

1. Opening: Kronecker modules.

1.1. The Fitting lemma.

We start with a typical result from Linear Algebra, namely the Fitting lemma for endomorphisms of finite dimensional vector spaces. Later, we will consider a more general version.

Fitting Lemma. *Let V be a finite dimensional vector space and $f: V \rightarrow V$ an endomorphism. Then there are subspaces V_0, V' of V with $V = V_0 \oplus V'$ such that $f(V_0) \subseteq V_0$, and $f(V') \subseteq V'$ and such that the restriction of f to V_0 is nilpotent, whereas the restriction of f to V' is injective.*

Moreover, these subspaces are uniquely determined. The following stronger property holds: *Let U be any subspace of V with $f(U) \subseteq U$. If the restriction of f to U is nilpotent, then $U \subseteq V_0$. If the restriction of f to U is injective, then $U \subseteq V'$.*

Proof: We have

$$f^{-1}0 \subseteq f^{-2}0 \subseteq \dots \subseteq f^{-t}0 \subseteq \dots$$

(For $x \in f^{-t}0$ we have $f^t(x) = 0$ and therefore $f^{t+1}x = 0$.) This is an ascending chain of subspaces of V and V is finite dimensional, thus there is some n such that $f^{-t}0 = f^{-n}0$ for all $t \geq n$. Similarly, we have

$$fV \supseteq f^2V \supseteq \dots \supseteq f^tV \supseteq \dots$$

(For $x \in f^{t+1}V$ we have $x = f^{t+1}y = f^t(fy)$ for some $y \in V$. Thus $x \in f^tV$.) This is a descending chain of subspaces of V and V is finite dimensional, thus there is some n' such that $f^{-t}0 = f^{-n'}0$ for all $t \geq n'$. Without loss of generality, we can assume $n' = n$.

Let $V_0 = f^{-n}0$ and $V' = f^nV$. We show the various assertions:

(1) $V_0 \cap V' = 0$. Proof. Let $x \in V_0 \cap V'$. Since $x \in V' = f^nV$, there is $y \in V$ with $x = f^ny$. Since $x \in V_0 = f^{-n}0$, we have $f^nx = 0$, thus $f^{2n}y = f^nx = 0$. This shows that $y \in f^{-2n}0 = f^{-n}0$, thus $x = f^ny = 0$.

(3) $V_0 + V' = V$. Proof. Let $v \in V$ and consider f^nv . This is an element of $f^nV = f^{2n}V$, thus there is some w with $f^nv = f^{2n}w$. But this implies that $f^n(v - f^nw) = 0$, thus $v - f^nw \in f^{-n}0$. Now $v = (v - f^nw) + f^nw$ shows that $v \in V_0 + V'$.

(3) $fV_0 \subseteq V_0$. Proof. Let $v \in V_0$, then $f^nv = 0$, thus $f^n(fv) = f^{n+1}v = 0$ and therefore $fv \in V_0$.

(4) $fV' \subseteq V'$. Proof. $f(V') = f(f^nV) = f^{n+1}V \subseteq f^nV = V'$.

(5) The restriction of f to V_0 is nilpotent. Proof: $f^n(V_0) = 0$.

(6) The restriction of f to V' is injective. Proof: We have $V' \cap f^{-1}0 \subseteq V' \cap f^{-n}0$, and the latter is zero by (1).

Now, let U be a subspace of V with $fU \subseteq U$.

(7) Assume that the restriction of f to U is nilpotent, say let $f^mU = 0$ for some m . Then $U \subseteq f^{-m}0 \subseteq f^{-n}0 = V_0$.

(8) Assume that the restriction of f to U is injective. Then $fU = U$, since U is finite dimensional. Therefore also $f^nU = U$. But then $U = f^nU \subseteq f^nV = V'$.

This completes the proof.

Exercise: In the setting of the Fitting lemma, show:

- (a) If $f^{-n}0 = f^{-n-1}0$, then $f^{-n-1}0 = f^{-n-2}0$ (and thus $f^{-t}0 = f^{-n}0$, for all $t \geq n$).
 (b) If $f^n V = f^{n+1}V$, then $f^{n+1}V = f^{n+2}V$ (and thus $f^t V = f^n V$, for all $t \geq n$).

1.2. What are Kronecker modules?

Definition: A *Kronecker module* is of the form $M = (M_1, M_2; M_\alpha, M_\beta) = (V, W, \alpha, \beta)$ where $V = M_1$ and $W = M_2$ are k -spaces, and $\alpha = M_\alpha$ and $\beta = M_\beta$ are k -linear maps $V \rightarrow W$. The *dimension* of M is by definition $\dim M = \dim M_1 + \dim M_2$. One special (but trivial) example should be mentioned right away: A Kronecker module $M = (M_1, M_2; M_\alpha, M_\beta)$ with M_1 and M_2 both being zero spaces (and thus M_α, M_β both being zero maps) is said to be a *zero* Kronecker module (or, sometimes, **the** zero module) and will be denoted just by $0 = (0, 0; 0, 0)$.

Two Kronecker modules M, M' are *isomorphic*, provided there are isomorphisms

$$f_1: M_1 \rightarrow M'_1 \quad \text{and} \quad f_2: M_2 \rightarrow M'_2$$

of vector spaces such that the following two equalities hold:

$$f_2 M_\alpha = M'_\alpha f_1, \quad \text{and} \quad f_2 M_\beta = M'_\beta f_1.$$

In this case, we say that $f = (f_1, f_2): M \rightarrow M'$ is an isomorphism (of Kronecker modules) and if such an isomorphism exists, then we write $M \cong M'$.

Show: (a) If $f = (f_1, f_2)$ is an isomorphism, then also (f_1^{-1}, f_2^{-1}) is an isomorphism.

(b) If $f = (f_1, f_2): M \rightarrow M'$ and $g = (g_1, g_2): M' \rightarrow M''$ are isomorphisms, then $gf = (g_1 f_1, g_2 f_2): M \rightarrow M''$ is an isomorphism.

Lemma. *Let $M = (M_1, M_2; M_\alpha, M_\beta)$ be a Kronecker module with M_α invertible. Then M is isomorphic to $M' = (M_1, M_1; 1, M_\alpha^{-1} M_\beta)$. Here, the map $M'_\alpha = 1$ is the identity map of M_1 .*

Proof: Define an isomorphism $f = (f_1, f_2): M' \rightarrow M$ by taking $f_1 = 1$, the identity map of M_1 , and $f_2 = M_\alpha$. Note that

$$f_2 M'_\alpha = M_\alpha = M_\alpha f_1, \quad \text{and} \quad f_2 M'_\beta = M_\alpha M_\alpha^{-1} M_\beta = M_\beta = M_\beta f_1.$$

We call a Kronecker module M a *band module* provided both M_α and M_β are invertible.

Corollary: *Any band module is isomorphic to a band module M' with $M'_1 = M'_2$ and M'_α being the identity map.*

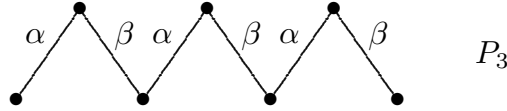
Proof: We only have to observe that with M_β also $M_\alpha^{-1}M_\beta$ is invertible, thus M' as given by the lemma is again a band module.

(Of course, any band module is also isomorphic to a band module M'' with $M_1'' = M_2''$ and M_β'' being the identity map.)

Recall that two endomorphisms ϕ, ϕ' of a vector space V are called *similar*, provided there is an automorphism f of V such that $f\phi = \phi'f$.

Exercise: Let V be a finite dimensional vector space, and ϕ, ϕ' endomorphisms of V . Then $(V, V, 1, \phi)$ is isomorphic to $(V, V, 1, \phi')$ if and only if ϕ and ϕ' are similar.

Some Kronecker modules can be visualized quite well, namely the so-called *string modules*. For example, consider the following labelled graph:



This means the following: The upper bullets symbolize the basis vectors of the vector space V , the lower ones the basis vectors of the vector space W , the edges are considered as arrows pointing downwards. There is just one arrow labelled α starting at the first upper bullet: this means that this vector is sent under α to the basis vector represented by the bullet where the arrow ends (here: the first bullet in the lower row). If we label the basis vectors of V and of W from left to right, we see that we obtain the following matrices:

$$\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This Kronecker module will be denoted by $P_3 = (k^3, k^4, \alpha, \beta)$. In general, we may define P_n for $n \geq 0$ as follows:

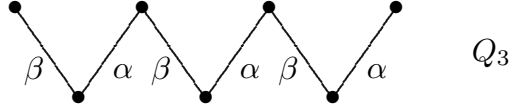
$$P_n = (k^n, k^{n+1}, \alpha, \beta),$$

where α, β are obtained by adding to the $(n \times n)$ -identity matrix one additional zero row: for α the additional row is added as the last row, for β as the first row:

$$\alpha = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \dots & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & \dots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

For reasons which will become apparent later, these Kronecker modules P_n (with $n \geq 0$) are said to be the *preprojective* ones. Note that $P_0 = (0, k; 0, 0)$.

Similarly, consider the following labelled graph:



Again, the upper bullets symbolize the basis vectors of the vector space V , the lower ones the basis vectors of the vector space W . Since there is no arrow labelled α starting at the first upper bullet, the corresponding vector is sent under α to zero, and so on. Thus, here we deal with the Kronecker module

$$Q_3 = (k^4, k^3; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}).$$

Again, in general, we may define Q_n for $n \geq 0$ as follows:

$$Q_n = (k^{n+1}, k^n, \alpha, \beta),$$

where α, β are obtained by adding to the $(n \times n)$ -identity matrix one additional zero column: for α the additional column is added as the first column, for β as the last column:

$$\alpha = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$

Again, it later will be clear why these Kronecker modules Q_n (with $n \geq 0$) are said to be the *preinjective* ones. Note that $Q_0 = (k, 0; 0, 0)$.

Two other sequences of string modules have to be mentioned, those of the form $R_0[m]$ and $R_\infty[m]$, and with $m \geq 1$. For example, for $m = 3$, these are given by the graphs



The general definition is as follows:

$$R_0[m] = (k^m, k^m, I_m, N(m)), \quad R_\infty[m] = (k^m, k^m, N(m), I_m),$$

where I_m is the $(m \times m)$ -identity matrix and $N(m)$ is the $(m \times m)$ -Jordan block with eigenvalue 0:

$$N(m) = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}.$$

Here is now the precise description of those Kronecker modules which will be called *string modules*: there are the preprojective ones P_n , the preinjective ones Q_n and the regular ones $R_0[m]$ and $R_\infty[m]$, with $n \geq 0$ and $m \geq 1$.

If two Kronecker modules M, M' are given, the *direct sum* $M \oplus M'$ is defined as follows:

$$M \oplus M' = (M_1 \oplus M'_1, M_2 \oplus M'_2; M_\alpha \oplus M'_\alpha, M_\beta \oplus M'_\beta).$$

(If V, V' are vector spaces, we denote their direct sum by $V \oplus V'$. If $f: V \rightarrow W, f': V' \rightarrow W'$ are linear transformations, we also can form the direct sum $f \oplus f': V \oplus V' \rightarrow W \oplus W'$; in terms of matrices F, F' , the direct sum operation is defined as the matrix $F \oplus F' = \begin{bmatrix} F & 0 \\ 0 & F' \end{bmatrix}$.)

Using induction, we define direct sums with an arbitrary finite number of summands: If $M^{(i)}$ with $1 \leq i \leq t$ are Kronecker modules, then we write $M^{(1)} \oplus \dots \oplus M^{(t)}$ or also $\bigoplus_{i=1}^t M^{(i)}$ for the direct sum of these Kronecker modules; this is defined using induction on t as follows: For $t = 0$, this direct sum is the zero module, and for $t \geq 1$, we put

$$M^{(1)} \oplus \dots \oplus M^{(t)} = \left(M^{(1)} \oplus \dots \oplus M^{(t-1)} \right) \oplus M^{(t)}.$$

In this way, we define finite direct sums of Kronecker modules. Of course, one may extend this definition in order to deal with infinite direct sums, as well.

Structure Theorem for Kronecker Modules. *If M is a finite dimensional Kronecker module, then M is isomorphic to a direct sum of string modules and a band module.*

The proof is not at all obvious! Before we start with the proof, let us make some comments.

1.3. Preliminary consideration.

Let $M = (M_1, M_2; M_\alpha, M_\beta)$ be a finite dimensional Kronecker module. By definition, the *defect* δM of M is

$$\delta M = \dim M_1 - \dim M_2.$$

If M is a band module, then $\delta M = 0$. If M is a string module, then we obtain the following values:

$$\delta(P_n) = -1, \quad \delta(R_0[m]) = 0, \quad \delta(R_\infty[m]) = 0, \quad \delta(Q_n) = 1,$$

for all $n \geq 0, m \geq 1$.

We need some more vocabulary concerning direct sums. First of all, given Kronecker modules N and M , we say that N is a *direct summand* of M provided M is isomorphic to $N \oplus N'$ for some Kronecker module N' . A Kronecker module M is said to be *indecomposable* provided M is non-zero, and satisfies the following condition: if M is isomorphic to the direct sum $M' \oplus M''$ of two Kronecker modules, then M' or M'' is zero.

It is obvious that any finite-dimensional Kronecker module M can be written as a direct sum of indecomposable Kronecker modules (as usual, this statement means for a

zero module that one considers it as a direct sum with no summand at all): if M is non-zero, then either M is indecomposable (and we consider it as a direct sum with just one summand) or else $M = M' \oplus M''$ with non-zero Kronecker modules M', M'' . By induction on the dimension, we can assume that M' and M'' are written as direct sums of indecomposable Kronecker modules and therefore this is also the case for M . A famous theorem which we will encounter later (the Krull-Remak-Schmidt theorem) asserts that such a direct sum decomposition is unique up to isomorphism.

Note that the structure theorem for Kronecker modules asserts, in particular, that any indecomposable Kronecker module has defect $-1, 0$, or $+1$.

It is worthwhile to spend some time looking at the two one-dimensional Kronecker modules $P_0 = (0, k; 0, 0)$ and $Q_0 = (k, 0; 0, 0)$. A Kronecker module M is said to be P_0 -free provided P_0 is not a direct summand of M . Similarly, we say that M is Q_0 -free provided Q_0 is not a direct summand of M .

Lemma. *Let $M = (M_1, M_2; M_\alpha, M_\beta)$ be a Kronecker module.*

- (a) *M is P_0 -free if and only if $M_\alpha M_1 + M_\beta M_1 = M_2$.*
- (b) *M is Q_0 -free if and only if $M_\alpha^{-1}0 \cap M_\beta^{-1}0 = 0$.*

Proof. (a) Consider a Kronecker module M which is not P_0 -free, thus M is isomorphic to $N \oplus P_0$ for some Kronecker module N . Now $N \oplus P_0 = (N_1, N_2 \oplus k; N_\alpha, N_\beta)$, thus the image $N_\alpha N_1$ of N_α as well as the image $N_\beta N_1$ of N_β both are contained in the subspace $N_2 \oplus 0$ of $N_2 \oplus k$, and this is a proper subspace. This shows that also $M_\alpha M_1 + M_\beta M_1$ is a proper subspace of M_2 .

On the other hand, start with an arbitrary Kronecker module M and choose a subspace C of M_2 such that $(M_\alpha M_1 + M_\beta M_1) \oplus C = M_2$. Then

$$\begin{aligned} M &= (M_1, M_2; M_\alpha, M_\beta) \\ &= (M_1, (M_\alpha M_1 + M_\beta M_1) \oplus C; M_\alpha, M_\beta) \\ &= (M_1, M_\alpha M_1 + M_\beta M_1; M_\alpha, M_\beta) \oplus (0, C; 0, 0). \end{aligned}$$

Of course, $(0, C; 0, 0)$ is a direct sum of copies of P_0 . Thus, in case $M_\alpha M_1 + M_\beta M_1$ is a proper subspace of M_2 , then C is non-zero, and thus $(0, C; 0, 0)$ has at least one direct summand P_0 .

(b) Now consider a Kronecker module M which is not Q_0 -free, thus M is isomorphic to $N \oplus Q_0$ for some Kronecker module N . Now $N \oplus Q_0 = (N_1 \oplus k, N_2; N_\alpha \oplus 0, N_\beta \oplus 0)$, and the kernel of $N_\alpha \oplus 0$ as well as that of $N_\beta \oplus 0$ both contain the subspace $0 \oplus k$ of $N_1 \oplus k$, and this is a proper subspace. This shows that $M_\alpha^{-1}0 \cap M_\beta^{-1}0$ is non-zero.

On the other hand, start with an arbitrary Kronecker module M and choose a subspace C of M_1 such that $(M_\alpha^{-1}0 \cap M_\beta^{-1}0) \oplus C = M_1$. Then

$$\begin{aligned} M &= (M_1, M_2; M_\alpha, M_\beta) \\ &= ((M_\alpha^{-1}0 \cap M_\beta^{-1}0) \oplus C, M_2; M_\alpha, M_\beta) \\ &= ((M_\alpha^{-1}0 \cap M_\beta^{-1}0), 0; 0, 0) \oplus (C, M_2; M_\alpha, M_\beta), \end{aligned}$$

and $(M_\alpha^{-1}0 \cap M_\beta^{-1}0, 0; 0, 0)$ is a direct sum of copies of Q_0 . Thus, in case $M_\alpha^{-1}0 \cap M_\beta^{-1}0$ is non-zero, there is at least one direct summand Q_0 .

Looking at the two maps $M_\alpha, M_\beta: M_1 \rightarrow M_2$ given by a Kronecker module, we may combine them to a single map in two different ways, namely as

$$\begin{aligned} [M_\alpha \quad M_\beta]: M_1 \oplus M_1 &\longrightarrow M_2, \\ \begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}: M_1 &\longrightarrow M_2 \oplus M_2. \end{aligned}$$

Note that the image of $[M_\alpha \quad M_\beta]$ is $M_\alpha M_1 + M_\beta M_1$, whereas the kernel of $\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}$ is $M_\alpha^{-1}0 \cap M_\beta^{-1}0$. It follows:

- (a) M is P_0 -free if and only if $[M_\alpha \quad M_\beta]$ is surjective.
- (b) M is Q_0 -free if and only if $\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}$ is injective.

1.4. The reflection functors σ_+ and σ_- .

Starting with a Kronecker module $M = (M_1, M_2; M_\alpha, M_\beta)$ we are going to define two new Kronecker modules σ_+M and σ_-M (these constructions are “functorial”, but this will be explained only later).

First, we define σ_+M . Let U be the kernel of the map

$$[M_\alpha \quad M_\beta]: M_1 \oplus M_1 \longrightarrow M_2,$$

with inclusion map $u: U \rightarrow M_1 \oplus M_1$. We denote the two canonical projections $M_1 \oplus M_1 \rightarrow M_1$ by p_1 and p_2 (to be precise: if (x, y) is an element of $M_1 \oplus M_1$, where $x, y \in M_1$, then $p_1(x, y) = x$, and $p_2(x, y) = y$), thus we can write $u = \begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix}: U \rightarrow M_1 \oplus M_1$. Let $\sigma_+M = (U, M_1; p_1 u, p_2 u)$, this is again a Kronecker module.

We will need the following property which is easy to verify: *If M and M' are isomorphic Kronecker modules, then also σ_+M and σ_+M' are isomorphic.*

(1) σ_+M is Q_0 -free. This follows immediately from the construction: the map $\begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix} = u$ is injective.

(2) $\sigma_+M = 0$ if and only if $M_1 = 0$, thus if and only if M is the direct sum of copies of P_0 . Proof: If $M_1 \neq 0$, then, by definition, $\sigma_+M \neq 0$. On the other hand, if $M_1 = 0$, then also $U = 0$, since U is a subspace of $M_1 \oplus M_1$, thus $\sigma_+M = 0$.

(3) If M is P_0 -free, then there is the following exact sequence:

$$0 \rightarrow U \xrightarrow{\begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix}} M_1 \oplus M_1 \xrightarrow{[M_\alpha \quad M_\beta]} M_2 \rightarrow 0.$$

(4) If M is P_0 -free, then

$$\dim \sigma_+ M = \dim M + 2\delta M, \quad \text{and} \quad \delta(\sigma_+ M) = \delta M.$$

Proof. Let $\dim M_1 = a$, $\dim M_2 = b$. Then $\dim U = 2a - b$, according to (3), thus

$$\dim \sigma_+ M = \dim U + \dim M_1 = 2a - b + a = (a + b) + 2(a - b) = \dim M + 2\delta M.$$

Also,

$$\delta(\sigma_+ M) = \dim U - \dim M_1 = (2a - b) - a = a - b = \delta M.$$

(5) If M, M' are Kronecker modules, then $\sigma_+(M \oplus M') = \sigma_+(M) \oplus \sigma_+(M')$. Proof: Again this is obvious. We have to construct the kernel of the map

$$[M_\alpha \oplus M'_\alpha \quad M_\beta \oplus M'_\beta] : (M_1 \oplus M'_1) \oplus (M_1 \oplus M'_1) \longrightarrow M_2 \oplus M'_2,$$

but this map is the direct sum of the two maps

$$[M_\alpha \quad M_\beta] : M_1 \oplus M_1 \longrightarrow M_2, \quad \text{and} \quad [M'_\alpha \quad M'_\beta] : M'_1 \oplus M'_1 \longrightarrow M'_2.$$

Next, we define $\sigma_- M$. Let Y be the cokernel of the map

$$\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix} : M_1 \longrightarrow M_2 \oplus M_2,$$

with canonical projection $q: M_2 \oplus M_2 \rightarrow Y$. We denote the two canonical inclusions $M_2 \rightarrow M_2 \oplus M_2$ by i_1 and i_2 (to be precise: for $z \in M_2$, let $i_1 z = (z, 0) \in M_2 \oplus M_2$ and $i_2 z = (0, z) \in M_2 \oplus M_2$), thus we can write $q = [qi_1 \quad qi_2] : M_2 \oplus M_2 \rightarrow Y$. Let $\sigma_- M = (M_2, Y; qi_1, qi_2)$. Again, this is a Kronecker module. Also: *If M and M' are isomorphic Kronecker modules, then also $\sigma_- M$ and $\sigma_- M'$ are isomorphic.*

(1*) $\sigma_- M$ is P_0 -free. This follows immediately from the construction: $[qi_1 \quad qi_2] = q$ is a surjective map.

(2*) $\sigma_- M = 0$ if and only if $M_2 = 0$, thus if and only if M is the direct sum of copies of Q_0 . Proof: If $M_2 \neq 0$, then, by definition, $\sigma_- M \neq 0$. On the other hand, if $M_2 = 0$, then also $Y = 0$, since Y is a factor space of $M_2 \oplus M_2$, and therefore $\sigma_- M = 0$.

(3*) If M is Q_0 -free, then there is the following exact sequence:

$$0 \rightarrow M_1 \xrightarrow{\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}} M_2 \oplus M_2 \xrightarrow{[qi_1 \quad qi_2]} Y \rightarrow 0.$$

(4*) If M is Q_0 -free, then

$$\dim \sigma_- M = \dim M - 2\delta M, \quad \text{and} \quad \delta(\sigma_- M) = \delta M.$$

Proof. Let $\dim M_1 = a, \dim M_2 = b$. Then $\dim Y = 2b - a$, thus

$$\dim \sigma_- M = \dim M_2 + \dim Y = b + 2b - a = (a + b) - 2(a - b) = \dim M - 2\delta M.$$

Also,

$$\delta(\sigma_- M) = \dim M_2 - \dim Y = b - (2b - a) = a - b = \delta M.$$

(5*) If M, M' are Kronecker modules, then $\sigma_-(M \oplus M') = \sigma_-(M) \oplus \sigma_-(M')$. Proof: Again, we only have to notice that we consider the cokernel of the map

$$\begin{bmatrix} M_\alpha \oplus M'_\alpha \\ M_\beta \oplus M'_\beta \end{bmatrix} : M_1 \oplus M'_1 \longrightarrow (M_2 \oplus M'_2) \oplus (M_2 \oplus M'_2),$$

but this map is the direct sum of the two maps

$$\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix} : M_1 \longrightarrow M_2 \oplus M_2, \quad \text{and} \quad \begin{bmatrix} M'_\alpha \\ M'_\beta \end{bmatrix} : M'_1 \longrightarrow M'_2 \oplus M'_2.$$

Now we combine the two constructions:

(6) If M is P_0 -free, then $\sigma_- \sigma_+ M \cong M$. Proof: We start with the exact sequence exhibited in (3), in this way, we obtain $\sigma_+ M$. Now we have to construct the cokernel Y of the map $\begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix} : U \rightarrow M_1 \oplus M_1$ in order to obtain $\sigma_- \sigma_+ M$. Thus, we have to deal with the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & U & \xrightarrow{\begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix}} & M_1 \oplus M_1 & \xrightarrow{\begin{bmatrix} M_\alpha & M_\beta \end{bmatrix}} & M_2 & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \uparrow f & & \\ 0 & \longrightarrow & U & \longrightarrow & M_1 \oplus M_1 & \xrightarrow{\begin{bmatrix} q_{i_1} & q_{i_2} \end{bmatrix}} & Y & \longrightarrow & 0 \\ & & & & \begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix} & & & & \end{array}$$

Here, the map f is obtained, since Y is the cokernel of $\begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix}$ and $\begin{bmatrix} M_\alpha & M_\beta \end{bmatrix} \begin{bmatrix} p_1 u \\ p_2 u \end{bmatrix} = 0$. Since $\begin{bmatrix} M_\alpha & M_\beta \end{bmatrix}$ is surjective, it follows that f is bijective. But this means that $(1, f)$ is an isomorphism $\sigma_- \sigma_+ M \rightarrow M$.

(6*) If M is Q_0 -free, then $\sigma_+ \sigma_- M \cong M$. Proof: Here, we start with the exact sequence exhibited in (3*), in this way, we obtain $\sigma_- M$. Now we have to construct the kernel U

of the map $[qi_1 \quad qi_2] : M_2 \oplus M_2 \rightarrow Y$ in order to obtain $\sigma_+\sigma_-M$. Thus, we have to deal with the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}} & M_2 \oplus M_2 & \xrightarrow{[qi_1 \quad qi_2]} & Y & \longrightarrow & 0 \\
& & \downarrow g & & \parallel & & \parallel & & \\
0 & \longrightarrow & U & \xrightarrow{\begin{bmatrix} p_1u \\ p_2u \end{bmatrix}} & M_2 \oplus M_2 & \xrightarrow{[qi_1 \quad qi_2]} & Y & \longrightarrow & 0
\end{array}$$

Here, the map g is obtained, since U is the kernel of $[qi_1 \quad qi_2]$ and $[qi_1 \quad qi_2] \begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix} = 0$.

Since $\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}$ is injective, it follows that g is bijective. But this means that $(g, 1)$ is an isomorphism $M \rightarrow \sigma_+\sigma_-M$.

(7) *If M is indecomposable and not isomorphic to P_0 , then σ_+M is indecomposable.*

Proof: Consider Kronecker modules N, N' with $\sigma_+M = N \oplus N'$. According to (6) and (5*), we have

$$M \cong \sigma_-\sigma_+M = \sigma_-(N \oplus N') = \sigma_-N \oplus \sigma_-N'.$$

Since we assume that M is indecomposable, one of the Kronecker modules on the right has to be zero, say $\sigma_-N' = 0$. But then, by (2*), N' is a direct sum of copies of Q_0 . On the other hand, (1) asserts that σ_+M and therefore also N' is Q_0 -free. This shows that $N' = 0$.

(7*) *If M is indecomposable and not isomorphic to Q_0 , then σ_-M is indecomposable.*

The proof is similar to the proof of (7).

Exercises.

(a) Isomorphic modules are sent under σ_+ (as well as under σ_-) to isomorphic modules.

(b) If M is a Kronecker module, then $M \cong (\sigma_-\sigma_+M) \oplus N$, where N is the direct sum of copies of P_0 .

(b*) If M is a Kronecker module, then $M \cong (\sigma_+\sigma_-M) \oplus N$, where N is the direct sum of copies of Q_0 .

(These assertions strengthen (6) and (6*.)

(c) If N, N' are Kronecker modules, then

$$\begin{aligned}
\dim(M \oplus M') &= \dim M + \dim M', \quad \text{and} \\
\delta(M \oplus M') &= \delta M + \delta M'.
\end{aligned}$$

1.5. Interated use of the reflection functors.

(8) *Let M be a finite dimensional Kronecker module. If M is indecomposable and $\delta M < 0$, then $M \cong (\sigma_-)^t(P_0)$ for some natural number t .*

Proof: We use induction on the dimension of M . Obviously, there are only two Kronecker modules with $\dim M = 1$, namely $M = P_0$ and $M = Q_0$. Since we assume that $\delta M < 0$, we see that $M = P_0$. Thus, assume that $\dim M > 1$. In particular, M is not isomorphic to P_0 , and since M is indecomposable, it is P_0 -free. According to (4), we see that

$$\dim \sigma_+ M = \dim M + 2\delta M < \dim M \quad \text{and} \quad \delta(\sigma_+ M) = \delta M < 0.$$

Also, according to (7), $\sigma_+ M$ is again indecomposable. Thus we can use induction in order to conclude that $\sigma_+ M \cong \sigma_-(P_0)$ for some natural number t . According to (6)

$$M \cong \sigma_- \sigma_+ M = \sigma_-(\sigma_-)^t(P_0) = (\sigma_-)^{t+1}(P_0).$$

This completes the proof.

(8*) *Let M be a finite dimensional Kronecker module. If M is indecomposable and $\delta M > 0$, then $M \cong (\sigma_+)^t(Q_0)$ for some natural number t .*

The proof is similar to the proof of (8).

(9) $\sigma_-(P_{t-1}) \cong P_t$ for any $t \geq 1$, and thus $P_t \cong (\sigma_-)^t(P_0)$.

Proof: Recall that $P_{t-1} = (k^{t-1}, k^t; \alpha, \beta)$ with α, β being matrices as specified above, thus

$$\begin{bmatrix} \alpha \\ \dots \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

(here, the upper and the lower blocks both are the $(t-1) \times (t-1)$ -identity matrix, altogether there are $2t$ rows and $t-1$ columns. We have to determine the cokernel q of the corresponding map $k^{t-1} \rightarrow k^{2t}$ given by this matrix. The cokernel q is the map $k^{2t} \rightarrow k^{t+1}$ with the following matrix

$$\begin{bmatrix} 0 & \dots & 0 & -1 & 0 \\ 1 & & 0 & & \ddots \\ & \ddots & & 0 & -1 \\ 0 & & 1 & 0 & 0 \end{bmatrix}$$

(here, on the left side one deals with a zero row above the $t \times t$ -identity matrix, on the right side one deals with a zero row below the negative of the $t \times t$ -identity matrix). It is

sufficient to check that the composition of the matrices is zero and that the new matrix has rank $t + 1$ (the latter is seen by looking just at the first $t + 1$ columns).

Now the matrix for the cokernel is not yet what we want — we have to construct new bases of k^t and k^{t+1} so that the maps $\alpha = qi_1, \beta = qi_2$ are given by the required matrices: we have to renumber the given bases and multiply any second element with -1 . Let us do this at least once explicitly! We denote the canonical basis of k^t by v_1, \dots, v_t , that of k^{t+1} by w_1, \dots, w_{t+1} . Then

$$\alpha(v_i) = w_{i+1}, \quad \text{and} \quad \beta(v_i) = -w_i.$$

Now, let

$$v'_i = (-1)^i v_{t+1-i}, \quad \text{and} \quad w'_i = (-1)^i w_{t+2-i}.$$

Then

$$\begin{aligned} \alpha(v'_i) &= \alpha((-1)^i v_{t+1-i}) = (-1)^i w_{t+2-i} = w'_i, \\ \beta(v'_i) &= \beta((-1)^i v_{t+1-i}) = (-1)^i (-w_{t+1-i}) = w'_{i+1}, \end{aligned}$$

thus with respect to the new bases, the matrices obtained for α and β are precisely the ones exhibited for P_t .

$$(9^*) \quad \sigma_+(Q_{t-1}) \cong Q_t \text{ for any } t \geq 1, \text{ and thus } Q_t \cong (\sigma_+)^t(Q_0).$$

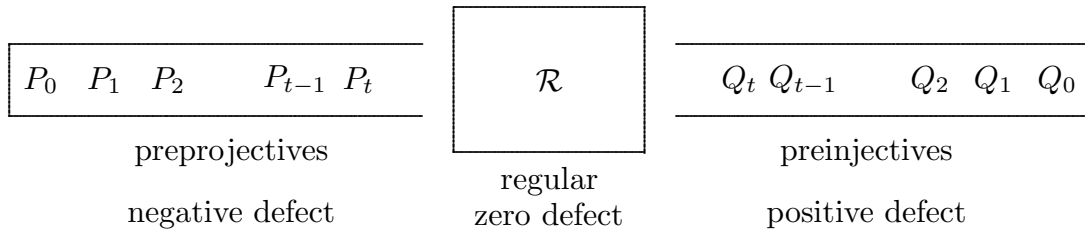
Again, this is the dual statement. For the proof one just can use the transposes of the matrices presented in the proof of (9).

Exercise: Show directly that P_t is indecomposable (for example, using matrix calculations). Then (8) provides another proof of (9).

Corollary. *Let M be a finite dimensional Kronecker module. If the defect of M is negative, then M has a direct summand isomorphic to some P_t . If its defect is positive, then M has a direct summand isomorphic to some Q_t .*

Thus the indecomposable Kronecker modules of non-zero defect are the preprojective and the preinjective string modules. In particular, δM is equal to $-1, 0$, or $+1$ for any indecomposable Kronecker module.

A finite dimensional Kronecker module is said to be *regular* provided any indecomposable direct summand has defect zero. We denote the class of regular Kronecker modules by \mathcal{R} . The picture which one always has to have in the mind, is the following:



here, the action of σ_+ on the preprojective part as well as on the preinjective part is the shift to the left, thus

$$\begin{aligned}\sigma_+(P_0) &= 0, \\ \sigma_+(P_t) &= P_{t-1} \quad \text{for } t \geq 1 \\ \sigma_+(Q_t) &= Q_{t+1} \quad \text{for } t \geq 0\end{aligned}$$

whereas σ_- is the corresponding shift to the right. On the regular part, both σ_+ and σ_- provide permutations of the isomorphism classes

(10) *Let M be indecomposable. The following conditions are equivalent:*

- (i) *M is not isomorphic to one of the Kronecker modules P_0, \dots, P_t .*
- (ii) *$(\sigma_-)^{t+1}M \neq 0$.*
- (iii) *$(\sigma_+)^{t+1}(\sigma_-)^{t+1}(M) \cong M$.*

(10*) *Let M be indecomposable. The following conditions are equivalent:*

- (i) *M is not isomorphic to one of the Kronecker modules Q_0, \dots, Q_t .*
- (ii) *$(\sigma_-)^{t+1}M \neq 0$.*
- (iii) *$(\sigma_+)^{t+1}(\sigma_-)^{t+1}(M) \cong M$.*

Before we proceed, we need a further concept, that of a **submodule**. Let $M = (M_1, M_2; M_\alpha, M_\beta)$ be a Kronecker module. Assume that there given subspaces $N_1 \subseteq M_1$, and $N_2 \subseteq M_2$ such that $M_\alpha N_1 \subseteq N_2$, and $M_\beta N_1 \subseteq N_2$. If we denote by N_α and N_β the restrictions of M_α , respectively M_β to N_1 , then clearly $N = (N_1, N_2; N_\alpha, N_\beta)$ is again a Kronecker module, and we say that N is a submodule of M and write $N \subseteq M$. Since the maps N_α, N_β are determined by M_α, M_β , it is sufficient to specify (N_1, N_2) . If N is isomorphic to a submodule of M , we say that N can be *embedded* into M .

In case N is a submodule of M , we define the corresponding *factor module* M/N as follows:

$$M/N = (M_1/N_1, M_2/N_2; \overline{M}_\alpha, \overline{M}_\beta),$$

where $\overline{M}_\alpha, \overline{M}_\beta$ are the linear transformations $M_1/N_1 \rightarrow M_2/N_2$ induced by M_α , and M_β , respectively.

(11) *If the Kronecker module M has a submodule isomorphic to Q_t for some t , then M has a direct summand of the form Q_s with $s \leq t$.*

Proof: We show this by induction on t . The case $t = 0$ is obvious, thus we assume that $t > 0$. Let $M' \subseteq M$ with $M' \cong Q_t$. We apply σ_- both to M and to M' , thus, we get

a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
M'_1 & \xrightarrow{\begin{bmatrix} M'_\alpha \\ M'_\beta \end{bmatrix}} & M'_2 \oplus M'_2 & \xrightarrow{q'} & Y' & \longrightarrow & 0 \\
\downarrow u_1 & & \downarrow u_2 \oplus u_2 & & \downarrow f & & \\
M_1 & \xrightarrow{\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}} & M_2 \oplus M_2 & \xrightarrow{q} & Y & \longrightarrow & 0
\end{array}$$

where u_1, u_2 are just the inclusion maps. Here we obtain f since Y' is the cokernel of $\begin{bmatrix} M'_\alpha \\ M'_\beta \end{bmatrix}$ and $q(u_2 \oplus u_2)$ vanishes on the image of $\begin{bmatrix} M'_\alpha \\ M'_\beta \end{bmatrix}$. Let Y'' be the image of f , and $u: Y'' \rightarrow Y$ the inclusion map. We obtain the following commutative diagram

$$\begin{array}{ccc}
M'_2 \oplus M'_2 & \xrightarrow{fq'} & Y'' \\
\downarrow u_2 \oplus u_2 & & \downarrow u \\
M_2 \oplus M_2 & \xrightarrow{q} & Y
\end{array}$$

which shows that the upper row yields a submodule N of $\sigma_- M$, namely $N_1 = M'_2$ and $N_2 = Y''$. Since Y'' is a factor space of Y' , we see that $\dim Y'' \leq \dim Y'$. We calculate the defect of N :

$$\delta N = \dim M'_2 - \dim Y'' \geq \dim M'_2 - \dim Y' = \delta(\sigma_- M') = \delta M' = 1$$

using (4*). It follows that N has a direct summand of the form Q_s , see the corollary of (8*). Since

$$2s + 1 = \dim Q_s \leq \dim N \leq \dim \sigma_- M' = \dim Q_{t-1} = 2(t-1) + 1,$$

it follows that $s \leq t-1$. Altogether, we have shown that $\sigma_- M$ has a submodule isomorphic to Q_s for some $s \leq t-1$.

By induction, $\sigma_- M \cong Q_r \oplus N'$, for some $r \leq s$. We have

$$\sigma_+ \sigma_- M \cong \sigma_+ Q_r \oplus \sigma_+ N' \cong Q_{r+1} \oplus \sigma_+ N',$$

using (5) and (9*). In this way, we see that $\sigma_+ \sigma_- M$ has a direct summand of the form Q_{r+1} and $r+1 \leq s+1 \leq t$. On the other hand, $\sigma_+ \sigma_- M$ itself is a direct summand of M , thus also M has a direct summand of the form Q_{r+1} .

(12) *Let M be an indecomposable Kronecker module and assume that M has a submodule isomorphic to Q_t for some t , then $M \cong Q_t$.*

Proof: According to (11), we know that M has a direct summand isomorphic to Q_s with $s \leq t$. But since M is indecomposable, it follows that $M \cong Q_s$. In particular, $\dim M = 2s+1$. But Q_t is a submodule of M and $\dim Q_t = 2t+1$, therefore $2t+1 \leq 2s+1$. The inequalities together yield $s = t$.

Exercises.

- (a) Find the dual statements for (12) and (12),
- (b) Show that in contrast to (12) the asserting of (11) cannot be improved to $s = t$. Namely, for any $t \geq 1$, let M be the direct sum of t copies of Q_1 . Define explicitly a subspace $N_1 \subseteq M_1$ such that $N = (N_1, M_2)$ is isomorphic to Q_t .
- (c) Show that for any vector space V , the Kronecker module

$$(V \oplus V, V, [1 \ 0], [0 \ 1])$$

is a direct sum of copies of Q_1 .

- (d) Show that any Q_0 -free Kronecker module $M = (M_1, M_2, M_\alpha, M_\beta)$ can be embedded into the Kronecker module

$$(M_2 \oplus M_2, M_2, [1 \ 0], [0 \ 1]).$$

(Assertions (c) and (d) together show: Any Q_0 -free Kronecker module can be embedded into a direct sum of copies of Q_1 .)

1.6. Regular Kronecker modules.

Submodule characterization of the regular Kronecker modules. *A finite dimensional Kronecker module M with zero defect is regular if and only if $\delta N \leq 0$ for any submodule N of M .*

Proof. First, consider the case of M being regular. If N is a submodule of M with positive defect, then M has a direct summand with positive defect, according to (8*) and (11), impossible. The reverse implication is trivial: If we assume that all submodules N of M satisfy $\delta N \leq 0$, and N is a direct summand of M , say $M \cong N \oplus N'$, then $0 = \delta M = \delta N + \delta N'$ implies that both $\delta N = 0 = \delta N'$.

Corollary. *Let M be a regular Kronecker module. Any submodule N of M of defect zero is regular.*

Proof: With M also N satisfies the submodule condition.

Let \mathcal{R}' be the class of finite dimensional Kronecker modules M with M_α being bijective; and \mathcal{R}_∞ the class of finite dimensional Kronecker modules with M_β bijective and $(M_\beta)^{-1}M_\alpha$ nilpotent. Of course, both $\mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{R}_\infty \subseteq \mathcal{R}$.

Proposition. *Any regular Kronecker module M is the direct sum of a Kronecker module in \mathcal{R}' and a Kronecker module in \mathcal{R}_∞ .*

The proof is by induction on the dimension of $M = (M_1, M_2, \alpha, \beta)$. If $\dim M = 0$, nothing has to be shown, since the zero module belongs both to \mathcal{R}_∞ as well as to \mathcal{R}' . Thus assume that M is not the zero module. If M_α is invertible, then M belongs to \mathcal{R}' . Thus we assume that α is not invertible. Since $\dim M_2 = \dim M_1$, this means that α is not surjective. Let N_2 be a subspace of codimension 1 of M_2 which contains the image of M_α . Let $N_1 = \beta^{-1}(N_2)$, this is a subspace of M_1 . Obviously, the map $\bar{\beta}$ induces a map

$$\bar{\beta}: M_1/N_1 \rightarrow M_2/N_2,$$

and β is injective, since $N_1 = \beta^{-1}(N_2)$. Now, by assumption, $\dim M_2/N_2 = 1$. Note that M is P_0 -free, thus $\alpha M_1 + \beta M_1 = M_2$, in particular βM_1 is not contained in N_2 and therefore $\bar{\beta} \neq 0$. This shows that $\dim M_1/N_1 = 1$. Therefore $\dim N_1 = \dim N_2$. Since by construction $\alpha N_1 \subseteq N_2, \beta N_2 \subseteq N_2$, we have constructed a submodule $N = (N_1, N_2)$ of M of defect zero. By the corollary above, N itself is regular.

By induction, we write $N = N' \oplus N''$ with $N' \in \mathcal{R}'$ and $N'' \in \mathcal{R}_\infty$. We claim that there exists $x \in M_1 \setminus N_1$ such that $\alpha x \in N_2''$. Namely, choose $x \in M_1 \setminus N_1$ and consider αx . Now $\alpha x \in N_2 = N_2' + N_2''$, thus we write $\alpha x = y' + y''$ with $y' \in N_2'$ and $y'' \in N_2''$. Since α is bijective for M' , there exists $x' \in N_1'$ with $\alpha(x') = y'$ and therefore $\alpha(x - x') = y'' \in N_2''$. Note that with x also $x - x'$ belongs to $M_1 \setminus N_1$. Thus replace x by $x - x'$.

Now, starting with an element $x \in M_1 \setminus N_1$ such that $\alpha x \in N_2''$, we define a submodule $M'' = (M_1'', M_2'')$ of M as follows:

$$M_1'' = N_1'' + kx, \quad \text{and} \quad M_2'' = N_2'' + k(\beta x)$$

(it is obvious that this is a submodule). Note that

$$(*) \quad N_1' + M_1'' = N_1' + N_1'' + kx = N_1 + kx = M_1.$$

It follows that $\beta x \notin N_2$, since otherwise (M_1, N_2) would be a submodule of positive defect. In particular, $\beta x \notin N_2''$, and therefore $\beta: M_1'' \rightarrow M_2''$ is invertible. Also, since $\beta^{-1}\alpha(x) \in N_1''$ and $\beta^{-1}\alpha$ is nilpotent on N_1'' , we see that $\beta^{-1}\alpha$ is nilpotent on M_1'' . This shows that M'' belongs to \mathcal{R}_∞ .

It follows from (*) and $\dim M_1 = \dim N_1' + \dim M_1''$, that $N_1' \cap M_1'' = 0$. Similar, we have

$$N_2' + M_2'' = N_2' + N_2'' + k(\beta x) = N_2 + k(\beta x) = M_2$$

and $\dim M_2 = \dim N_2' + \dim M_2''$, thus $N_2' \cap M_2'' = 0$.

Altogether we see that $M = N' \oplus M''$, where $N' \in \mathcal{R}'$ and $M'' \in \mathcal{R}_\infty$. This concludes the proof.

Let \mathcal{R}_0 be the class of finite dimensional Kronecker modules with M_α bijective and $(M_\alpha)^{-1}M_\beta$ nilpotent, this is a subclass of \mathcal{R}' .

Theorem. *Any Kronecker module in \mathcal{R}' is the direct sum of a Kronecker module in \mathcal{R}_0 and a band.*

Proof: When we deal with a Kronecker module $M = (M_1, M_2; \alpha, \beta)$ in \mathcal{R}' , we can assume up to isomorphism that α is the identity map, thus we can assume that $M = (V, V; 1, \beta)$ where V is a finite dimensional vector space and β an endomorphism of V . Now we just have to apply the Fitting Lemma.

Theorem.

- (a) *The indecomposable Kronecker modules in \mathcal{R}_∞ are the Kronecker modules $R_\infty[m]$ with $m \geq 1$.*
- (b) *The indecomposable Kronecker modules in \mathcal{R}_0 are the Kronecker modules $R_0[m]$ with $m \geq 1$.*

Proof. (b) As we have mentioned, when dealing with a Kronecker module $M = (M_1, M_2; \alpha, \beta)$ in \mathcal{R}' , we can assume up to isomorphism that α is the identity map, thus we can assume that $M = (V, V; 1, \beta)$ where V is a finite dimensional vector space and β an endomorphism of V . In case M belongs to \mathcal{R}_0 , the map β is nilpotent and we can use the Jordan normal form for nilpotent endomorphisms.

The proof of (a) is obtained by interchanging α and β .

This completes the proof of the Structure Theorem for Kronecker modules.

1.7. Homomorphisms.

We did not yet consider homomorphisms of Kronecker modules in general, in particular we did not yet consider endomorphism rings of Kronecker modules. The only homomorphisms of Kronecker modules which were discussed in some detail were the isomorphisms as well as the submodule inclusions.

Let $M = (M_1, M_2; M_\alpha, M_\beta)$ and $M' = (M'_1, M'_2; M'_\alpha, M'_\beta)$ be Kronecker modules. A *homomorphism* $f: M \rightarrow M'$ is a pair $f = (f_1, f_2)$ of linear transformations $f_1: M_1 \rightarrow M'_1$ and $f_2: M_2 \rightarrow M'_2$ such that

$$M'_\alpha f_1 = f_2 M_\alpha, \quad \text{and} \quad M'_\beta f_1 = f_2 M_\beta.$$

We denote by

$$\text{Hom}(M, M') = \{f: M \rightarrow M' \mid f \text{ is a homomorphism}\}.$$

the set of homomorphisms from M to M' . Claim: This is a vector space (over our base field k). Namely, recall that for any pair of vector spaces V, V' , the set of linear transformations $V \rightarrow V'$ is considered as a vector space using the so-called “pointwise” addition and scalar multiplication: If $\phi, \psi: V \rightarrow V'$ are linear transformations, and $\lambda \in k$, then $\phi + \psi$ and $\lambda\phi$ are defined by

$$\begin{aligned} (\phi + \psi)(v) &= \phi(v) + \psi(v) \\ (\lambda\phi)(v) &= \lambda\phi(v). \end{aligned}$$

Now, if $f = (f_1, f_2)$ and $g = (g_1, g_2)$ belong to $\text{Hom}(M, M')$, then, by definition

$$\begin{aligned} f + g &= (f_1 + g_1, f_2 + g_2) \\ \lambda f &= (\lambda f_1, \lambda f_2), \end{aligned}$$

note that these are again elements of $\text{Hom}(M, M')$. Also, one verifies without difficulties that in this way $\text{Hom}(M, M')$ becomes a vector space. Note that the zero element of $\text{Hom}(M, M')$ is the pair $(0, 0)$ (it will be denoted just by 0).

Of course, if M'' is a further Kronecker module, then we can compose homomorphisms $f = (f_1, f_2): M \rightarrow M'$ and $f' = (f'_1, f'_2): M' \rightarrow M''$ as follows

$$f'f = (f'_1f_1, f'_2f_2).$$

The obvious associativity and distributivity rules hold; also, for $\lambda \in k$, we have $(\lambda f')f = f'(\lambda f)$.

Homomorphisms $f: M \rightarrow M$ are called *endomorphisms* and instead of $\text{Hom}(M, M)$ we write $\text{End}(M)$. The identity maps 1_{M_1} and 1_{M_2} combine to what is called the identity homomorphism $1_M = (1_{M_1}, 1_{M_2}): M \rightarrow M$. The notion of an isomorphism $f = (f_1, f_2): M \rightarrow M'$ has already been introduced: these are precisely the homomorphisms with both f_1, f_2 being invertible. If $f: M \rightarrow M$ is an isomorphism, then we can consider $f^{-1} = (f_1^{-1}, f_2^{-1}): M \rightarrow M$, this is again an isomorphism and we have $f^{-1}f = 1_M$, and $ff^{-1} = 1_{M'}$. In case M is not the zero Kronecker module, the scalar multiples $\lambda 1_M$ of 1_M form a one-dimensional subspace of $\text{End}(M)$. Thus, if we write $\text{End}(M) = k$, this just means that any endomorphism of M is given by scalar multiplication.

When dealing with Kronecker modules, the following assertion is of great importance:

Theorem. *Let M, N be Kronecker modules. Then there is a vector space isomorphism*

$$\text{Hom}(\sigma_- M, N) \cong \text{Hom}(M, \sigma_+ N).$$

The experienced reader will see that this means that the pair of functors (σ_-, σ_+) is an adjoint pair.

Proof: Let $f = (f_1, f_2): \sigma_- M, N$ be a homomorphism. Consider the following diagram

$$\begin{array}{ccccccc} M_1 & \xrightarrow{\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}} & M_2 \oplus M_2 & \xrightarrow{q} & Y & \longrightarrow & 0 \\ & & \downarrow f_1 \oplus f_1 & & \downarrow f_2 & & \\ 0 & \longrightarrow & U & \xrightarrow{u} & N_1 \oplus N_1 & \xrightarrow{\begin{bmatrix} M_\alpha & M_\beta \end{bmatrix}} & N_2 \longrightarrow 0 \end{array}$$

Since u is the kernel of $[M_\alpha \ M_\beta]$, it follows that there exists a linear transformation $f_0: M_1 \rightarrow U$ such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}} & M_2 \oplus M_2 \\ \downarrow f_0 & & \downarrow f_1 \oplus f_1 \\ U & \xrightarrow{u} & N_1 \oplus N_1 \end{array}$$

but this just means that $(f_0, f_1): M \rightarrow \sigma_+ N$ is a homomorphism. It is easily checked that the mapping $(f_1, f_2) \mapsto (f_0, f_1)$ commutes with addition and scalar multiplication, thus we obtain in this way a linear transformation

$$\eta: \text{Hom}(\sigma_- M, N) \longrightarrow \text{Hom}(M, \sigma_+ N).$$

Conversely, assume there is given a homomorphism $g = (g_1, g_2)$ in $\text{Hom}(M, \sigma_+ N)$. Then we look at

$$\begin{array}{ccccccc} M_1 & \xrightarrow{\begin{bmatrix} M_\alpha \\ M_\beta \end{bmatrix}} & M_2 \oplus M_2 & \xrightarrow{q} & Y & \longrightarrow & 0 \\ \downarrow g_1 & & \downarrow g_2 \oplus g_2 & & & & \\ 0 & \longrightarrow & U & \xrightarrow{u} & N_1 \oplus N_1 & \xrightarrow{\begin{bmatrix} N_\alpha & N_\beta \end{bmatrix}} & N_2 \longrightarrow 0 \end{array}$$

Now we use that the map q is a cokernel map, thus we obtain $g_3: Y \rightarrow N_2$ such that the following diagram commutes:

$$\begin{array}{ccc} M_2 \oplus M_2 & \xrightarrow{q} & Y \\ \downarrow g_2 \oplus g_2 & & \downarrow g_3 \\ N_1 \oplus N_1 & \xrightarrow{\begin{bmatrix} N_\alpha & N_\beta \end{bmatrix}} & N_2 \end{array}$$

but this just means that $(g_2, g_3): \sigma_- M \rightarrow N$ is a homomorphism. It is obvious that the two constructions $(f_1, f_2) \mapsto (f_0, f_1)$ and $(g_1, g_2) \mapsto (g_2, g_3)$ are inverse to each other, thus η is a vector space isomorphism.

Theorem. *If M is an indecomposable Kronecker module and $\text{Hom}(Q_t, M) \neq 0$, then $M \cong Q_s$ for some $s \leq t$.*

Proof: Assume that M is not isomorphic to Q_s for $0 \leq s \leq t$. Then, according to (10*), $(\sigma_+)^{t+1}(\sigma_-)^{t+1}M \cong M$ and therefore

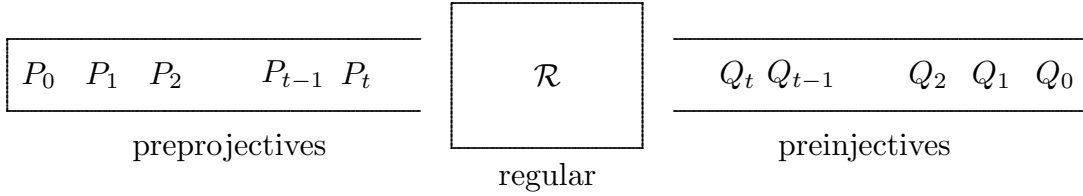
$$\begin{aligned} \text{Hom}(Q_t, M) &\cong \text{Hom}(Q_t, (\sigma_+)^{t+1}(\sigma_-)^{t+1}M) \\ &\cong \text{Hom}((\sigma_-)^{t+1}Q_t, (\sigma_-)^{t+1}M) \\ &= \text{Hom}(0, (\sigma_-)^{t+1}M) = 0. \end{aligned}$$

. As a corollary, we get a new proof of (11): Assume that the Kronecker module M has a submodule N isomorphic to Q_t for some t . If we compose an isomorphism $Q_t \rightarrow N$ and the inclusion map $N \rightarrow M$, we obtain a non-zero homomorphism $\text{Hom}(Q_t, M)$. Write M as a direct sum of indecomposable modules $M^{(i)}$, then $\text{Hom}(Q_t, M^{(i)})$ for some i . The theorem now asserts that $M \cong Q_s$ for some $s \leq t$.

There is the dual assertion:

Theorem *. *If M is an indecomposable Kronecker module and $\text{Hom}(M, P_t) \neq 0$, then $M \cong P_s$ for some $s \leq t$.*

Looking again at the picture shown above:



we now see that it also reflects the existence of (non-zero) homomorphisms: Non-zero homomorphisms essentially go from left to right: theorem asserts that there are no non-zero homomorphisms from a preinjective Kronecker module to a Kronecker module which is preprojective or regular. The dual theorem excludes in addition also non-zero maps from a regular Kronecker module to a preprojective one. In addition, looking at the sequences P_0, P_1, P_2, \dots and \dots, Q_2, Q_1, Q_0 , we also see that non-zero maps cannot go from right to left.

Let us denote by \mathcal{B} the class of band modules, thus, as we know, any indecomposable regular Kronecker module belongs to one of the classes $\mathcal{R}_0, \mathcal{B}, \mathcal{R}_\infty$.

Exercise: If M, N are indecomposable regular Kronecker modules and do not belong both to \mathcal{R}_0 , or both to \mathcal{B} or both to \mathcal{R}_∞ , then $\text{Hom}(M, N) = 0$.

Endomorphism rings. If M is Q_0 -free, then we see:

$$\text{Hom}(\sigma_- M, \sigma_- M) \cong \text{Hom}(M, \sigma_+ \sigma_- M) \cong \text{Hom}(M, M),$$

thus $\text{End}(M) = k$ if and only if $\text{End}(\sigma_- M) = k$. If M is P_0 -free, then similarly, we have

$$\text{Hom}(\sigma_+ M, \sigma_+ M) \cong \text{Hom}(\sigma_- \sigma_+ M, M) \cong \text{Hom}(M, M),$$

and therefore $\text{End}(M) = k$ if and only if $\text{End}(\sigma_+ M) = k$.

Corollary. $\text{End}(P_t) = k$ and $\text{End}(Q_t) = k$ for all t .

Proof by induction on t . For $t = 0$, the assertions are obvious.

Remark. In contrast, the endomorphism rings of the regular string modules are usually bigger: One easily verifies that $\text{End}(R_0[1]) = k$ and $\text{End}(R_\infty[1]) = k$, whereas for $m \geq 2$, one has $\dim \text{End}(R_0[m]) \geq 2$ and $\dim \text{End}(R_\infty[m]) \geq 2$.

Exercise: show that any $R_0[m]$ has an endomorphism with image $R_0[1]$, and similarly, that any $R_\infty[m]$ has an endomorphism with image $R_\infty[1]$.

1.7. Linear relations on a vector space.

The concept of a “relation” is very basic in mathematics. Modern mathematics is usually formulated in terms of sets and maps between sets, but actually the (set-theoretical) maps are defined as special relations. Recall that a *relation* ρ between two sets W_1 and W_2 is just a subset of $W_1 \times W_2$ and such a relation ρ is called a *map* provided the following two conditions are satisfied:

- (R1) For any $w_1 \in W_1$, there exists $w_2 \in W_2$ such that $(w_1, w_2) \in \rho$.
- (R2) If $w_1 \in W_1$ and $w_1, w'_2 \in W_2$ such that $(w_1, w_2), (w_1, w'_2) \in \rho$, then $w_2 = w'_2$.

If these conditions are satisfied, one usually writes $\rho: W_1 \rightarrow W_2$, and, for $(w_1, w_2) \in \rho$, one writes $w_2 = \rho(w_1)$. Note that in this way, one identifies a map $f: W_1 \rightarrow W_2$ with what one calls its graph: the subset $\{(w, f(w)) \mid w \in W_1\}$. Of special interest seems to be the case when $W_1 = W_2 = W$. A relation $\rho \subseteq W \times W$ will just be called a *relation on* W .

Similarly, in the linear world, we should look not only at linear transformations, but more generally at “linear relations”, a *linear relation* between two vector spaces W_1, W_2 is by definition a subspace ρ of $W_1 \oplus W_2$. Note that such a subspace $\rho \subseteq W_1 \oplus W_2$ is (the graph of) a linear transformation precisely if ρ as a subset of $W_1 \times W_2$ satisfies the conditions (R1) and (R2).

Thus, let us consider the linear relations of a vector space W . As we have said, these are just the subspaces of $W \oplus W$.

Proposition. *The linear relations on a vector space are nothing else than the Q_0 -free Kronecker modules.*