The Relevance and the Ubiquity of Prüfer Modules

Claus Michael Ringel

Abstract: Let R be a ring. An R-module M is called a Prüfer module provided there exists a locally nilpotent, surjective endomorphism of M with kernel of finite length. We want to outline the relevance, but also the ubiquity of Prüfer modules. The main assertion will be that any Prüfer module which is not of finite type gives rise to a generic module, thus to infinite families of indecomposable modules with fixed endo-length (here we are in the setting of the second Brauer-Thrall conjecture). In addition, we will report on a construction procedure which yields a wealth of Prüfer modules. Unfortunately, we do not know which modules obtained in this way are of finite type.

This is the written account of a lecture given at 4th International Conference on Representation Theory (ICRT-IV), Lhasa, July 16–20, 2007. Section 1 recalls the definition of a Prüfer module as introduced in [R6] and provides some examples, in section 2 we show that the degeneration theory of modules concerns certain Prüfer modules of finite type, whereas section 4 provides a construction of Prüfer modules using pairs of monomorphisms $U_0 \to U_1$; these two sections 2 and 4 are reports on some of the results of [R6]. In the last section 6 we discuss the question how to search for pairs of monomorphisms $U_0 \to U_1$. This is an announcement of results of the forthcoming paper [R9], where "take-off categories" are introduced. The central part is section 3. There, we show that the existence of a Prüfer module which is not of finite type implies the existence of a generic module, thus of infinite families of indecomposable modules with fixed (and arbitrarily large) endolength. This has not yet appeared in print (but see [R8]) and has been announced under the title: *Prüfer modules which are not of finite type.* Also section 5 is new, here we show that given a tame hereditary algebra, a finite length module N can generate a non-finite-type module M only in case N has a preprojective direct summand which is sincere. This gives an indication why it seems to be reasonable to look for module embeddings in take-off categories.

1. Prüfer modules.

(1.1) Let R be any ring. We deal with (left) R-modules. An R-module M is called a *Prüfer module* provided there exists an endomorphism ϕ of M with the following properties:

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ϕ is locally nilpotent, surjective, and the kernel W of ϕ is non-zero and of finite length. The module W is called the *basis* of M. Let $W[n]$ be the kernel of ϕ^n, then $M = \bigcup_n W[n]$, thus we also may write $M = W[\infty]$.

(1.2) The classical example. Let $R = \mathbb{Z}$ and p a prime number. Let $S = \mathbb{Z}[p^{-1}]$ the subring of $\mathbb{Q}$ generated by p^{-1}. Then

$$S/R = \varinjlim \mathbb{Z}/\mathbb{Z}p^n$$

is the Prüfer group for the prime p. These Prüfer groups are all the indecomposable $\mathbb{Z}$-modules which are Prüfer modules.

(1.3) More generally, let R be a Dedekind ring. Then any indecomposable R-module of finite length is of the from $W[n]$, where W is the basis of a Prüfer module and $n \in \mathbb{N}$. In particular, this applies to $R = \mathbb{Z}$, but also say to the polynomial ring $R = k[T]$ in one variable with coefficients in the field k. Note that a $k[T]$-module is just a pair (V, f), where V is a k-space and f is a linear operator on V. If $\operatorname{char} k = 0$, then the pair $(k[T], \frac{\mathrm{d}}{\mathrm{d}\,t})$ is a Prüfer module.

As Atiyah [A] has shown, a corresponding assertion holds for the coherent sheaves over an elliptic curve: any indecomposable coherent sheaf of finite length is of the from $W[n]$, where W is the basis of a Prüfer module and $n \in \mathbb{N}$.

(1.4) Example. Consider the Kronecker algebra $\Lambda = kQ$, this is the path algebra of the quiver Q

$$\circ \leftleftarrows \circ$$

The embedding functor

$$\operatorname{mod} k[T] \longrightarrow \operatorname{mod} k\Lambda \qquad\qquad (V, f) \mapsto V \overset{1}{\underset{f}{\leftleftarrows}} V$$

preserves Prüfer modules. Using this functor, we obtain all indecomposable Prüfer modules for the Kronecker algebra with one exception, the remaining one is of the form

$$V \overset{f}{\underset{1}{\leftleftarrows}} V$$

where (V, f) is the indecomposable Prüfer $k[T]$-module such that f has 0 as eigenvalue.

(1.5) We also should mention a famous theorem of Crawley-Boevey [CB1]: Let Λ be a finite-dimensional k-algebra and k an algebraically closed field. If Λ is **tame,** and $d \in \mathbb{N}$, then almost all indecomposable Λ-modules of length d are of the from $W[n]$, where $W[\infty]$ is an indecomposable Prüfer module.

(1.6) Warning. The Prüfer modules as defined above do not have to be indecomposable: for example the countable direct sum $W^{(\mathbb{N})}$ of copies of W with the shift endomorphism $(w_1, w_2, \dots) \mapsto (w_2, w_3, \dots)$ is a Prüfer module: the *trivial* Prüfer module with basis W. Less trivial examples will be seen in the next section.

(1.7) Lemma. *Let M be a Prüfer module with basis W. If the endomorphism ring of W is a division ring, then either $M = W^{(\mathbb{N})}$ or else M is indecomposable.*

Proof: This is an immediate consequence of the process of simplification [R1].

A module M is of *finite type* provided it is the direct sum of finitely generated modules and such that there are only finitely many isomorphism classes of indecomposable direct summands. Our main concern will be Prüfer modules which are not of finite type. But first we consider some Prüfer modules of finite type.

2. Degenerations of modules

In this and the next section we consider artin algebras (these are rings which are module finite over the center, the center being artinian).

(2.1) In order to motivate the notion of a degeneration, let us consider first the case of Λ being a finite-dimensional k-algebra where k is an algebraically closed field.

Assume that Λ is generated as a k-algebra by $a_1, a_2, \ldots, a_t$ subject to relations ρ_i. For $d \in \mathbb{N}$, we consider the variety

$$\mathcal{M}(d) = \{(A_1, \ldots, A_t) \in M(d \times d, k)^t \mid \rho_i(A_1, \ldots, A_t) = 0 \text{ for all } i\},$$

of d-dimensional Λ-modules: its elements are the d-dim Λ-modules with underlying vector space k^d (thus, up to isomorphism, all d-dim Λ-modules). The group $\mathrm{GL}(d, k)$ operates on $\mathcal{M}(d)$ by simultaneous conjugation. Elements of $\mathcal{M}(d)$ belong to the same orbit if and only if they are isomorphic.

Theorem (Zwara). *Let X, Y be d-dimensional Λ-modules. Then Y is in the orbit closure of X if and only if there exists a finitely generated Λ-module U and an exact sequence*

$$0 \to U \to X \oplus U \to Y \to 0.$$

such a sequence should be called a *Riedtmann-Zwara sequence.*

(2.2) Now let Λ be an arbitrary artin algebra and X, Y Λ-modules of finite length. We call Y a *degeneration* of X provided there exists a finitely generated Λ-module U and an exact sequence

$$0 \to U \to X \oplus U \to Y \to 0$$

Proposition (Zwara). *Y is a degeneration of X if and only if there is a Prüfer module M with basis Y such that $Y[t+1] \simeq Y[t] \oplus X$ for some t, or, equivalently, for almost all t.*

Proof: See [Z] and [R6]

Note that an isomorphism $Y[t+1] \simeq Y[t] \oplus X$ yields directly a Riedtmann-Zwara sequence as well as a co-Riedtmann-Zwara sequence, using the canonical exact sequences

$$0 \to Y[t] \to Y[t+1] \to Y \to 0\ ,$$
$$0 \to Y \to Y[t+1] \to Y[t] \to 0$$

and replacing the middle term by $Y[t] \oplus X$.

(2.3) Note that the Prüfer modules $Y[\infty]$ obtained in (2.2) satisfy $Y[\infty] \simeq Y[t] \oplus X^{(\mathbb{N})}$ for some t. In particular, these are modules of finite type.

3. Prüfer modules and the second Brauer-Thrall conjecture

(3.1) Let Λ be an artin algebra. The Krull-Remak-Schmidt Theorem asserts that any finitely generated Λ-module can be written as a direct sum of indecomposable modules, and such a decomposition is unique up to isomorphism.

The artin algebra Λ is called *representation-infinite* provided there are infinitely many isomorphism classes of indecomposable Λ-modules, otherwise *representation-finite.*

(3.2) The first Brauer-Thrall conjecture was solved by Roiter in 1968:

Theorem (Roiter [Ro]): *If Λ is a representation-infinite artin algebra, then there are indecomposable modules of arbitrarily large finite length.*

(3.3) The second Brauer-Thrall conjecture has been solved only for finite dimensional k-algebras, where k is an algebraically closed (or at least perfect) field:

Theorem (Bautista [Ba], Bongartz [Bo]): *If Λ is a representation-infinite k-algebra, where k is an infinite perfect field, then there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of length d.*

It has been conjectured by Brauer-Thrall that the assertion holds for any infinite field. For finite fields, or, more generally, for an arbitrary artin algebra λ, one may conjecture the following: if Λ is representation-infinite, then there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of endo-length d. (The *endo-length* of a module M is the length of M when M is considered as a module over its endomorphism ring).

(3.3) Let λ be an artin algebra. A Λ-module M is said to be *generic* provided M is indecomposable, of infinite length, but of finite endo-length.

Theorem (Crawley-Boevey [CB2]): *Let Λ be a finite-dimensional k-algebra (k a field). Let M be a generic Λ-module. Then there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of endo-length d.*

(3.4) Prüfer modules yield generic modules.

Theorem. *Let M be a Prüfer module. The following conditions are equivalent:*

(i) *M is not of finite type.*

(ii) *There is an infinite index I set such that the product module M^I has a generic direct summand.*

(iii) *For every infinite index I set, the product module M^I has a generic direct summand.*

Proof: The implications (iii) $\implies$ (ii) is trivial. Also (ii) $\implies$ (i) is obvious: If M is of finite type, then all product modules M^I are of finite type. We only have to show (i)

$\Longrightarrow$ (iii). (It is sufficient to consider $I = \mathbb{N}$ in (iii), since any infinite index set I can be written as the disjoint union of $\mathbb{N}$ and some other index set I', and then $M^I = M^{\mathbb{N}} \oplus M^{I'}$, however, there is no problem to work in general.)

Now assume that I is an infinite index set and that M^I has no indecomposable direct summand which is endo-finite and of infinite length. Since M is a Prüfer module, there is a surjective, locally nilpotent endomorphism ϕ with kernel $W = W[1]$ non-zero and of finite length. Let $W[n]$ be the kernel of ϕ^n. Thus

$$M[1] \subset M[2] \subset \cdots \subset \bigcup_n M[n] = M$$

is a filtration of M with finite length modules $M[n]$. We obtain a corresponding chain of inclusions

$$M[1]^I \subset M[2]^I \subset \cdots \subset \bigcup_n M[n]^I = M'.$$

It has been shown in [R3] (see also [K]) that M' is isomorphic to a direct sum of copies of M and itself a direct summand of M^I; there is an endo-finite submodule E of M^I such that

$$M^I = M' \oplus E.$$

Any endo-finite module E can be written as a direct sum of copies of finitely many indecomposable endo-finite modules, say $E_1, \dots, E_t$. By assumption, all these modules E_i are of finite length. A well-known lemma of Auslander asserts that any indecomposable direct summand of M^I of finite length is a direct summand of M itself, thus the modules $E_1, \dots, E_t$ occur as direct summands of M.

Since M is artinian as a module over its endomorphism ring, M is Σ-algebraic compact, thus it is a direct sum of indecomposable modules with local endomorphism ring. Write $M = A \oplus B$, where A is a direct sum of copies of the various E_i and B has no direct summand of the form E_i, for any i. We want to show that B is of finite length. This then shows that M is of finite type.

The modules A, B are also filtered, with $A_n = A \cap M[n],\ B_n = B \cap M[n]$ (it is obvious that $A = \bigcup_n A_n,\ B = \bigcup_n B_n$). For any n there is some n' with $M[n] \subseteq A_{n'} \oplus B_{n'}$. (Namely, let $x \in M[n]$, write $x = a + b$ with $a \in A,\ b \in B$. Then there is some n' with $a, b \in M_{n'}$, thus $a \in A_{n'},\ b \in B_{n'}$.)

We write $A' = \bigcup_i A_i^I$ and $B' = \bigcup_i B_i^I$. Then

$$M' = A' \oplus B'$$

(the inclusion $\supseteq$ is obvious, the other follows from $M[n]^I \subseteq (A_{n'} \oplus B_{n'})^I = A_{n'}^I \oplus B_{n'}^I \subseteq A' \oplus B'$.). We see that

$$(A^I/A') \oplus (B^I/B') = M^I/M' = E,$$

thus $A^I/A' = E_A$ and $B^I/B' = E_B$ with $E = E_A \oplus E_B$. In particular, E_A and E_B are direct sums of copies of $E_1, \dots, E_t$. Since the direct sum of the inclusion maps

$$A' \to A^I \quad \text{and} \quad B' \to B^I$$

is a split monomorphism, the maps themselves are split monomorphisms, thus

$$A^I \simeq A' \oplus E_A \quad \text{and} \quad B^I \simeq B' \oplus E_B.$$

Consider the last isomorphism. If E_i is a direct summand of E_B, then it is a direct summand of B (Auslander Lemma), impossible. This shows that $E_B = 0$. But then $B' = B^I$ implies that $B = B_n$ for some n, thus $B \subseteq M[n]$. This shows that B is of finite length.

May-be one should record: *Assume that M^I/M' is the direct sum of copies of indecomposable modules $E_1, \dots, E_t$ of finite length, then M is the direct sum of a finite length module B and of copies of the modules E_i.*

(3.5) As we have mentioned, the second Brauer-Thrall conjecture claims the following: if Λ is a representation-infinite algebra, then there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of endo-length d.

Note that this is an assertion which concerns only modules of finite length. But it seems that it may be reasonable to look for a solution using modules of infinite length. As we have seen, it will be sufficient to show that a representation-infinite algebra has a Prüfer module which is not of finite tye, since this implies the existence of a generic module and thus the existence of infinitely many indecomposable Λ-modules of endo-length d.

4. The ladder construction of Prüfer modules

We return to rings and modules in general.

(4.1) This construction was exhibited in [R6], let us recall here the essential steps: We start with a proper inclusion $U_0 \subset U_1$ (say with cokernel W) and a map $v_0 \colon U_0 \to U_1$, and we form the pushout of w_0 and v_0

$$\begin{tikzcd} 0 \arrow[r] & U_0 \arrow[r, "w_0"] \arrow[d, "v_0"] & U_1 \arrow[r] \arrow[d, "v_1"] & W \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U_1 \arrow[r, "w_1"] & U_2 \arrow[r] & W \arrow[r] & 0 \end{tikzcd}$$

we obtain a module U_2, as well as a monomorphism $w_1 \colon U_1 \subset U_2$ (again with cokernel W) and a map $v_1 \colon U_1 \to U_2$.

Using induction, we obtain in this way modules U_i, monomorphisms $w_i \colon U_i \subset U_{i+1}$ (all with cokernel W) as well as maps $v_i \colon U_i \to U_{i+1}$ such that $v_{i+1}w_i = w_{i+1}v_i$ for all $i \geq 0$. This means that we obtain the following ladder of commutative squares:

$$\begin{tikzcd} U_0 \arrow[r, "w_0"] \arrow[d, "v_0"'] & U_1 \arrow[r, "w_1"] \arrow[d, "v_1"'] & U_2 \arrow[r, "w_2"] \arrow[d, "v_2"'] & U_3 \arrow[r, "w_3"] \arrow[d, "v_3"'] & \cdots \\ U_1 \arrow[r, "w_1"] & U_2 \arrow[r, "w_2"] & U_3 \arrow[r, "w_3"] & U_4 \arrow[r, "w_4"] & \cdots \end{tikzcd}$$

We form the inductive limit $U_\infty = \bigcup_i U_i$ (along the maps w_i). Since all the squares commute, the maps v_i induce a map $U_\infty \to U_\infty$ which we denote by v_∞:

$$\begin{tikzcd}
U_0 \arrow[r, "w_0"] \arrow[d, "v_0"'] & U_1 \arrow[r, "w_1"] \arrow[d, "v_1"'] & U_2 \arrow[r, "w_2"] \arrow[d, "v_2"'] & U_3 \arrow[r, "w_3"] \arrow[d, "v_3"'] & \cdots & \bigcup_i U_i = U_\infty \arrow[d, "v_\infty"] \\
U_1 \arrow[r, "w_1"] & U_2 \arrow[r, "w_2"] & U_3 \arrow[r, "w_3"] & U_4 \arrow[r, "w_4"] & \cdots & \bigcup_i U_i = U_\infty
\end{tikzcd}$$

We also may consider the factor modules U_∞/U_0 and U_∞/U_1. The map $v_\infty\colon U_\infty \to U_\infty$ maps U_0 into U_1, thus it induces a map

$$\overline{v}\colon U_\infty/U_0 \longrightarrow U_\infty/U_1.$$

and *this map $\overline{v}$ is an isomorphism.* Namely, there are the commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & U_{i-1} \arrow[r, "w_{i-1}"] \arrow[d, "v_{i-1}"] & U_i \arrow[r] \arrow[d, "v_i"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U_i \arrow[r, "w_i"] & U_{i+1} \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

which means that the cokernel $U_i/U_{i-1} = W$ of w_{i-1} is mapped under the restriction $\overline{v}_i$ of $\overline{v}$ isomorphically onto the cokernel $U_{i+1}/U_i = W$ of w_i. Thus, we see that the map $\overline{v}$ is a map from a filtered module with factors U_i/U_{i-1} (where $i \geq 1$) to a filtered module with factors U_{i+1}/U_i (again with $i \geq 1$), and the maps $\overline{v_i}$ are just those induced on the factors. Since all the maps $\overline{v_i}$ are isomorphisms, also $\overline{v}$ itself is an isomorphism.

It follows: The composition of maps

$$U_\infty/U_0 \xrightarrow{\ p\ } U_\infty/U_1 \xrightarrow{\ \overline{v}^{-1}\ } U_\infty/U_0$$

(p the projection map) is an epimorphism ϕ with kernel U_1/U_0. It is easy to see that ϕ is locally nilpotent.

Proposition. *The module U_∞/U_0 is a Prüfer module with respect to the endomorphism $\phi = \overline{v}^{-1} \circ p$, its basis is $W = U_1/U_0$.*

This shows that starting with a proper inclusion $U_0 \subset U_1$ and a map $v_0\colon U_0 \to U_1$, the ladder construction yields a Prüfer module U_∞/U_0 with bases U_1/U_0.

(4.2) In case also v_0 is injective, we obtain a second Prüfer module. Namely, there is the following chessboard:

$$\begin{tikzcd}
U_0 \arrow[r, "w_0"] \arrow[d, "v_0"'] & U_1 \arrow[r, "w_1"] \arrow[d, "v_1"'] & U_2 \arrow[r, "w_2"] \arrow[d, "v_2"'] & U_3 \arrow[r, "w_3"] \arrow[d, "v_3"'] & \cdots \\
U_1 \arrow[r, "w_1"] \arrow[d, "v_1"'] & U_2 \arrow[r, "w_2"] \arrow[d, "v_2"'] & U_3 \arrow[r, "w_3"] \arrow[d, "v_3"'] & \cdots & \\
U_2 \arrow[r, "w_2"] \arrow[d, "v_2"'] & U_3 \arrow[r, "w_3"] \arrow[d, "v_3"'] & \cdots & & \\
U_4 \arrow[r, "w_3"] \arrow[d, "v_3"'] & \cdots & & & \\
\cdots & & & &
\end{tikzcd}$$

We see both horizontally as well as vertically ladders: the horizontal ladders yield U_∞ and its endomorphism v_∞; the vertical ladders yield U'_∞ with an endomorphism w_∞.

(4.3) Examples. First, let us show that the ordinary Prüfer groups (as considered in abelian group theory) are obtained in this way. Let $R = \mathbb{Z}$ be the ring of integers. Module homomorphisms $\mathbb{Z} \to \mathbb{Z}$ are given by the multiplication with some integer n, thus we denote such a map just by n. Let $U_0 = U_1 = \mathbb{Z}$ and $w_0 = 2$, $v_0 = n$. *If n is odd, then the Prüfer module U_∞/U_0 is just the Prüfer group for the prime* 2 (and $U_\infty(2, n) = \mathbb{Z}[\frac{1}{2}]$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{2}$). Note that if n is even, then the Prüfer module U_∞/U_0 is an elementary abelian 2-group.

Second, let $R = K(2)$ be the Kronecker algebra over some field k. Let U_0 be simple projective, U_1 indecomposable projective of length 3 and $w_0 \colon U_0 \to U_1$ a non-zero map with cokernel H (one of the indecomposable modules of length 2). For any map $v_0 \colon U_0 \to U_1$, we obtain a Prüfer module $M = U_\infty/U_0$. In case $v_0 \notin kw_0$, this module M is indecomposable (and it is the Prüfer module $H[\infty]$ as considered in [R2]), otherwise M it is a direct sum of copies of H.

(4.4) Lemma. *The modules U_∞ as well as U_∞/U_0 are generated by U_1.*

Proof: We only have to consider $U_\infty = \bigoplus_i U_i$. The pushout construction shows that for $i \geq 2$, the module U_i is a factor module of $U_{i-1} \oplus U_{i-1}$, thus by induction U_i is generated by U_1.

(4.5) A self-extension $0 \to W \to W[2] \to W \to 0$ is called a *ladder extension* provided there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & U_0 \arrow[r, "w_0"] \arrow[d, "\alpha"'] & U_1 \arrow[r, "q"] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W \arrow[r] & W[2] \arrow[r] & W \arrow[r] & 0
\end{tikzcd}$$

such that $\alpha = qv_0$ for some $v_0\colon U_0 \to U_1$. In this case,the given self-extension is the $W[2]$ part of the Prüfer module $W[\infty]$ which is obtained from the maps w_0, v_0 using the ladder construction.

One should note that not every self-extension of a module is a ladder extension (for example, if S is a simple R-module, where R is artinian, then no non-trivial self-extension of S is a ladder extension). On the other hand, for R hereditary, every self-extension is a ladder extension [R6].

(4.5) Assume that Λ is a finite-dimensional hereditary k-algebra. (For example, Λ may be the path algebra kQ of a finite quiver Q without oriented cycles.)

Recall that the Euler characteristic

$$\sum\nolimits_{i\geq 0}(-1)^i \dim \operatorname{Ext}^i(M, M')$$

yields a quadratic form q on the Grothendieck group $K_0(\Lambda)$ and q is positive definite if and only if Λ is representation-finite. (In the quiver case, this means that Q is the disjoint union of quivers of Dynkin type A_n, D_n, E_6, E_7, E_8.)

We see: *Any Λ-module M with* $\operatorname{End}(M)$ *a division ring and* $q([M]) \leq 0$ *is the basis of an indecomposable Prüfer module. The Prüfer module is unique if and only if* $q([M]) = 0$.

5. The search for pairs of embeddings

Let Λ be a representation-infinite artin algebra. The aim is to find pairs of embeddings $w_0, v_0\colon U_0 \to U_1$ such that the corresponding Prüfer module U_∞/U_0 is not of finite type.

(5.1) For dealing with Prüfer modules obtained using the ladder construction, it seems to be of interest to relate the finite type properties of U_∞ and U_∞/U_0.

Lemma. *U_∞/U_0 is of finite type iff U_∞ is of finite type.*

Proof. First, assume that U_∞ is of finite tye, say $U_\infty = \bigoplus_{i\in I} M_i$ with all M_i indecomposable of finite length, and with only finitely many isomorphism classes of modules involved. Now $U_0 \subseteq \bigoplus_{i\in I'} M_i = M'$ with I' a finite subset of I. Then

$$U_\infty/U_0 = M'/U_0 \oplus \bigoplus\nolimits_{i\in I\setminus I'} M_i,$$

is a direct sum of indecomposable modules of finite length (one has to decompose M'/U_0) and only finitely many isomorphism classes are involved.

The converse follows from Roiter's extension argument, see for example [R5].

(5.2) Proposition. *Assume that Λ is tame hereditary (or tame concealed). Let M, N be a Λ-modules such that N is of finite length and generates M. If M is not of finite type, then N has a direct summand which is sincere and preprojective.*

Note that we cannot claim that N has an indecomposable direct summand which is sincere and preprojective. A typical example will be $N = {}_\Lambda\Lambda$, this module generates all the Λ-modules, but usually has no indecomposable sincere direct summand.

Proof of Proposition. Let k be the center of Λ, this is a field and Λ is a finite-dimensional k-algebra. We will assume that Λ is hereditary (the case when Λ is concealed requires only few modifications). Let $N = P \oplus R \oplus Q$ with P preprojective, R regular and Q preinjective. Assume P is not sincere, thus there is a simple Λ-module S which does not occur as a composition factor of P.

We can order the indecomposable preprojective modules $P_1, P_2, \dots$ and the indecomposable preinjective modules $\dots, Q_2, Q_1$ such that $\operatorname{Hom}(P_i, P_j) = 0$ for $i > j$ and $\operatorname{Hom}(Q_i, Q_j) = 0$ for $i < j$. There is an index n such that all the modules P_i with $i > n$ are sincere.

Note that there is a bound b such that $\dim_k \operatorname{Hom}(R, Q_i) \le b$ for all i. Namely, $Q(i) = \tau^t I(y)$ for some $t \ge 0$ and some vertex y in the quiver of Λ. Since R is regular, it is τ-periodic with period at most 6, thus $\dim_k \operatorname{Hom}(R, Q_i)$ is bounded by the maximum of the numbers $\dim_k \operatorname{Hom}(R, \tau^t I(y)$ with $0 \le t \le 5$ and y a vertex of the quiver.

This implies the following: If Q_i is generated by R, then there is a surjective map $R^b \to Q_i$, and therefore the multiplicity $[Q_i : S]$ of S as a composition factor of Q_i is bounded by $b[R : S]$. There are only finitely many Q_i with $[Q_i : S] \le b[R : S]$, thus there is some m such that $[Q_i : S] > b[R : S]$ for all $i > m$. But this implies that a module Q_i with $i > m$ cannot be generated by $P \oplus R$ (the trace of R in Q_i is a submodule with at most $b[R : S]$ composition factors S and the trace of P in Q_i does not provide any such composition factor).

We can assume in addition that m is chosen in such a way that all the indecomposable direct summands of Q are of the form Q_i with $i \le m$. Then we see that the modules a module Q_i with $i > m$ cannot be generated by $N = P \oplus R \oplus Q$.

Now let M be a (not necessarily finitely generated) module which is generated by N. We want to show that M is of finite type. According to [R2], we can write $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is a direct sum of modules of the form P_i with $1 \le i \le n$, where M_3 is a direct sum of modules of the form Q_i with $1 \le i \le m$, and where M_2 has no direct summand of the form P_i with $1 \le i \le n$, or Q_i with $1 \le i \le m$. With M also M_2 is generated by N, and we want to see that M_2 is of finite type (then also M is of finite type).

Thus we see that we can assume that $M = M_2$, this means that we consider a module generated by N which has no direct summand of the form P_i with $1 \le i \le n$, or Q_i with $1 \le i \le m$. First of all, M cannot have any indecomposable preprojective direct summand M'. Namely, $\operatorname{Hom}(R \oplus Q, M') = 0$, thus M' would be generated by P, but P is not sincere, whereas M' is sincere. Second, we note that M cannot have any indecomposable preinjective direct summand. Namely, it would be generated by N, but an indecomposable preinjective module Q_i which is generated by N satisfies $i \le m$.

This means that M is regular (as defined in [R2]). Also, we see that $\operatorname{Hom}(Q, M) = 0$, thus M is generated by $P \oplus R$. Let M' be the trace of R in M. Since R is a regular module of finite length, it follows that M' is regular and of finite type (it is a direct sum of copies of the regular factor modules of R). Now M/M' is generated by P, thus it is not sincere and therefore of finite type. Write M/M' as a direct sum of indecomposable modules, and collect these modules according to the property of being preprojective, regular or preinjective. Thus $M/M' = P' \oplus R' \oplus Q'$, where P' is a direct sum of modules of finite length modules which are preprojective, R' is a direct sum of modules of finite length which are regular, and Q' is a direct sum of modules of finite length which are preinjective.

Now $P' = 0$, since otherwise M would have a proper factor module which is preprojective (and therefore a direct summand). Also, $Q' = 0$, since otherwise M cannot be regular.

This shows that M is an extension of M' by $M/M' = R'$. As we have seen, M' is a direct sum of copies of the regular factor modules of R. whereas R' is a direct sum of finite length modules which are regular and do not contain the composition factor S. But this implies that M is regular and is of finite type.

6. Take-off subcategories.

Let Λ be a representation-infinite artin algebra and $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules.

A full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is said to be a *take-off* subcategory, provided the following conditions are satisfied:
(1) $\mathcal{C}$ is closed under direct sums and under submodules.
(2) $\mathcal{C}$ contains infinitely many isomorphism classes of indecomposable modules.
(3) No proper subcategory of $\mathcal{C}$ satisfies (1) and (2).

(6.1) Theorem. *Any subcategory satisfying* (1) *and* (2) *contains a take-off subcategory.*

In particular, this means that the module category of any representation-infinite artin algebra has at least one take-off subcategory: take-off subcategories always do exist!

(6.2) Examples: If Λ is a connected hereditary algebra which is representation-infinite, then the preprojective modules form a take-off subcategory.

In general, there may be several take-off subcategories: For example, if Λ has several minimal representation-infinite factor algebras, then any such factor algebra yields a take-off subcategory of $\operatorname{mod}\Lambda$.

Remark. Observe that the existence of take-off subcategories is in sharp contrast to the usual characterization of "infinity" (a set is infinite iff it contains proper subsets of the same cardinality)!

(6.3) Properties of a take-off subcategory $\mathcal{C}$

Let $\mathcal{C}$ be a take-off subcategory of $\operatorname{mod}\Lambda$.

(1) *For any d, there are only finitely many isomorphism classes of modules of length d which belong to $\mathcal{C}$.*

Thus: *$\mathcal{C}$ contains indecomposable modules of arbitrarily large finite length.*

Let $\overline{\mathcal{C}}$ be the class of all Λ-modules M such that any finitely generated submodule of M belongs to $\mathcal{C}$.

(2) *There are indecomposable modules M in $\overline{\mathcal{C}}$ of infinite length.*

(3) *If M is an indecomposable modules M in $\overline{\mathcal{C}}$ of infinite length, then any indecomposable module N in $\mathcal{C}$ embeds into M — even a countable direct sum $N^{(\mathbb{N})}$ embeds into M.*

(6.4) We have seen that $\operatorname{mod}\Lambda$ contains take-off subcategories $\mathcal{C}$, such a subcategory $\mathcal{C}$ contains indecomposable modules M of arbitrarily large finite length, and thus indecomposable modules with arbitrarily large socle.

Conjecture. *Let U be an indecomposable Λ-module belonging to a take-off subcategory. If there is a simple module S such that S^7 embeds into U, then there are two embeddings $w_0, v_0\colon S \to U$ such that the corresponding Prüfer module is not of finite type.*

Remark. The bound 7 cannot be lowered, as the path algebra kQ of the $\widetilde{E}_8$-quiver Q with subspace orientation shows. Note that kQ is a tame hereditary algebra, and for a tame hereditary algebra, the preprojective modules form the unique take-off subcategory. Now consider the indecomposable representation U with dimension vector

$$\begin{array}{ccccccc} & 5 & 4 & 3 & 2 & 0 \\ 6 & 4 & 2 & & & \\ & 3 & & & & \end{array}$$

is preprojective and S^6 embeds into U, where S is the simple projective kQ-module. Note that U is not faithful, thus the Prüfer modules constructed by pairs $w_0, v_0\colon S \to U$ are also not faithful. But all non-faithful kQ-modules are of finite type.

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