hom}(B, C) = 1$ for each bristle B, we see that f maps any submodule of C which is a bristle into itself, thus it maps any one-dimensional subspace of C_2 into itself. This shows that $f|C_2$ is a scalar multiplication. But this implies that f itself is a scalar multiplication. $\square$

Remark. The map $d\colon \mathbb{P}C_2 \to \mathbb{P}C_1$ is the degree 2 Veronese embedding.

(2.2) *There is a natural bijection between quadratic forms ϕ on C_2 and homomorphisms $\phi'\colon C \to S(1)$ of quiver representations,* it is given as follows: We denote by $k[x_1, \dots, x_n]$ the polynomial ring in n variables, and we evaluate these polynomials on C_2 via $x_r(\sum_i \lambda_i c_i) = \lambda_r$); the space of quadratic forms on C_2 is the set of quadratic homogeneous polynomials, this space has the basis $x_r x_s$ with $1 \le r \le s \le n$. We may evaluate this polynomial $x_r x_s$ on C_1 in the obvious way: $x_r x_s(\sum_{i \le j} \lambda_{ij} c_{ij}) = \lambda_{rs}$, this yields a linear transformation $\phi'\colon C_1 \to k$, and we may interpret this linear transformation as a homomorphism $C \to S(1)$ of quiver representations which we also denote by ϕ'.

The maximal submodules of C are just the kernels of homomorphisms $C \to S(1)$. Recall that a submodule U of a module M of finite length is said to be an *essential* submodule provided that U contains the socle of M. Since C is a non-simple indecomposable $K(n)$-module, the socle of C is equal to the radical of C, thus the essential submodules of C are just the intersections of maximal submodules, or, equivalently, the intersections of the kernels of a set of homomorphisms $C \to S(1)$. The relationship between the vanishing set for ϕ and the kernel of ϕ' is given as follows:

(2.3) *Let ϕ be a quadratic form on C_2 and $c \in C_2$. Then $\phi(c) = 0$ if and only if $d(c)$ belongs to the kernel of ϕ'.*

Proof: This is a tautological statement, since we have $\phi = \phi' \circ d$. Indeed, let us apply ϕ and $\phi' \circ d$ to $c = \sum_i \lambda_i c_i$. If ϕ is a basis element, say $\phi = x_r x_s$, then $x_r x_s(c) = \lambda_r \lambda_s$, and $(\phi' \circ d)(c) = \phi'(d(c)) = x_r x_s(\sum_{i,j} \lambda_i \lambda_j c_{ij}) = \lambda_r \lambda_s$. In general, ϕ is a linear combination of such basis elements, and ϕ' is the corresponding linear combination. $\square$

(2.4) The representation C of $K(n)$ can be obtained also by looking at the corresponding Beilinson algebra $A(n)$, this is the path algebra with quiver $\Delta(n)$ with 3 vertices, say $1, 2, 3$, with n arrows $1 \to 2$ labeled $\alpha_1, \dots, \alpha_n$ as well as n arrows $2 \to 3$, also labeled $\alpha_1, \dots, \alpha_n$, modulo the "commutativity" relations $\alpha_i \alpha_j = \alpha_j \alpha_i$ (whenever this makes sense).

$$\begin{tikzcd} 1 \arrow[r, shift left=2, "\alpha_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\alpha_n"'] & 2 \arrow[r, shift left=2, "\alpha_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\alpha_n"'] & 3 \end{tikzcd}$$

For $i = 1, 2, 3$, let $I_{A(n)}(i)$ be the injective envelope and $P_{A(n)}(i)$ the projective cover of the $A(n)$-module $S(i)$. If Z is an $A(n)$-module, we denote by $Z|[1,2]$ the restriction of Z to the subquiver of type $K(n)$ with vertices $1, 2$. Then

$$C = I_{A(n)}(3)|[1,2].$$

Proof: Let $k[x_1, \dots, x_n]$ be the polynomial ring in n variables $x_1, \dots, x_n$. We may identify the $A(n)$-module $P_{A(n)}(1)$ with the subspace of $k[x_1, \dots, x_n]$ generated by the monomials of degree $i - 1$, for $1 \le i \le 3$. This shows that $P_{A(n)}(1)$ has dimension vector $(1, n, \binom{n+1}{2})$. Using duality, we see that the dimension vector of $I_{A(n)}(3)$ is $(\binom{n+1}{2}, n, 1)$.

On the other hand, we may extend C to an $A(n)$-module by setting $C_3 = k$, with maps $\alpha_i : C_2 \to C_3$ defined by $\alpha_i(c_j) = 1$ if $i = j$ and zero otherwise (obviously, the commutativity relations are satisfied). This extended module has simple socle, namely the simple module $S(3)$, thus it can be embedded into $I_{A(n)}(3)$, and the dimension vectors show that the extended module is equal to $I_{A(n)}(3)$.

3. Proof of Theorem 1′.

We assume now that k is algebraically closed. We want to show that any projective variety $\mathcal{V}$ occurs as $\beta(M)$ for a reduced Kronecker module M. It is well-known that we can realize $\mathcal{V}$ as a closed subset of a projective space, say $\mathbb{P}^{n-1}$, defined by a finite set of quadratic homogeneous polynomials, say $q_1, \dots, q_m$. We consider the corresponding linear maps $q_i'\colon C \to S(1)$, where C is the canonical $K(n)$-module and put $M = \bigcap_i \operatorname{Ker}(q_i')$.

Thus, let $\mathcal{V}$ be the closed subset of the projective space $\mathbb{P}^{n-1}$, defined by the vanishing of homogeneous polynomials $q_1, \dots, q_m$ of degree 2. Let Λ' be the factor algebra of the Beilinson algebra $A(n)$ taking the elements $q_1, \dots, q_m$ as additional relations (obviously, these elements may be considered as linear combinations of paths of length 2 in the quiver $\Delta(n)$). The injective envelope $I_{\Lambda'}(3)$ of $S(3)$ as a Λ'-module is just the submodule of $I_{A(n)}(3)$ which is obtained as the kernel of the linear maps $q_1', \dots, q_m'\colon I_{A(n)}(3) \to S(1)$.

Since C is reduced, also its submodule M is reduced. This completes the proof.

4. Open problems and remarks.

(4.1) Problems. Looking at Theorem 1 (and 1′), a lot of questions may be asked. Let us mention just a few.

(a) Theorem 1 asserts that any projective variety can be realized as the eigenvector variety of some matrix pencil $P = (a, b; \alpha_1, \dots, \alpha_n)$, or, equivalently, as the bristle variety of an n-Kronecker module. The construction of P given in the proof shows that even for quite innocent projective varieties, the number n may be large. Thus, we may ask *which projective varieties occur as the eigenvector variety of a matrix pencil with a fixed number n of matrices,* or, equivalently, as the bristle variety of an n-Kronecker module with n fixed.

For $n = 1$, the reduced matrix pencils $P = (a, b; \alpha)$ are given by an arbitrary injective linear transformation $\alpha\colon k^a \to k^b$. It follows that any non-zero element of k^a is an eigenvector for P, thus $\epsilon(P) = \mathbb{P}(k^a)$ is a projective space.

For $n = 2$, one can use the well-known classification of the indecomposable matrix pencils (see [4] or also [9]) in order to determine all possible eigenvector varieties $\epsilon(P)$. Let us mention here which varieties occur as eigenvector varieties $\epsilon(P)$ where P is indecomposable (and different from $S(1)$). The dimension vector of an indecomposable matrix pencil P is of the form (a, b) with $|a - b| \le 1$. If $a < b$, then $\epsilon(P) = \emptyset$. If $a = b$, then $\epsilon(P)$ consists of at most one point (if k is algebraically closed, then $\epsilon(P)$ is non-empty). Finally, if $a > b$, then $\epsilon(P)$ is a non-singular quadratic curve.

Thus, the first case of real interest is the case $n = 3$: which projective varieties occur as the eigenvector variety of a pencil of 3 matrices?

(b) If we realize the projective variety $\mathcal{V}$ as $\beta(M)$ for some reduced Kronecker module M, the elements of $\mathcal{V}$ are the bristle submodules of M. Now certain elements of $\mathcal{V}$ are of particular interest, for example the singular points. One may ask about specific properties of the corresponding bristle submodules of M.

(c) Also, we may fix the dimension of $\mathcal{V}$. Already the case of dimension 1 seems to be of interest. Thus, one may ask: *For which matrix pencils P is $\epsilon(P)$ a projective curve?*

(4.2) Square matrices. The main result of this note may be strengthened as follows: *Let k be an algebraically closed field. Any projective variety occurs as the eigenvector variety of a reduced matrix pencil $P = (a, b; \alpha_1, \dots, \alpha_n)$ with square matrices α_i.*

Proof. Again, the proof will be given in terms of n-Kronecker modules. We will show that for any reduced n-Kronecker module M say with dimension vector $\mathbf{dim}\, M = (a, b)$ there is a reduced n-Kronecker module N with dimension vector $\mathbf{dim}\, N = (a', a')$ such that $\beta(M) = \beta(N)$ and $a' \le a$. The proof relies on the following lemma.

Lemma. *Let M be any n-Kronecker module, say with dimension vector* $\mathbf{dim}\, M = (a, b)$*. If M is the sum of its bristle submodules, then M is the sum of a bristle submodules. As a consequence, we have $a \ge b$.*

Proof. For the proof of the first assertion, we use induction on a. Let M be an n-Kronecker module with dimension vector (a, b) and assume that M is a sum of bristle submodules. If $a = 0$, then also $b = 0$. Thus, assume that $a > 0$. Since M has finite length, M is the sum of finitely many bristle submodules, say of the bristle submodules $U_1, \dots, U_m$, and we assume that m is minimal. Of course, we must have $a \le m$. Let $M' = \sum_{i=1}^{m-1} U_i$ and $\mathbf{dim}\, M' = (a', b')$. Assume that $a' = a$. Choose a vector space complement V of M'_2 in M_2, thus $\dim V = b - b'$. Then $M'' = (0, V)$ is a submodule of M and $M = M' \oplus M''$, thus M'' is a direct summand of M. But $\mathrm{Hom}(B, M'') = 0$ for any bristle. Since M is generated by bristle submodules, M'' is generated by the images of maps $B \to M''$, where B are bristles. It follows that $M'' = 0$, thus $b' = b$, and therefore $M' = M$. But this contradicts the minimality of m. This shows that we must have $a' < a$. By induction, M' is the sum of a' bristle submodules. Since M is the sum of M' and U_m, we see that M is the sum of $a' + 1$ bristle submodules. The minimality of m implies that $m \le a' + 1$, thus $m \le a' + 1 \le a$. Altogether, we see that $m = a$, thus M is the sum of a bristle submodules. This completes the proof of the first assertion.

In order to show the second assertion, we note that the inclusion maps $U_i \to M$ yield a surjective map $\bigoplus_{i=1}^{a} U_i \to M$. Therefore

$$(a, a) = \mathbf{dim} \bigoplus\nolimits_{i=1}^{a} U_i \ \ge\ \mathbf{dim}\, M = (a, b).$$

This shows that $a \ge b$. $\square$

Now, let M be any reduced n-Kronecker module. Let us denote by M' the sum of all bristle submodules of M. If $\mathbf{dim}\, M' = (a', b')$, then $a' \le a$ (and $b' \le b$). Since M is reduced, also its submodule M' is reduced, and, of course, we have

$$\beta(M') = \beta(M).$$

According to the Lemma, we have $a' \geq b'$. Let $N = M' \oplus S(2)^{a'-b'}$. Since M' is reduced, also N is reduced. Since $\mathrm{Hom}(B, S(2)) = 0$ for any bristle B, we see that any bristle submodule of N is contained in $M' \oplus 0$, thus

$$\beta(N) = \beta(M').$$

On the other hand, we have $\mathbf{dim}\, N = (a', a')$. □

(4.3) Non-reduced matrix pencils. It seems worthwhile to have a short look also at non-reduced matrix pencils and non-reduced n-Kronecker modules. Any n-Kronecker module M can be written as a direct sum $M = M' \oplus M''$, where M' is a direct sum of copies of $S(1)$ and M'' is reduced; the submodule M' is uniquely determined and, of course, $M'' = M/M'$ is the maximal reduced factor module of M. In [11] we have proposed to call $\eta(M) = \beta(M'')$ the *bristle variety of* M. In this way, the bristle variety of any n-Kronecker module is defined and is a projective variety.

Looking at an arbitrary n-Kronecker module M, we also may consider the set $\beta(M)$ of bristle submodules of M, this is an (open) subset of $\mathbb{G}_{(1,1)}(M)$. As we have mentioned above, we have $\beta(M) = \mathbb{G}_{(1,1)}(M)$ in case M is reduced. This equality also holds true (for trivial reason) in case $M_2 = 0$, since then both sets $\beta(M)$ and $\mathbb{G}_{(1,1)}(M)$ are empty.

Let us assume now that M is not reduced and that $M_2 \neq 0$. In this case, M has a submodule isomorphic to $S(1) \oplus S(2)$, thus $\beta(M)$ is a proper subset of $\mathbb{G}_{(1,1)}(M)$. In case $\beta(M)$ is not empty, it is not closed in $\mathbb{G}_{(1,1)}(M)$ (thus not a projective variety). Namely, let U be a bristle submodule of M. Since we assume that M is not reduced, M has a submodule V isomorphic to $S(1)$ and, of course, $U \cap V = 0$. The submodule $U \oplus V$ of M shows that $\mathrm{soc}(U \oplus V)$ is a degeneration of U (and $\mathrm{soc}(U \oplus V)$ belongs to $\mathbb{G}_{(1,1)}(M)$, but not to $\beta(M)$).

(4.4) Wild acyclic quivers. The result presented in this paper can be used in order to show: *Let k be an algebraically closed field. If Q is any connected wild acyclic quiver with at least 3 vertices, then any projective variety arises as a quiver Grassmannian of some representation of Q.* The proof will be given in [12]. It is an open question whether the condition that Q has at least 3 vertices can be omitted (thus, whether any projective variety arises as a quiver Grassmannian of some 3-Kronecker quiver).

5. Appendix. On the existence of eigenvectors.

The aim of this appendix is to provide a translation of the main result of [11] to the matrix pencil language, namely to reformulate this result as an existence assertion for eigenvectors. We assume that $n \geq 3$.

(5.1) We will say that a reduced matrix pencil $P = (a, b; \alpha_1, \ldots, \alpha_n)$ has *sufficiently many eigenvectors* provided the eigenvectors of P generate the vector space k^a (or, equivalently, provided the corresponding Kronecker module is bristled). What we will see is that there is a huge class of matrix pencils with sufficiently many eigenvectors.

If v is an eigenvector of the reduced matrix pencil $P = (a, b; \alpha_1, \ldots, \alpha_n)$, and w is a non-zero vector in $\langle \alpha_1(v), \ldots, \alpha_n(v) \rangle$, let $\alpha_i(v) = \lambda_i w$, for $1 \leq i \leq n$. The element

$\langle(\lambda_1,\dots,\lambda_n)\rangle \in E = \mathbb{P}(k^n)$ does not depend on the choice of w and will be called the *eigenvalue of* v. Of course, equivalent eigenvectors have the same eigenvalue.

If E' is a subset of E, we say that the reduced matrix pencil $P = (a, b; \alpha_1, \dots, \alpha_n)$ has *sufficiently many E'-eigenvectors* provided the eigenvectors with eigenvalue in E' generate k^a. For $n \geq 3$, we are going to exhibit a finite (!) subset E_0 of E such that there is a huge class of matrix pencils with sufficiently many E_0-eigenvectors.

As before, we will denote by $e_1, \dots, e_n$ the canonical basis of k^n. If $1 \leq i, j \leq n$ and $i \neq j$, let $e_{i,j} = e_i + e_j$. We denote by E_0 the subset of E given by the following elements: first, $\langle e_{n-1}\rangle$ and $\langle e_n\rangle$, second $\langle e_{i,i+1}\rangle$ for $1 \leq i < n$, and finally $\langle e_{1,n}\rangle$. Note that E_0 is a set of cardinality $n+2$.

Let us now define the reflection functor σ as introduced by Bernstein-Gelfand-Ponomarev. We recall that matrix pencils $P = (a, b; \alpha_1, \dots, \alpha_n)$ and $P' = (a', b'; \alpha'_1, \dots, \alpha'_n)$ are said to be *equivalent* provided $a = a', b = b'$ and there are invertible linear transformations $\beta\colon k^a \to k^a$ and $\gamma\colon k^b \to k^b$ such that $\gamma\alpha_i\beta = \alpha'_i$ for all $1 \leq i \leq n$. The reflection functor σ sends the (equivalence class of the) matrix pencil $P = (a, b; \alpha_1, \dots, \alpha_n)$ to the following equivalence class σP of matrix pencils: Let U be the set of elements $u = (u_1, \dots, u_{na})$ in k^{na} such that $\sum_{i=1}^n \alpha_i(u_{(i-1)a+1}, \dots, u_{(i-1)a+a}) = 0$. Let z be the dimension of U. Write the inclusion map $U \to k^{na} = k^a \times \cdots \times k^a$ in terms of some basis of U and the canonical basis of $k^a \times \cdots \times k^a$; we obtain in this way a sequence of n $(a \times z)$-matrices $\omega_1, \dots, \omega_n$. Then σP is the equivalence class of $(z, a; \omega_1, \dots, \omega_n)$.

Theorem 2. *Let $n \geq 3$. If $P = (a, b; \alpha_1, \dots, \alpha_n)$ is any matrix pencil, then, for $t \gg 1$, the matrix pencil $\sigma^t P$ has sufficiently many E_0-eigenvectors.*

Let us stress that Theorem 2 is essentially a reformulation of part of the main theorem in [11]. We note the following: if $B(\lambda)$ is a bristle n-Kronecker module, then any non-zero vector in $B(\lambda)_1$ is an eigenvector with eigenvalue λ, and, conversely, any eigenvector of a matrix pencil P with eigenvalue λ gives rise to an embedding of $B(\lambda)$ into P, considered as a Kronecker module. In particular, the set $\mathcal{B}_0$ of bristles which is used in [11] corresponds bijectively to the set $E_0 \subset \mathbb{P}(k^n)$. In the paper [11], instead of the functor σ the Auslander-Reiten translation τ of the category of n-Kronecker modules was used. But according to Gabriel [3], we have $\tau = \sigma^2$. Thus it is sufficient to to apply (1.3)(b) of [11] both for $M(P)$ and $\sigma M(P)$.

Remark. The reader should be aware that for $n = 2$, the corresponding assertion is not valid, typical examples are the matrix pairs $\left(2, 2; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right)$ with $\lambda \in k$.

(5.2) Let us mention under what condition one may recover P from $\sigma^t P$, where t is a natural number. Following [9], we say that an indecomposable matrix pencil P is *preprojective* provided $\sigma^t P = 0$ for some natural number t. If $P = (a, b; \alpha_1, \dots, \alpha_n)$ is preprojective, then $a < b$ and $a^2 + b^2 - nab = 1$ and any indecomposable matrix pencil with this dimension vector (a, b) is equivalent to P (also, any pair (a, b) of natural numbers with $a < b$ and $a^2 + b^2 - nab = 1$ arises in this way). It follows that for $n \geq 2$, there are countably many equivalence classes of preprojective matrix pencils, and it is easy to see that for $n \geq 2$, *the preprojective matrix pencils have no eigenvectors.*

If P is an indecomposable matrix pencil which is not preprojective, then we can recover P from $\sigma^t P$ by using a corresponding functor σ^{-t} (the left adjoint for σ^t): namely, P and $\sigma^{-t}\sigma^t P$ are equivalent matrix pencils. Since Theorem 2 asserts that $\sigma^t P$ has sufficiently many eigenvectors for $t \gg 1$, we see that *any indecomposable matrix pencil which is not preprojective is of the form* $\sigma^{-t}P'$ *for some* t, *where* P' *is a matrix pencil with sufficiently many eigenvectors.*

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