The Self-Injective Cluster-Tilted Algebras

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Abstract. We are going to determine all the self-injective cluster-tilted algebras. All are of finite representation type and special biserial. There are two different classes. The first class are the self-injective serial (or Nakayama) algebras with $n \geq 3$ simple modules and Loewy length $n-1$. The second class of algebras has an even number $2m$ of simple modules; m indecomposable projective modules have length 3, the remaining m have length $m+1$.

Let k be a field. The algebras we will deal with are finite-dimensional associative k-algebras with 1. Given such an algebra A, the modules considered will usually be (finite-dimensional) left A-modules. The k-duality will be denoted by $D = \mathrm{Hom}(-, k)$.

The class of cluster-tilted algebras has been introduced by Buan, Marsh and Reiten [BMR], at least in case k is algebraically closed. The cluster-tilted algebras are the endomorphism rings of the cluster-tilting objects in a cluster category. According to Zhu [Z] and Assem, Brüstle and Schiffler [ABS], they also can be defined as the semi-direct extensions of B by the B-B-bimodule $\mathrm{Ext}^2_B(D(B_B), {}_BB)$, where B is a tilted algebra. The aim of this note is to single out those cluster-tilted algebras which are self-injective.

The occurence of self-injective algebras in the context of tilting hereditary algebras may on first sight come as a surprise, but it is not! Already the smallest example of a cluster-tilted algebra which is not hereditary is self-injective: it is a serial algebra with 3 simple modules and radical square zero, the Dynkin type of the corresponding cluster category is A_3. If one wants to see other self-injective cluster-tilted algebras, one does not find any further example when looking at the Dynkin types A_n with $n \geq 4$. As we will show, one finds such examples by dealing with the Dynkin types D_n (and A_3 fits into this scheme, since $D_3 = A_3$). It may be reasonable to stress here that cluster-tilted algebras have quite different properties than tilted algebras: in particular, they are Gorenstein algebras of Gorenstein dimension at most 1, as Keller and Reiten [KR] have shown. Note that the global dimension of a Gorenstein algebra of Gorenstein dimension at most 1 is 0, 1 or infinite (whereas the proper tilted algebras have global dimension 2).

It is rather easy to see that self-injective cluster-tilted algebras have to be of finite representation type and we will see that all are obtained from special biserial k-algebras by scalar extension. There are two different classes. The first class are serial (or Nakayama) algebras with $n \geq 3$ simple modules such that all indecomposable projective modules are of length $n-1$. For the second class of algebras, there are two kinds of indecomposable projective modules: half of them are serial of length 3, for the other half, the heart $\mathrm{rad}\, P/\mathrm{soc}\, P$ of any indecomposable projective module P is the direct sum of a serial module and a simple module. Here is the precise description:

Theorem. *A finite-dimensional basic connected k-algebra A is a non-simple self-injective cluster-tilted algebra if and only if $A = A' \otimes_k D$ where D is a division k-algebra and A' is either*

(i) *the path algebra of a cyclic quiver with* $n \geq 3$ *vertices modulo the ideal generated by the paths of length* $n-1$, *or else*
(ii) *for* $m \geq 3$ *the path algebra of a quiver with* $n = 2m$ *vertices labeled* $1, 2, \ldots, m$ *and* $1', 2', \ldots, m'$, *with arrows* $\alpha_i : i \to i{+}1$, $\beta_i : i \to i'$, $\beta'_i : i' \to m{+}i{-}1$ *and relations*

$$\beta_{i+1}\alpha_i, \quad \alpha_{m+i-1}\beta'_i, \quad \beta'_i\beta_i = \alpha_{m+i-1}\cdots\alpha_{i+1}\alpha_i,$$

with $1 \leq i \leq m$ *(and calculations modulo* m*).*

Let us stress that in all cases the Dynkin type of the corresponding cluster category is D_n with $n \geq 3$ (where $D_3 = A_3$).

1. Criterion.

Let H be a connected hereditary finite-dimensional k-algebra. Let n be the number of isomorphism classes of simple H-modules. We denote by $\mathcal{C}(H)$ the corresponding cluster category; by definition, this is the orbit category of the derived category $D^b(\operatorname{mod} H)$ with respect to the functor $F = \tau_d^{-1}[1]$, where $[1]$ is the shift functor and τ_d is the Auslander-Reiten translation functor in the derived category. We denote by τ_c the Auslander-Reiten translation functor in the cluster category; both τ_d and τ_c are invertible. The Auslander-Reiten translation in $\operatorname{mod} H$ itself will be denoted by τ_H.

Let T be a multiplicity-free tilting H-module with endomorphism ring $B = B(T)$. We may consider T as an object in the cluster category, and we denote by $A = A(T)$ its endomorphism ring in the cluster category. The algebra B is a tilted algebra and A is the semi-direct extensions of B by $\operatorname{Ext}^2_B(D(B_B), {}_BB)$. Note that all cluster-tilted algebras A are obtained in this way, see [Z] and [ABS] (but also [R2]).

The question which we consider here is: when is $A(T)$ self-injective?

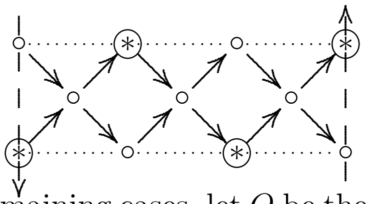
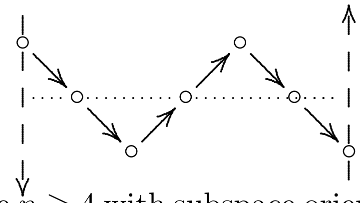
Lemma. *The cluster-tilted algebra* $A(T)$ *is self-injective if and only if* $\tau_c^2 T$ *is isomorphic to* T.

Proof: Write $T = \bigoplus_{i=1}^n T_i$ with T_i indecomposable. The category $\operatorname{mod} A$ is the factor category of $\mathcal{C}(H)$ modulo the ideal generated by the objects $\tau_c T_i$; in this factor category, the objects T_i are the indecomposable projective ones, the objects $\tau_c^2 T_i$ the indecomposable injective ones. In order for A to be self-injective we just need that any indecomposable injective A-module is projective, but this means that any $\tau_c^2 T_i$ is isomorphic to some T_j, so that $\tau_c^2 T$ is isomorphic to T.

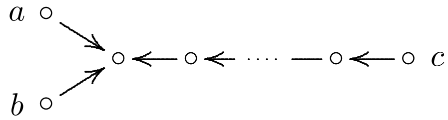
2. Some tilting modules, tilted algebras and the corresponding cluster-tilted algebras.

First, let us consider path algebras kQ with Q of type D_n, $n \geq 3$, and exhibit tilting modules T such that the corresponding cluster-tilted algebra $A(T)$ is self-injective.

2.1. Case $D_3 = A_3$. Let Q be the linearly oriented quiver of type A_3, and $H = kQ$ its path algebra. Let T be the unique multiplicity-free tilting H-module such that its endomorphism ring is not hereditary: T is the direct sum of the indecomposable projective-injective module and the two simple modules which are projective or injective. It is easy to see (and a well-known example in the literature) that the algebra $A(T)$ is serial with 3 simple modules and the Loewy length of any indecomposable projective module is 2. Here are the Auslander-Reiten quivers of $\mathcal{C}(H)$ and of $\operatorname{mod} A$. Note that in both cases the left hand boundary has to be identified with the right hand side using a twist in order to form a Möbius strip. The summands T_i are marked by a star $*$.

$\mathcal{C}(H)$  $\operatorname{mod} A$ 

For the remaining cases, let Q be the D_n-quiver where $n \geq 4$ with subspace orientation (this means that the branching vertex is the unique sink). Let a, b be two sources which are neighbors of the sink, and c the remaining source:



2.2. The case D_n with $n \geq 4$, using two short arms. If $n = 2m$ is even, let

$$T = \Big(\bigoplus_{i=0}^{m-1} \tau^{-2i} P(a)\Big) \oplus \Big(\bigoplus_{i=0}^{m-1} \tau^{-2i-1} P(b)\Big),$$

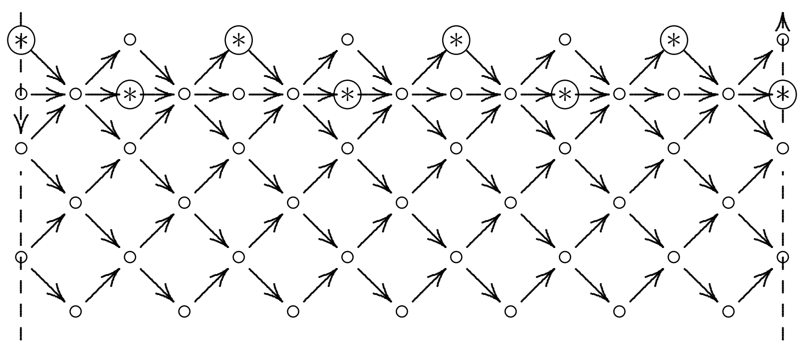
whereas if $n = 2m+1$ is odd, let

$$T = \Big(\bigoplus_{i=0}^{m} \tau^{-2i} P(a)\Big) \oplus \Big(\bigoplus_{i=0}^{m-1} \tau^{-2i-1} P(b)\Big).$$

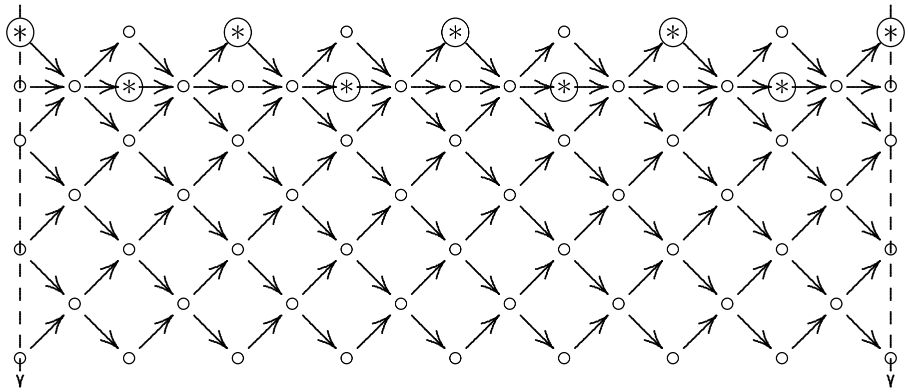
It is easy to see that both $B(T)$ and $A(T)$ are serial algebras and that all the indecomposable projective $A(T)$-modules have Loewy length $n-1$. Thus, the quiver of $A(T)$ is a cycle with n vertices, and all the paths of length $n-1$ are zero relations. If we label all the arrows by α, then the relations for $A(T)$ can be written just in the form $\alpha^{n-1} = 0$. It is not difficult to see that these relations generate the ideal of relations.

Note that the A_3 case may be considered as part of this sequence of serial algebras. For a better intuition, we are going to exhibit the cases $n = 7$ and $n = 8$.

Here is the case $n = 7$. In this case, the upper part, given by the two short arms, provides a Möbius strip. In general, the lower part given by the long arm is always a cylinder, whereas for n odd, the short arm part of the Auslander-Reiten quiver yields a Möbius strip.



And here is the case $n = 8$. Here, the left hand boundary has to be identified with the right hand side without any twist. In general, for n even, the Auslander-Reiten quiver of $\mathcal{C}(H)$ is just $\mathbb{Z}D_n/\tau^n$.



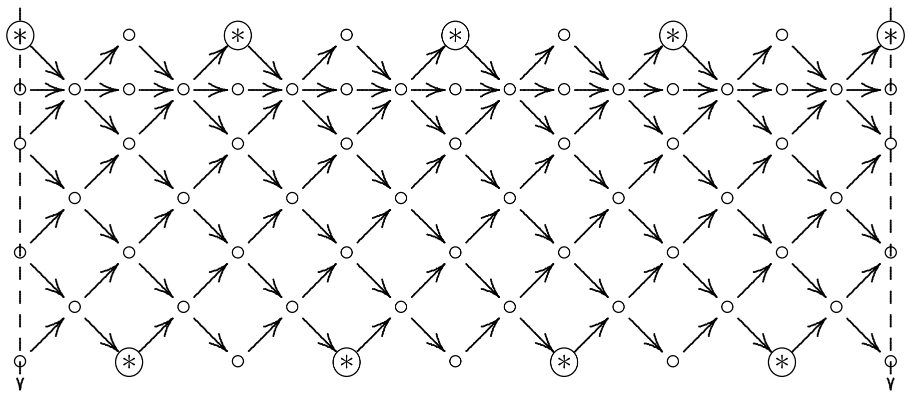
The cluster-tilted algebra $A = A(T)$ obtained from a hereditary algebra of type D_n by dealing with a tilting module which uses two short arms, is a serial algebra. The only difference between the cases n even or odd concerns the Nakayama permutation: In the even case, it has two orbits, in the odd case only one.

2.3. The case D_{2m} with $m \geq 3$, using the long arm and one short arm. Let

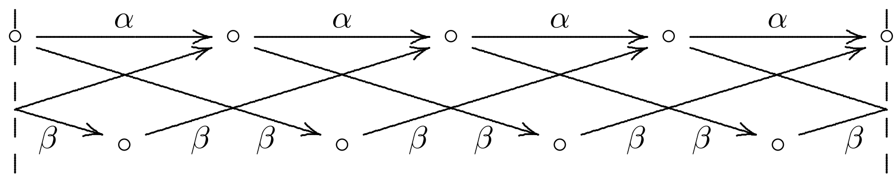
$$T = \Big(\bigoplus_{i=0}^{m-1} \tau^{-2i} P(a)\Big) \oplus \Big(\bigoplus_{i=0}^{m-1} \tau^{-2i-1} P(c)\Big),$$

this is a tilting module and the algebra $A(T)$ can be presented by quiver and relation as described in (ii) of Theorem. (This can be verified directly, but the calculation can be shortened by using the multiplicative basis theorem [BGRS] for representation finite k-algebras A with $A/\operatorname{rad} A$ being a product of copies of k).

Here we present the case D_{2m} with $m = 4$.



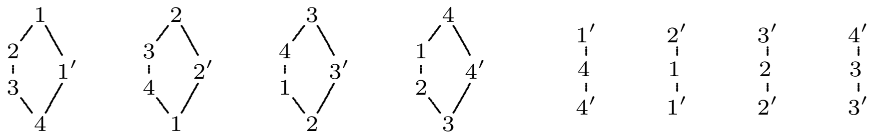
The quiver of A has the following shape (again, the right hand side has to be identified with the left hand side):



with relations

$$\alpha\beta = \beta\alpha = 0, \qquad \alpha^{m-1} = \beta^2.$$

The shape of the indecomposable projective A-modules is as follows:



The first indecomposable projective modules are those corresponding to the modules in the short arm, the last ones are those corresponding to the modules in the long arm. For arbitrary m, the indecomposable projective modules P corresponding to the direct summands of T in a short arm are of length $m+1$ and the heart $\operatorname{rad} P/\operatorname{soc} P$ is the direct sum of a serial module of length $m-2$ and a simple module, whereas those indecomposable projective modules P corresponding to vertices in the long arm are of length 3 (and of course serial).

In this way, we have obtained certain self-injective cluster-tilted k-algebras A such that $A/\operatorname{rad} A$ is a product of copies of k. If D is a division k-algebra, then the tensor algebra $A \otimes_k D$ is still self-injective, but also a cluster-tilted algebra.

In general, let H be a finite-dimensional hereditary k-algebra with quiver Q being a tree and such that the valuation of all edges of the quiver is $(1,1)$ (this means that for any pair of simple H-modules S, T, the bimodule $\operatorname{Ext}^1_H(T,S)$ is of length at most 1 when considered as a left $\operatorname{End}(T)$-module as well as when considered as a right $\operatorname{End}(S)$-module). If D is the endomorphism ring of a simple H-module, then it follows that $H = H' \otimes_k D$ where $H' = kQ$, the path algebra of Q, thus H is obtained from H' by scalar extension. According to [R1], also all the tilting H-modules T are obtained from tilting H'-modules T' by scalar extension: $T = T' \otimes_k D$. It follows, that the corresponding tilted algebras, as well as the cluster-tilted algebras are obtained by scalar extension.

3. Self-injective cluster-tilted algebras are representation-finite.

We return to the general situation discussed in section 1. Thus, H is a connected hereditary finite-dimensional k-algebra with n simple H-modules and $\mathcal{C}(H)$ is the corresponding cluster category. Let $T = \bigoplus_i T_i$ be a multiplicity-free tilting H-module with indecomposable direct summands T_i.

Assume that T and $\tau_c^2 T$ are isomorphic. Now T has only finitely many indecomposable direct summands (namely n); this means that any object T_i has to be τ_c-periodic. If H is wild, then $\mathcal{C}(H)$ has no τ_c-periodic objects at all. Thus H cannot be wild. If H is tame, then the τ_c-periodic objects in $\mathcal{C}(H)$ are actually τ_H-periodic modules. But it is well-known that the number of τ_H-periodic indecomposable direct summands of a tilting H-module is at most $n-2$. (A stable tube of rank r can have only $r-1$ indecomposable direct summands which are pairwise Ext-orthogonal, and the sum of these numbers $r-1$ over all the tubes is $n-2$, see [DR].) This shows that any self-injective cluster-tilted algebra A has to be representation-finite.

Note that *the τ_c-orbit of any summand T_i has to have an even number of elements, and then half of these elements are direct summands of T.* Namely, if the τ_c-orbit of T_i would have odd cardinality, then all the objects in this τ_c-orbit would be direct summands of T, but neighboring elements of a τ_c-orbit have non-trivial extensions.

4. The various Dynkin types.

We have to consider now the various cases $A_n, B_n, \ldots, F_4, G_2$ in detail. Let Δ be the quiver of H (we consider it as a valued quiver (see [DR]), if we deal with one of the cases B_n, C_n, F_4 and G_2). Let a be a vertex of Δ. Denote by $P(a)$ the indecomposable projective H-module corresponding to the vertex a. We are looking for natural numbers t such that $\operatorname{Hom}_H(P(a), \tau_H^{-t}P(a)) \neq 0$. If $\tau_H^{-t}P(a)$ is not injective, then we see that $\operatorname{Ext}^1_H(\tau_H^{-t-1}P(a), P(a)) \neq 0$. Now, if t is odd, then $-t-1$ is even: We conclude that in this case neither $P(a)$ nor any other element of the τ_c-orbit of $P(a)$ can be a direct summand of T. In order to decide whether $\operatorname{Hom}_H(P(a), \tau_H^{-t}P(a))$ is zero or not, one just has to calculate the hammock function starting at $P(a)$, see [G] or [RV].

Let us assume now that T has an indecomposable direct summand T_i which is in the τ_H-orbit of $P(a)$.

(1) *The vertex a is a boundary vertex of Δ.* Assume to the contrary that a is an interior vertex. We have $\operatorname{Hom}_H(P(a), \tau^{-1}P(a)) \neq 0$. If Δ is different from A_3, then $\tau^{-1}P(a)$ is not injective, thus we get a contradiction. If Δ is of type A_3, then the τ_c-orbit of a has length 3, thus is of odd length, again a contradiction.

(2) *The edge between a and its (unique) neighbor has valuation* $(1,1)$. Here again we see that otherwise $\operatorname{Hom}_H(P(a), \tau^{-1}P(a)) \neq 0$, and in this case $\tau^{-1}P(a)$ cannot be injective. This immediately excludes G_2. Also, together with (1) it excludes B_n and C_n: Namely, all the indecomposable summands of T would have to belong to a single τ_H-orbit — but since the simple H-modules do have two different kinds of endomorphism rings, the same is true for any tilting H-module ([R1], alternatively, we also could argue that the orbit in question does not have enough elements).

Assume that Δ is of type D_n or E_n, and that $a = a_1, a_2, \ldots, a_p$ is the minimal path from a to the branching vertex a_p. If necessary, we write $p = p(a)$.

(3) *The number p is even.* (We have $\operatorname{Hom}_H(P(a), \tau^{-p}P(a)) \neq 0$, and $\tau^{-p}P(a)$ is not injective.)

(4) *If $\Delta = E_n$, then $p > 2$.* (Again, $\operatorname{Hom}_H(P(a), \tau^{-3}P(a)) \neq 0$, and $\tau^{-3}P(a)$ is not injective.)

The assertions (1), (3) and (4) exclude immediately the cases E_6, E_8, and they show that in case E_7 all the indecomposable direct summands of T have to belong to the τ_c-orbit which contains the module $P(a)$ with $p(a) = 4$. But any τ_c-orbit of $\mathcal{C}(H)$ with H of type E_7 is of length 10, thus only 5 indecomposable direct summands of T can belong to this orbit. Thus also E_7 is excluded. Similarly, we see that in case H is of type D_n, and $a(p) > 2$, then $n = a(p) + 2$ has to be even.

In order to exclude the case F_4, we only have to notice that $\operatorname{Hom}_H(P(a), \tau^{-3}P(a)) \neq 0$ for any of the two boundary vertices.

Consider now the cases A_n. The two τ_H-orbits of H yield a single τ_c-orbit containing $n+3$ elements. These are the only possible summands of T, thus we must have $\frac{1}{2}(n+3) = n$, therefore $n = 3$.

Besides A_3 also the cases D_n remain. Assume we are in case D_n and $p(a) > 2$. The τ_c-orbit of $P(a)$ contains precisely n elements, this shows that n has to be even. For $n \geq 5$, there are two τ_H-orbits a with $a(p) = 2$; for $n = 4$, there are three such τ_H-orbits. In case n is odd, the two τ_H-orbits with $a(p) = 2$ combine to form part of a single τ_c-orbit with $2n$ elements. In case n is even, the τ_H-orbits with $a(p) = 2$ are contained in two (or, for $n = 4$, three) separate τ_c-orbits, each having n elements. All these τ_c-orbits actually occur, as we have shown in section 2.

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