On radical square zero rings.

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Let Λ be a connected left artinian ring with radical square zero and with n simple modules. If Λ is not self-injective, then we show that any module M with $\mathrm{Ext}^i(M,\Lambda) = 0$ for $1 \leq i \leq n+1$ is projective. We also determine the structure of the artin algebras with radical square zero and n simple modules which have a non-projective module M such that $\mathrm{Ext}^i(M,\Lambda) = 0$ for $1 \leq i \leq n$.

Xiao-Wu Chen [C] has recently shown: given a connected artin algebra Λ with radical square zero then either Λ is self-injective or else any CM module is projective. Here we extend this result by showing: If Λ is a connected artin algebra with radical square zero and n simple modules then either Λ is self-injective or else any module M with $\mathrm{Ext}^i(M,\Lambda) = 0$ for $1 \leq i \leq n+1$ is projective. Actually, we will not need the assumption on Λ to be an artin algebra; it is sufficient to assume that Λ is a left artinian ring. And we show that for artin algebras the bound $n+1$ is optimal by determining the structure of those artin algebras with radical square zero and n simple modules which have a non-projective module M such that $\mathrm{Ext}^i(M,\Lambda) = 0$ for $1 \leq i \leq n$.

From now on, let Λ be a left artinian ring with radical square zero, this means that Λ has an ideal I with $I^2 = 0$ (the radical) such that Λ/I is semisimple artinian. We also assume that Λ is connected (the only central idempotents are 0 and 1). The modules to be considered are usually finitely generated left Λ-modules. Let n be the number of (isomorphism classes of) simple modules.

Given a module M, we denote by PM a projective cover, by QM an injective envelope of M. Also, we denote by ΩM a syzygy module for M, this is the kernel of a projective cover $PM \to M$. Since Λ is a ring with radical square zero, all the syzygy modules are semisimple. Inductively, we define $\Omega_0 M = M$, and $\Omega_{i+1}M = \Omega(\Omega_i M)$ for $i \geq 0$.

Lemma 1. *If M is a non-projective module with $\mathrm{Ext}^i(M,\Lambda) = 0$ for $1 \leq i \leq d+1$ (and $d \geq 1$), then there exists a simple non-projective module S with $\mathrm{Ext}^i(S,\Lambda) = 0$ for $1 \leq i \leq d$.*

Proof: We have $\mathrm{Ext}^i(M,\Lambda) \simeq \mathrm{Ext}^{i-1}(\Omega M,\Lambda)$, for all $i \geq 2$. Since M is not projective, $\Omega M \neq 0$. Now ΩM is semisimple. If all simple direct summands of ΩM are projective, then also ΩM is projective, but then the condition $\mathrm{Ext}^1(M,\Lambda) = 0$ implies that $\mathrm{Ext}^1(M,\Omega M) = 0$ in contrast to the existence of the exact sequence $0 \to \Omega M \to PM \to M \to 0$. Thus, let S be a non-projective simple direct summand of ΩM.

Lemma 2. *If S is a non-projective simple module with $\mathrm{Ext}^1(S,\Lambda) = 0$, then PS is injective and ΩS is simple and not projective.*

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Proof: First, we show that PS has length 2. Otherwise, ΩS is of length at least 2, thus there is a proper decomposition $\Omega S = U \oplus U'$ and then there is a canonical exact sequence

$$0 \to PS \to PS/U \oplus PS/U' \to S \to 0,$$

which of course does not split. But since $\operatorname{Ext}^1(S,\Lambda) = 0$, we have $\operatorname{Ext}^1(S,P) = 0$, for any projective module P. Thus, we obtain a contradiction.

This shows also that ΩS is simple. Of course, ΩS cannot be projective, again according to the assumption that $\operatorname{Ext}^1(S,P) = 0$, for any projective module P.

Now let us consider the injective envelope Q of ΩS. It contains PS as a submodule (since PS has ΩS as socle). Assume that Q is of length at least 3. Take a submodule I of Q of length 2 which is different from PS and let $V = PS + I$, this is a submodule of Q of length 3. Thus, there are the following inclusion maps u_1, u_2, v_1, v_2:

$$\begin{array}{ccc} \Omega S & \xrightarrow{u_1} & PS \\ {\scriptstyle v_1}\big\downarrow & & \big\downarrow{\scriptstyle u_2} \\ I & \xrightarrow{v_2} & V \end{array}$$

The projective cover $p\colon PI \to I$ has as restriction a surjective map $p'\colon \operatorname{rad} PI \to \Omega S$. But $\operatorname{rad} PI$ is semisimple, thus p' is a split epimorphism, thus we obtain a map $w\colon \Omega S \to PI$ such that $pw = v_1$. We consider the exact sequence induced from the sequence $0 \to \Omega S \to PS \to S \to 0$ by the map w:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega S & \xrightarrow{u_1} & PS & \xrightarrow{e_1} & S & \longrightarrow & 0 \\ & & {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle w'} & & \big\| & & \\ 0 & \longrightarrow & PI & \xrightarrow{u_1'} & N & \xrightarrow{e_1'} & S & \longrightarrow & 0 \end{array}$$

Here, N is the pushout of the two maps u_1 and w. Since we know that $u_2u_1 = v_2v_1 = v_2pw$, there is a map $f\colon N \to V$ such that $fu_1' = v_2p$ and $fw' = u_2$. Since the map $[\, v_2p \quad u_2 \,]: PI \oplus PS \to V$ is surjective, also f is surjective.

But recall that we assume that $\operatorname{Ext}^1(S,\Lambda) = 0$, thus $\operatorname{Ext}^1(S,PI) = 0$. This means that the lower exact sequence splits and therefore the socle of $N = PI \oplus S$ is a maximal submodule of N (since I is a local module, also PI is a local module). Now f maps the socle of N into the socle of V, thus it maps a maximal submodule of N into a simple submodule of V. This implies that the image of f has length at most 2, thus f cannot be surjective. This contradiction shows that Q has to be of length 2, thus $Q = PS$ and therefore PS is injective.

Lemma 3. *If S is a non-projective simple module with $\operatorname{Ext}^i(S,\Lambda) = 0$ for $1 \le i \le d$, then the modules $S_i = \Omega_i S$ with $0 \le i \le d$ are simple and not projective, and the modules $P(S_i)$ are injective for $0 \le i < d$.*

The proof is by induction. If $d \ge 2$, we know by induction that the modules S_i with $0 \le i \le d-1$ are simple and not projective, and that the modules $P(S_i)$ are injective for

$0 \leq i < d-1$. But $\mathrm{Ext}^1(\Omega_{d-1}S, \Lambda) \simeq \mathrm{Ext}^d(S, \Lambda) = 0$, thus Lemma 2 asserts that also S_d is simple and not projective and that $P(S_{d-1})$ is injective.

Lemma 4. *Let $S_0, S_1, \dots, S_b$ be simple modules with $S_i = \Omega_i(S_0)$ for $1 \leq i \leq b$. Assume that there is an integer $0 \leq a < b$ such that the modules S_i with $a \leq i < b$ are pairwise non-isomorphic, whereas S_b is isomorphic to S_a. In addition, we asssume that the modules $P(S_i)$ for $a \leq i < b$ are injective. Then $S_a, \dots, S_{b-1}$ is the list of all the simple modules and Λ is self-injective.*

Proof: Let $\mathcal{S}$ be the subcategory of all modules with composition factors of the form S_i, where $a \leq i < b$. We claim that this subcategory is closed under projective covers and injective envelopes. Indeed, the projective cover of S_i for $a \leq i < b$ has the composition factors S_i and S_{i+1} (and $S_b = S_a$), thus is in $\mathcal{S}$. Similarly, the injective envelope for S_i with $a < i < b$ is $Q(S_i) = P(S_{i-1})$, thus it has the composition factors S_{i-1} and S_i, and $Q(S_a) = Q(S_b) = P(S_{b-1})$ has the composition factors S_{b-1} and S_a. Since we assume that Λ is connected, we know that the only non-trivial subcategory closed under composition factors, extensions, projective covers and injective envelopes is the module category itself. This shows that $S_a, \dots, S_{b-1}$ are all the simple modules. Since the projective cover of any simple module is injective, Λ is self-injective.

Theorem 1. *Let Λ be a connected left artinian ring with radical square zero. Assume that Λ is not self-injective. If S is a non-projective simple module such that $\mathrm{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, then the modules $S_i = \Omega_i S$ with $0 \leq i \leq d$ are pairwise non-isomorphic simple and non-projective modules and the modules $P(S_i)$ are injective for $0 \leq i < d$.*

Proof. According to Lemma 3, the modules S_i (with $0 \leq i \leq d$) are simple and non-projective, and the modules $P(S_i)$ are injective for $0 \leq i < d$. If at least two of the modules $S_0, \dots, S_d$ are isomorphic, then Lemma 4 asserts that Λ is self-injective, but this we have excluded.

Theorem 2. *Let Λ be a connected left artinian ring with radical square zero and with n simple modules. The following conditions are equivalent:*

(i) *Λ is self-injective, but not a simple ring.*
(ii) *There exists a non-projective module M with $\mathrm{Ext}^i(M, \Lambda) = 0$ for $1 \leq i \leq n+1$.*
(iii) *There exists a non-projective simple module S with $\mathrm{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq n$.*

Proof. First, assume that Λ is self-injective, but not simple. Since Λ is not semisimple, there is a non-projective module M. Since Λ is self-injective, $\mathrm{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$. This shows the implication (i) $\implies$ (ii). The implication (ii) $\implies$ (iii) follows from Lemma 1. Finally, for the implication (iii) $\implies$ (ii) we use Theorem 1. Namely, if Λ is not self-injective, then Theorem 1 asserts that the simple modules $S_i = \Omega_i S$ with $0 \leq i \leq n$ are pairwise non-isomorphic. However, these are $n+1$ simple modules, and we assume that the number of isomorphism classes of simple modules is n. This completes the proof of Theorem 2.

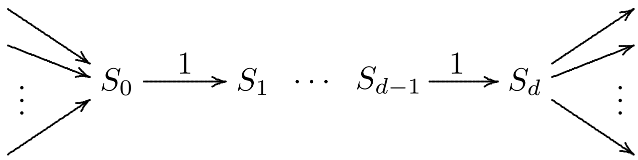
Note that the implication (ii) $\implies$ (i) in Theorem 2 asserts in particular that either Λ is self-injective or else that any CM module is projective, as shown by Chen [C]. Let us recall that a module M is said to be a CM module provided $\mathrm{Ext}^i(M, \Lambda) = 0$ and

$\operatorname{Ext}^i(\operatorname{Tr} M, \Lambda) = 0$, for all $i \geq 1$ (here Tr denotes the transpose of the module); these modules are also called Gorenstein-projective modules, or totally reflexive modules, or modules of G-dimension equal to 0. Note that in general there do exist modules M with $\operatorname{Ext}^i(M, \Lambda) = 0$ for all $i \geq 1$ which are not CM modules, see [JS].

We also draw the attention to the generalized Nakayama conjecture formulated by Auslander-Reiten [AR]. It asserts that for any artin algebra Λ and any simple Λ-module S there should exist an integer $i \geq 0$ such that $\operatorname{Ext}^i(S, \Lambda) \neq 0$. It is known that this conjecture holds true for algebras with radical square zero. The implication (iii) $\Longrightarrow$ (i) of Theorem 2 provides an effective bound: *If n is the number of simple Λ-modules, and S is simple, then* $\operatorname{Ext}^i(S, \Lambda) \neq 0$ *for some* $0 \leq i \leq n$.. Namely, in case S is projective or Λ is self-injective, then $\operatorname{Ext}^0(S, \Lambda) \neq 0$. Now assume that S is simple and not projective and that Λ is not self-injective. Then there must exist some integer $1 \leq i \leq n$ with $\operatorname{Ext}^i(S, \Lambda) \neq 0$, since otherwise the condition (iii) would be satisfied and therefore condition (i).

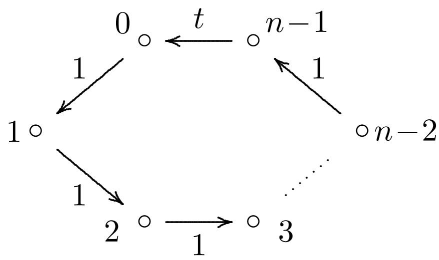
Theorem 1 may be interpreted as a statement concerning the Ext-quiver of Λ. Recall that the Ext-*quiver* $\Gamma(R)$ of a left artinian ring R has as vertices the (isomorphism classes of the) simple R-modules, and if S, T are simple R-modules, there is an arrow $T \to S$ provided $\operatorname{Ext}^1(T, S) \neq 0$, thus provided that there exists an indecomposable R-module M of length 2 with socle S and top T. We may add to the arrow $\alpha\colon T \to S$ the number $l(\alpha) = ab$, where a is the length of $\operatorname{soc} PT$ and b is the length of QS/soc (note that b may be infinite). The properties of $\Gamma(R)$ which are relevant for this note are the following: the vertex S is a sink if and only if S is projective; the vertex S is a source if and only if S is injective; finally, if R is a radical square zero ring and S, T are simple R-modules then $PT = QS$ if and only if there is an arrow $\alpha\colon T \to S$ with $l(\alpha) = 1$ and this is the only arrow starting at T and the only arrow ending in S.

Theorem 1 assert the following: Let Λ be a connected left artinian ring with radical square zero. Assume that Λ is not self-injective. Let S be a non-projective simple module such that $\operatorname{Ext}^i(S, \Lambda) = 0$ for $1 \leq i \leq d$, and let $S_i = \Omega_i S$ with $0 \leq i \leq d$. Then the local structure of $\Gamma(\Lambda)$ is as follows:



such that there is at least one arrow starting in S_d (but may-be no arrow ending in S_0). To be precise: the picture is supposed to show all the arrows starting or ending in the vertices $S_0, \dots, S_d$ (and to assert that the vertices $S_0, \dots, S_d$ are pairwise different).

Let us introduce the quivers $\Delta(n, t)$, where n, t are positive integers. The quiver $\Delta(n, t)$ has n vertices and also n arrows, namely the vertices labeled $0, 1, \dots, n-1$, and arrows $i \to i+1$ for $0 \leq i \leq n-1$ (modulo n) (thus, we deal with an oriented cycle); in addition, let $l(\alpha) = t$ for the arrow $\alpha\colon n{-}1 \to 0$ and let $l(\beta) = 1$ for the remaining arrows β:



Note that the Ext-quiver of a connected self-injective left artinian ring with radical square zero and n vertices is just $\Delta(n, 1)$. Our further interest lies in the cases $t > 1$.

Theorem 3. *Let Λ be a connected left artinian ring with radical square zero and with n simple modules.*

(a) *If there exists a non-projective simple modules S with $\operatorname{Ext}^i(S, \Lambda) = 0$ for $1 \le i \le n{-}1$, or if there exists a non-projective module M with $\operatorname{Ext}^i(M, \Lambda) = 0$ for $1 \le i \le n$, then $\Gamma(\Lambda)$ is of the form $\Delta(n, t)$ with $t > 1$.*

(b) *Conversely, if $\Gamma(\Lambda) = \Delta(n, t)$ and $t > 1$, then there exists a unique simple module S with $\operatorname{Ext}^i(S, \Lambda) = 0$ for $1 \le i \le n - 1$, namely the module $S = S(0)$ (and it satisfies $\operatorname{Ext}^n(S, \Lambda) \ne 0$).*

(c) *If $\Gamma(\Lambda) = \Delta(n, t)$ and $t > 1$, and if we assume in addition that Λ is an artin algebra, then there exists a unique indecomposable module M with $\operatorname{Ext}^i(M, \Lambda) = 0$ for $1 \le i \le n$, namely $M = \operatorname{Tr} D(S(0))$ (and it satisfies $\operatorname{Ext}^{n+1}(M, \Lambda) \ne 0$).*

Here, for Λ an artin algebra, D denotes the k-duality, where k is the center of Λ (thus $D = \operatorname{Hom}_k(-, E)$, where E is a minimal injective cogenerator in the category of k-modules); thus $D\operatorname{Tr}$ is the Auslander-Reiten translation and $\operatorname{Tr} D$ the reverse.

Proof of Theorem 3. Part (a) is a direct consequence of Theorem 1, using the interpretation in terms of the Ext-quiver as outlined above. Note that we must have $t > 1$, since otherwise Λ would be self-injective.

(b) We assume that $\Gamma(\Lambda) = \Delta(n, t)$ with $t > 1$. For $0 \le i < n$, let $S(i)$ be the simple module corresponding to the vertex i, let $P(i)$ be its projective cover, $I(i)$ its injective envelope. We see from the quiver that all the projective modules $P(i)$ with $0 \le i \le n{-}2$ are injective, thus $\operatorname{Ext}^j(-, \Lambda) = \operatorname{Ext}^j(-, P(n{-}1))$ for all $j \ge 1$. In addition, the quiver shows that $\Omega S(i) = S(i{+}1)$ for $0 \le i \le n{-}2$. Finally, we have $\Omega S(n{-}1) = S(0)^a$ for some positive integer a dividing t and the injective envelope of $P(n{-}1)$ yields an exact sequence

$$(*) \qquad 0 \to P(n{-}1) \to I(P(n{-}1)) \to S(n{-}1)^{t-1} \to 0$$

(namely, $I(P(n{-}1)) = I(\operatorname{soc} P(n{-}1)) = I(S(0)^a) = I(S(0))^a$ and $I(S(0))/\operatorname{soc}$ is the direct sum of b copies of $S(n{-}1)$, where $ab = t$; thus the cokernel of the inclusion map $P(n{-}1) \to I(P(n{-}1))$ consists of $t{-}1$ copies of $S(n{-}1)$).

Since $t > 1$, the exact sequence $(*)$ shows that $\operatorname{Ext}^1(S(n{-}1), P(n{-}1)) \ne 0$. It also implies that $\operatorname{Ext}^1(S(i), P(n{-}1)) = 0$ for $0 \le i \le n{-}2$, and therefore that

$$\begin{aligned}\operatorname{Ext}^i(S(0), P(n{-}1)) &= \operatorname{Ext}^1(\Omega_{i-1} S(0), P(n{-}1))\\ &= \operatorname{Ext}^1(S(i{-}1), P(n{-}1))\\ &= 0\end{aligned}$$

for $1 \le i \le n-1$.

Since $\Omega_{n-i-1}S(i) = S(n-1)$ for $0 \le i \le n-1$, we see that

$$\begin{aligned}\mathrm{Ext}^{n-i}(S(i), P(n-1)) &= \mathrm{Ext}^1(\Omega_{n-i-1}S(i), P(n-1)) \\ &= \mathrm{Ext}^1(S(n-1), P(n-1)) \\ &\neq 0\end{aligned}$$

for $0 \le i \le n-1$. Thus, on the one hand, we have $\mathrm{Ext}^n(S(0), \Lambda) \neq 0$, this concludes the proof that $S(0)$ has the required properties. On the other hand, we also see that $S = S(0)$ is the only simple module with $\mathrm{Ext}^i(S, \Lambda) = 0$ for $1 \le i \le n-1$. This completes the proof of (b).

(c) Assume now in addition that Λ is an artin algebra. As usual, we denote the Auslander-Reiten translation $D\,\mathrm{Tr}$ by τ. Let M be a non-projective indecomposable module with $\mathrm{Ext}^i(M, \Lambda) = 0$ for $1 \le i \le n$. The shape of $\Gamma(\Lambda)$ shows that $\Omega M = S^c$ for some simple module S (and we have $c \ge 1$), also it shows that no simple module is projective. Now $\mathrm{Ext}^i(S, \Lambda) = 0$ for $1 \le i < n$, thus according to (b) we must have $S = S(0)$. It follows that PM has to be a direct sum of copies of $P(n-1)$, say of d copies. Thus a minimal projective presentation of M is of the form

$$P(0)^c \to P(n-1)^d \to M \to 0,$$

and therefore a minimal injective copresentation of τM is of the form

$$0 \to \tau M \to I(0)^c \to I(n-1)^d.$$

In particular, $\mathrm{soc}\,\tau M = S(0)^c$ and $(\tau M)/\mathrm{soc}$ is a direct sum of copies of $S(n-1)$.

Assume that $\tau M \neq S(0)$, thus it has at least one composition factor of the form $S(n–1)$ and therefore there exists a non-zero map $f\colon P(n–1) \to \tau M$. Since τM is indecomposable and not injective, any map from an injective module to τM maps into the socle of τM. But the image of f is not contained in the socle of τM, therefore f cannot be factored through an injective module. It follows that

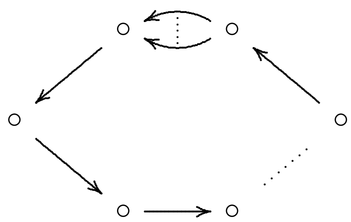
$$\mathrm{Ext}^1(M, P(n-1)) \simeq D\overline{\mathrm{Hom}}(P(n-1), \tau M) \neq 0,$$

which contradicts the assumption that $\mathrm{Ext}^1(M, \Lambda) = 0$. This shows that $\tau M = S(0)$ and therefore $M = \mathrm{Tr}\,DS(0)$.

Of course, conversely we see that $M = \mathrm{Tr}\,DS(0)$ satisfies $\mathrm{Ext}^i(M, P(n-1)) = 0$ for $1 \le i \le n$, and $\mathrm{Ext}^{n+1}(M, P(n-1)) \neq 0$.

Remarks. (1) The module $M = \mathrm{Tr}\,DS(0)$ considered in (c) has length $t^2 + t - 1$, thus the number t (and therefore $\Delta(n,t)$) is determined by M.

(2) If Λ is an artin algebra with Ext-quiver $\Delta(n,t)$, the number t has to be the square of an integer, say $t = m^2$. A typical example of such an artin algebra is the path algebra of the following quiver



with altogether $n + m - 1$ arrows, modulo the ideal generated by all paths of length 2. Of course, if Λ is a finite-dimensional k-algebra with radical square zero and Ext-quiver $\Delta(n, m^2)$, and k is an algebraically closed field, then Λ is Morita-equivalent to such an algebra.

Also the following artin algebras with radical square zero and Ext-quiver $\Delta(1, m^2)$ may be of interest: the factor rings of the polynomial ring $\mathbb{Z}[T_1, \dots, T_{m-1}]$ modulo the square of the ideal generated by some prime number p and the variables $T_1, \dots, T_{m-1}$.

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