Gorenstein-projective and semi-Gorenstein-projective modules

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Abstract: Let A be an artin algebra. An A-module M will be said to be semi-Gorenstein-projective provided that $\mathrm{Ext}^i(M, A) = 0$ for all $i \geq 1$. All Gorenstein-projective modules are semi-Gorenstein-projective and only few and quite complicated examples of semi-Gorenstein-projective modules which are not Gorenstein-projective are known. The aim of the paper is to provide conditions on A such that all semi-Gorenstein-projective modules are Gorenstein-projective (we call such an algebra weakly Gorenstein). In particular, we show that in case there are only finitely many isomorphism classes of indecomposable modules which are both semi-Gorenstein-projective and torsionless, then A is weakly Gorenstein. On the other hand, we exhibit a 6-dimensional algebra Λ with a semi-Gorenstein-projective module M which is not torsionless (thus not Gorenstein-projective). Actually, also the Λ-dual module M^* is semi-Gorenstein-projective module and all the syzygy-modules of M and M^* are 3-dimensional, thus the example can be visualized quite easily.

1. Introduction.

1.1. Notations and definitions. Let A be an artin algebra. All modules will be finitely generated. Usually, the modules we are starting with will be left modules, but some constructions then yield right modules. Let $\mathrm{mod}\, A$ be the category of all finitely generated left A-modules and $\mathrm{proj}(A)$ the full subcategory of all projective modules.

If M is a module, let PM be a projective cover of M, and ΩM the kernel of the canonical map $PM \to M$. The modules $\Omega^t M$ with $t \geq 0$ are called the *syzygy* modules of M. A module M is said to be Ω-*periodic* provided that there is some $t \geq 1$ with $\Omega^t M = M$.

The right module $M^* = \mathrm{Hom}(M, A)$ is called the A-*dual* of M. Let $\phi_M : M \to M^{**}$ be defined by $\phi_M(m)(f) = f(m)$ for $m \in M,\ f \in M^*$. A module M is said to be *torsionless* provided that M is a submodule of a projective module, or, equivalently, provided that ϕ_M is injective. A module M is called *reflexive* provided that ϕ_M is bijective.

Let $\mathrm{Tr}\, M$ be the cokernel of f^*, where f is a minimal projective presentation of M (this is the canonical map $P(\Omega M) \to PM$). Note also here, that $\mathrm{Tr}\, M$ is a right A-module, called the *transpose of* M.

Recall that a map $f\colon M \to M'$ is said to be *left minimal* provided that any map $h\colon M' \to M'$ with $hf = f$ is an automorphism [AR]. A left $\mathrm{proj}(A)$-approximation will be called *minimal* provided that it is left minimal. We denote by $\mho M$ the cokernel of a minimal left $\mathrm{proj}(A)$-approximation of M.

A complete projective resolution is a (double infinite) exact sequence

$$P^\bullet : \qquad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

of projective left A-modules, such that $\operatorname{Hom}_A(P^\bullet, A)$ is again exact. A module M is *Gorenstein-projective*, if there is a complete projective resolution $P^\bullet$ with M isomorphic to the image of d^0.

A module M is said to be *semi-Gorenstein-projective* provided that $\operatorname{Ext}^i(M, A) = 0$ for all $i \geq 1$. Note that all Gorenstein-projective modules are semi-Gorenstein-projective. If M is semi-Gorenstein-projective, then also ΩM is semi-Gorenstein-projective. We denote by $\operatorname{gp}(A)$ the class of all Gorenstein-projective modules and by ${}^\perp A$ the class of all semi-Gorenstein-projective modules. As we mentioned already, $\operatorname{gp}(A) \subseteq {}^\perp A$. We propose to call an artin algebra A *weakly Gorenstein* provided that ${}^\perp A = \operatorname{gp}(A)$, thus provided that any semi-Gorenstein-projective module is Gorenstein-projective.

1.2. It is well-known that a semi-Gorenstein-projective module M is Gorenstein-projective if and only if $\operatorname{Tr} M$ is semi-Gorenstein-projective, if and only if M is reflexive and M^* is semi-Gorenstein-projective (we will use our approach in order to include a proof, see 4.5). We want to give various characterizations of the weakly Gorenstein algebras.

Theorem. *Let A be an artin algebra. The following statements are equivalent:*
(1) *A is weakly Gorenstein.*
(2) *Any semi-Gorenstein-projective module is torsionless.*
(3) *Any semi-Gorenstein-projective module is reflexive.*
(4) *For any semi-Gorenstein-projective module M, the map ϕ_M is surjective.*
(5) *For any semi-Gorenstein-projective module M, the module M^* is semi-Gorenstein projective.*
(6) *Any semi-Gorenstein-projective module M satisfies* $\operatorname{Ext}^1(M^*, A_A) = 0$.
(7) *Any semi-Gorenstein-projective module M satisfies* $\operatorname{Ext}^1(\operatorname{Tr} M, A_A) = 0$.

1.3. The second result concerns artin algebras with finitely many semi-Gorenstein-projective modules or with finitely many torsionless modules.

Theorem. *If the number of isomorphism classes of indecomposable modules which are both semi-Gorenstein-projective and torsionless is finite, then A is weakly Gorenstein and any indecomposable non-projective semi-Gorenstein-projective module is Ω-periodic.*

This combines two different directions of thoughts. First of all, Yoshino [Y] has shown that for certain commutative rings R (in particular all artinian commutative rings) the finiteness of the number of isomorphism classes of indecomposable semi-Gorenstein-projective R-modules implies that R is weakly Gorenstein. Here we show the corresponding assertion for artin algebras. Second, according to Marczinzik [M1], all torsionless-finite artin algebras (these are the artin algebras with only finitely many isomorphism classes of torsionless indecomposable modules) are weakly Gorenstein (note that a lot of interesting classes of artin algebras are known to be torsionless-finite, see 3.6).

1.4. A (full) subcategory $\mathcal{C}$ of $\operatorname{mod} A$ is said to be *resolving* provided that it contains all the projective modules and is closed under extensions, direct summands and kernels of surjective maps. The embedding of $\mathcal{C}$ into $\operatorname{mod} A$ provides an exact structure on a resolving subcategory; this exact structure will be called its *canonical* exact structure (for the basic properties of exact structures, see for example the Appendix A of [K]).

An exact category $\mathcal{F}$ is called a *Frobenius category* provided that it has sufficiently many projective and sufficiently many injective objects and the projective objects in $\mathcal{F}$ are just the injective objects in $\mathcal{F}$.

Theorem. *Let A be an artin algebra and $\mathcal{F}$ a resolving subcategory of* $\operatorname{mod} A$ *with* ${}^{\perp}A \subseteq \mathcal{F}$. *Assume that $\mathcal{F}$ with its canonical exact structure is a Frobenius subcategory. Then* $\operatorname{gp}(A) = {}^{\perp}A = \mathcal{F}$.

Note that ${}^{\perp}A$ and $\operatorname{gp}(A)$ are resolving subcategories and that $\operatorname{gp}(A)$ with its canonical exact structure is always Frobenius. Thus, an immediate consequence of Theorem 1.4 is a further characterization of the weakly Gorenstein algebras.

Corollary. *An artin algebra A is weakly Gorenstein if and only if ${}^{\perp}A$ with its canonical exact structure is a Frobenius subcategory.*

1.5. The first example of a semi-Gorenstein-projective module which is not Gorenstein-projective was constructed by Jorgensen and Şega [JS] in 2006, for a commutative algebra A of dimension 8. Recently, Marczinzik [M2] constructed some non-commutative algebras with semi-Gorenstein-projective modules which are not Gorenstein-projective (using the Liu-Schulz example [LS,R1] and gendo-symmetric algebras [FK]). In 6.1, we will exhibit a class of 6-dimensional k-algebras $\Lambda(q)$ with parameter $q \in k \setminus \{0\}$ and a family $M(\alpha)$ of 3-dimensional indecomposable $\Lambda(q)$-modules (with $\alpha \in k$) in order to find new examples:

Theorem. *Let $\Lambda(q)$ be the algebra defined in 6.1. If the multiplicative order of q is infinite, then the modules $M(q)$ and $M(q)^*$ both are semi-Gorenstein-projective, but $M(q)$ is not torsionless, thus not Gorenstein-projective; all the syzygy modules $\Omega^t M(q)$ and $\Omega^t(M(q)^*)$ with $t \geq 0$ are* 3*-dimensional and indecomposable; the module $M(q)^{**} \simeq \Omega M(1)$ is also* 3*-dimensional, but decomposable.*

Addendum. *For any q, the modules $M(\alpha)$ with $\alpha \in k \setminus q^{\mathbb{Z}}$ are Gorenstein-projective. Thus, if k is infinite, there are infinitely many isomorphism classes of* 3*-dimensional Gorenstein-projective modules.*

As a consequence, we immediately see the independence (as pointed out in [JŞ]) of the following three conditions: (G1) The module is semi-Gorenstein-projective; (G2) the A-dual of the module is semi-Gorenstein-projective; (G3) the modules is reflexive. Namely, since the A-module $M = M(q)$ satisfies (G1) and (G2), but not (G3), the module $\Omega^2 M$ satisfies (G1),(G3), but not (G2), and the $\Lambda(q)^{\mathrm{op}}$-module $N = (\Omega^2 M)^*$ satisfies (G2), (G3), but not (G1), see 4.6. Actually, for our example $\Lambda(q)$, there is also a $\Lambda(q)$-module M' which satisfies (G2), (G3), but not (G1), namely the module $M' = M(1)$, see 7.3.

1.6. Outline of the paper. The proofs of theorem 1.2, 1.3 and 1.4 will be given in sections 2, 3 and 5, respectively. The main tool will be what we call approximation sequences: these are the exact sequences $0 \to X \to Y \to Z \to 0$ with Y projective and $\operatorname{Ext}^1(Z, A) = 0$, see section 2. Of special interest are the approximation sequences which are indecomposable sequences with both X and Z being non-zero: we call them $\Omega\mho$-sequences, see section 3. In section 4 we will introduce the $\Omega\mho$-quiver of A. Its vertices are the isomorphism classes $[M]$ of the indecomposable non-projective A-modules M and there is an arrow from $[Z]$ to $[X]$ provided that $X = \Omega Z$ and $\operatorname{Ext}^1(Z, A) = 0$, thus provided that there exists an $\Omega\mho$-sequence $0 \to X \to Y \to Z \to 0$. The components of the $\Omega\mho$-quiver are linearly directed quivers and directed cycles (see 4.1) and they allow to identify certain types of modules, in particular the torsionless and the reflexive modules, as well as the Gorenstein projective and the semi-Gorenstein-projective modules.

Section 5 will be devoted to resolving subcategories of $\operatorname{mod} A$.

In sections 6 and 7, we present a 6-dimensional algebra $\Lambda(q)$ and analyze some 3-dimensional representations which we will denote by $M(\alpha)$ with $\alpha \in k$. The essential properties of the modules $M(\alpha)$ can be found in sections 6.3 and 6.5 and are labeled (1) to (9). The properties (1) to (5) in 6.3 are those which are needed in order to exhibit a module (namely $M(q)$) which is semi-Gorenstein-projective, but not torsionless, see 6.4. The remaining properties (6) to (9) show, in particular, that also the A-dual $M(q)^*$ of $M(q)$ is semi-Gorenstein-projective. Altogether, the proof of Theorem 5.1 and its Addendum is given in 6.7 and 6.8. In 7.1 and 7.2, one finds a complete description of some of the components of the $\Omega\mho$-quiver of the algebra $\Lambda(q)$ and its opposite.

The final section 8 mentions four open questions.

The main tool used throughout the paper are left $\operatorname{proj}(A)$-approximations, in particular we stress the relevance of the cokernel $\mho M$ of a minimal left $\operatorname{proj}(A)$-approximation of a module M. The notation $\mho M$ (pronounced "agemo" M) should be a reminder that for the present considerations the construction $\mho$ has to be considered as a kind of inverse of Ω, as the $\Omega\mho$-sequences show. An essential ingredient in this setting seems to be Lemma 4.4 which identifies the kernel of the canonical map $\mho^{i-1}M \to (\mho^{i-1}M)^{**}$ with $\operatorname{Ext}^i(\operatorname{Tr} M, A)$.

1.7. Terminology. We end the introduction with some remarks concerning the terminology and its history. The usual reference for the introduction of the class of Gorenstein-projective modules are the Memoirs by Auslander and Bridger [AB] from 1969, where they appear under the name *modules of Gorenstein dimension zero.* Actually, Bridger, in his 1967 thesis [Br], attributes the concept of the Gorenstein dimension to Auslander: In January 1967, Auslander gave four lectures at the Séminaire Piere Samuel [MPS] where he discussed the class $G(A)$ of all reflexive modules M such that both M and M^* are semi-Gorenstein-projective modules (see [MPS], Definition 3.2.2), thus the class $G(A)$ of the Gorenstein-projective modules. In [AB], Proposition 3.8, it is shown that a module M belongs to $G(A)$ if and only if both M and $\operatorname{Tr} M$ are semi-Gorenstein-projective. Of course, these investigations concern finitely generated modules over a **commutative** noetherian ring A, however all the essential consideration in [MPS, Br, AB] are formulated for general abelian categories with enough projectives. Enochs and Jenda [EJ1,EJ2] reformulated the definition of Gorenstein-projective modules in terms of complete projective resolutions. Several other names for the Gorenstein-projective modules are in use, they are also called "totally reflexive" modules [AM] and "maximal Cohen-Macaulay" modules [Buch, Bel].

We should apologize that we propose a new name for the modules M with $\operatorname{Ext}^i(M, A) = 0$, namely "semi-Gorenstein-projective". These modules have been called for example Cohen-Macauley modules and stable modules. However, in our opinion, the name "Cohen-Macaulay module" is in conflict with its established use for commutative rings, and the wording "balanced" may be too vague as a proper identifier. We hope that the name "semi-Gorenstein-projective" describes well what is going on: that there is something like a half of a complete projective resolution ("semi" means "half"). We also propose the name "weakly Gorenstein" for an algebra A with $\operatorname{gp}(A) = {}^{\perp}A$ (in contrast to "nearly Gorenstein" in [M2]). Of course, a Gorenstein algebra A satisfies $\operatorname{gp}(A) = {}^{\perp}A$, but the algebras with $\operatorname{gp}(A) = {}^{\perp}A$ seem to be quite far away from being Gorenstein.

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2. Approximation sequences. Proof of Theorem 1.2.

2.1. Lemma. *Let* $\epsilon\colon 0 \to X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \to 0$ *be an exact sequence with* Y *projective. Then the following conditions are equivalent:*

(i) ω *is a left* $\operatorname{proj}(A)$*-approximation.*

(ii) $\operatorname{Ext}^1(Z, A) = 0$.

(iii) *The* A*-dual sequence* ϵ^* *of* ϵ *is exact.*

A sequence $0 \to X \to Y \to Z \to 0$ with Y projective satisfying the equivalent properties will be called an *approximation sequence.*

Proof of the equivalence of the properties. Since Y is projective, applying $\operatorname{Hom}(-, A)$ to ϵ we get the exact sequence $0 \to Z^* \xrightarrow{\pi^*} Y^* \xrightarrow{\omega^*} X^* \to \operatorname{Ext}^1(Z, A) \to 0$. Note that ω is a left $\operatorname{proj}(A)$-approximation if and only if ω^* is surjective. From this we get the equivalence of (i) and (ii) and the equivalence of (ii) and (iii). $\square$

Typical examples of approximation sequences: If M is torsionless, with minimal left $\operatorname{proj}(A)$-approximation $\omega\colon X \to P$, then

$$0 \to M \xrightarrow{\omega} P \to \mho M \to 0$$

is an approximation sequence (these are the approximation sequences $0 \to X \to Y \to Z \to 0$ such that Z has to non-zero projective direct summand). If $\operatorname{Ext}^1(M, A) = 0$, then

$$0 \to \Omega M \to PM \to M \to 0$$

is an approximation sequence (these are the approximation sequences $0 \to X \to Y \to Z \to 0$ such that X has to non-zero projective direct summand).

2.2. The following basic lemma is well-known (see, for example [R2]).

Lemma. *Let* $P_{-1} \xrightarrow{f} P_0 \xrightarrow{g} P_1$ *be an exact sequence of projective modules and let* $g = up$ *be a factorization with* $p : P_0 \to I$ *epi and* $u\colon I \to P_1$ *mono. Then* $P_{-1}^* \xleftarrow{f^*} P_0^* \xleftarrow{g^*} P_1^*$ *is exact if and only if* u *is a left* $\operatorname{proj}(A)$*-approximation.*

For the convenience of the reader, we insert the proof. Since $f^*g^* = (gf)^* = 0$, we have $\operatorname{Im} g^* \subseteq \operatorname{Ker} f^*$. Assume now that u is a left $\operatorname{proj}(A)$-approximation and let $h \in \operatorname{Ker} f^*$, thus $hf = 0$. Since p is a cokernel of f, there is h' with $h = h'p$. Since u is a left add$(A($-approximation, there is h'' with $h' = h''u$. Thus $h = h'p = h''up = h''g = g^*(h'')$ belongs to the image of g^*, there also $\operatorname{Ker} f^* \subseteq \operatorname{Im} g^*$.

Conversely, we assume that $\operatorname{Im} g^* = \operatorname{Ker} f^*$ and let $h\colon I \to A$ be a map. Then $hpf = 0$, so that $f^*(hp) = 0$. Therefore hp belongs to $\operatorname{Ker} f^*$, thus to $\operatorname{Im} g^*$. There is $h'' \in P_1^*$ with $hp = g^*(h'') = h''g = h''up$, and therefore $h = h''u$. $\square$

This Lemma will be used in various settings, see 4.3.

2.3. *A semi-Gorenstein-projective and* Ω*-periodic module is Gorenstein-projective.*

Proof. Let M be semi-Gorenstein-projective and assume that $\Omega^t M = M$ for some $t \geq 1$. Let $\cdots \to P_i \to \cdots \to P_0 \to M \to 0$ be a minimal projective resolution of M. Then

$$(+) \qquad 0 \to \Omega^t M \to P_{t-1} \to \cdots \to P_0 \to M \to 0$$

is the concatenation of approximation sequences. Since $\Omega^t M = M$, we can concatenate countably many copies of (+) in order to obtain a double infinite acyclic chain complex of projective modules. As a concatenation of approximation sequences, it is a complete projective resolution. Therefore, M is Gorenstein projective. $\square$

2.4. Here are two essential observations.

(a) *Let $0 \to X \to Y \to Z \to 0$ be an approximation sequence. Then ϕ_X is surjective if and only if Z is torsionless.* We can also say: *X is reflexive if and only if Z is torsionless.*

(b) *Let $0 \to X \to Y \to Z \to 0$ be an approximation sequence. Then* $\mathrm{Ext}^1(X^*, A) = 0$ *if and only if ϕ_Z is surjective.*

Proof of (a) and (b). Since $0 \to X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \to 0$ is an approximation sequence, it follows that

$$0 \longrightarrow X^* \xrightarrow{\pi^*} Y^* \longrightarrow Z^* \longrightarrow 0$$

is an exact sequence of right A-modules. This induces an exact sequence

$$0 \longrightarrow X^{**} \longrightarrow Y^{**} \xrightarrow{\pi^{**}} Z^{**} \longrightarrow \mathrm{Ext}^1_A(X^*, A) \longrightarrow 0$$

of left A-modules, and the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{\pi} & Z & \longrightarrow & 0 & & \\ & & \downarrow{\phi_X} & & \| & & \downarrow{\phi_Z} & & & & \\ 0 & \longrightarrow & X^{**} & \longrightarrow & Y^{**} & \xrightarrow{\pi^{**}} & Z^{**} & \longrightarrow & \mathrm{Ext}^1(X^*, A) & \longrightarrow & 0 \end{array}$$

By the Snake Lemma, the kernel of ϕ_Z is isomorphic to the cokernel of ϕ_X, Thus ϕ_Z is a monomorphism if and only if ϕ_X is an epimorphism. Since X is torsionless, X is reflexive if and only if ϕ_X is surjective. This is (a).

By the commutative diagram above, we see that ϕ_Z is epic if and only if so is π^{**}, and if and only if $\mathrm{Ext}^1_A(X^*, A) = 0$. This is (b). $\square$

Corollary. *A module X is reflexive if and only if both X and $\mho X$ are torsionless.*

Proof. If X is reflexive, then it is torsionless. Thus we may assume from the beginning that X is torsionless. Any minimal left $\mathrm{add}(A)$-approximation $X \to Y$ is injective and its cokernel is $\mho X$. The exact sequence $0 \to X \to Y \to \mho X \to 0$ is an approximation sequence, and 2.4 (a) asserts that X is reflexive iff $\mho X$ is torsionless. $\square$

2.5. Lemma. *Let M be an indecomposable non-projective module with* $\mathrm{Ext}^i(M, A) = 0$ *for $i = 1, 2$. Then* $\mathrm{Tr}\, M = (\Omega^2 M)^*$.

Proof: Let $\pi\colon PM \to M$ and $\pi'\colon P\Omega M \to \Omega M$ be projective covers with inclusion maps $\omega\colon \Omega M \to PM$ and $\omega'\colon \Omega^2 M \to P\Omega M$. Then $\omega\pi'$ is a minimal projective presentation of M. By definition, $\mathrm{Tr}\, M$ is the cokernel of $(\omega\pi')^*$. Since $\mathrm{Ext}^i(M, A) = 0$ for $i = 1, 2$, the exact sequences

$$0 \to \Omega^2 M \xrightarrow{\omega'} P\Omega M \xrightarrow{\pi'} \Omega M \to 0, \quad 0 \to \Omega M \xrightarrow{\omega} PM \xrightarrow{\pi} M \to 0$$

are approximation sequences. As a consequence, the corresponding A-dual sequences

$$0 \leftarrow (\Omega^2 M)^* \xleftarrow{(\omega')^*} (P\Omega M)^* \xleftarrow{(\pi')^*} (\Omega M)^* \leftarrow 0, \quad 0 \leftarrow (\Omega M)^* \xleftarrow{\omega^*} (PM)^* \xleftarrow{\pi^*} M^* \leftarrow 0$$

are exact. The concatenation

$$0 \leftarrow (\Omega^2 M)^* \xleftarrow{(\omega')^*} (P\Omega M)^* \xleftarrow{(\omega\pi')^*} (PM)^* \xleftarrow{\pi^*} M^* \leftarrow 0$$

shows that also $(\Omega^2 M)^*$ is a cokernel of $(\omega\pi')^*$, thus $\operatorname{Tr} M = (\Omega^2 M)^*$. □

2.6. Proof of Theorem 1.

(1) implies (2) to (7): This follows directly from well-known properties of Gorenstein-projective modules. Namely, assume (1) and let M be Gorenstein-projective. Then M is reflexive, this yields (3), but, of course, also (2) and (4). Second, M^* is Gorenstein-projective, thus semi-Gorenstein-projective, therefore we get (5) and (6). Finally, $\operatorname{Tr} M$ is Gorenstein-projective, thus semi-Gorenstein-projective, therefore we get (7).

Both (3) and (4) imply (2): Let M be semi-Gorenstein-projective. Consider the approximation sequence $0 \to \Omega M \to PM \to M \to 0$ and note that ΩM is again semi-Gorenstein-projective. If (3) or just (4) holds, we know that $\phi_{\Omega M}$ is surjective, thus by 2.4 (a), M is torsionless.

Both (6) and (7) imply (2): Let M be semi-Gorenstein-projective. Consider the approximation sequences $0 \to \Omega M \to PM \to M \to 0$ and $0 \to \Omega^2 M \to PM \to \Omega M \to 0$. Since M is semi-Gorenstein-projective, also $\Omega^2 M$ is semi-Gorenstein-projective. If (6) holds, we use (6) for $\Omega^2 M$ in order to see that $\operatorname{Ext}^1((\Omega^2 M)^*, A) = 0$. If (7) holds, we use (7) for M in order to see that $\operatorname{Ext}^1(\operatorname{Tr} M, A) = 0$. According to 2.5, we see that $\operatorname{Tr} M = (\Omega^2 M)^*$. Thus in both cases (6) and (7), we have $\operatorname{Ext}^1((\Omega^2 M)^*, A) = 0$. According to 2.4 (b), it follows from $\operatorname{Ext}^1((\Omega^2 M)^*, A) = 0$ that $\phi_{\Omega M}$ is surjective. By 2.4 (a), M is torsionless.

Trivially, (5) implies (6). Altogether we have shown that any one of the assertions (3) to (7) implies (2).

It remains to show that (2) implies (1). Let M be semi-Gorenstein-projective and torsionless. We want to show that M is Gorenstein-projective. Let $M_i = \mho^i M$ for all $i \geq 0$ (with $M_0 = M$). Since M_0 is torsionless, there is an approximation sequence $0 \to M_0 \to P_1 \to M_1 \to 0$, and M_1 is again semi-Gorenstein-projective. By assumption, M_1 is again torsionless. Inductively, starting with a torsionless module M_i, we obtain an approximation sequence $\epsilon_i \colon 0 \to M_i \to P_{i+1} \to M_{i+1} \to 0$, we conclude that with M_i also M_{i+1} is semi-Gorenstein-projective. By (2) we see that M_{i+1} is torsionless, again. Concatenating a minimal projective resolution of M with these approximation sequences ϵ_i, for $0 \leq i$, we obtain a complete projective resolution of M. □

3. $\Omega\mho$-sequences. Proof of theorem 1.3.

3.1. An approximation sequence $0 \to X \to Y \to Z \to 0$ will be called an $\Omega\mho$*-sequence* provided that both X and Z are indecomposable and not projective (the relevance of such sequences was stressed already in [RX]).

Lemma. *An approximation sequence is the direct sum of $\Omega\mho$-sequences and of sequences of the form $0 \to P \xrightarrow{1} P \to 0 \to 0$ and $0 \to 0 \to P \xrightarrow{1} P \to 0$ with P indecomposable projective.*

Proof: Let $0 \to X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \to 0$ be an approximation sequence. Since Y is projective and π is surjective, a direct decomposition $Z = Z_1 \oplus Z_2$ yields a direct sum decomposition of the sequence.

Since ω is a left $\mathrm{proj}(A)$-approximation, there is also the corresponding assertion: If $X = X_1 \oplus X_2$, then $X \xrightarrow{\omega} Y$ is the direct sum of two maps $X_1 \to Y_1$ and $X_2 \to Y_2$, thus again we obtain a direct sum decomposition of the sequence. This shows that for an indecomposable approximation sequence $0 \to X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \to 0$, the modules X and Z are indecomposable or zero (and, of course, not both can be zero).

If Z is indecomposable and projective, then the sequence $0 \to X \to Y \to Z \to 0$ splits off $0 \to 0 \to Z \xrightarrow{1} Z \to 0$, thus $X = 0$. Similarly, if X is indecomposable and projective, then the sequence $0 \to X \to Y \to Z \to 0$ splits off $0 \to X \xrightarrow{1} X \to 0 \to 0$, thus $Z = 0$.

It remains that $0 \to X \to Y \to Z \to 0$ is an approximation sequence with both X and Z being indecomposable and non-projective. □

3.2. Lemma. *Let ϵ: $0 \to X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \to 0$ be an exact sequence. The following conditions are equivalent:*

(i) *ϵ is an $\Omega\mho$-sequence.*

(ii) *X and Z are indecomposable and not projective, ω is a minimal left* $\mathrm{proj}(A)$*-approximation, π is a projective cover, $X = \Omega Z$, $Z = \mho X$.*

(iii) *X is indecomposable and not projective, ω is a minimal left* $\mathrm{proj}(A)$*-approximation.*

(iv) *Z is indecomposable and not projective, π is a projective cover, and* $\mathrm{Ext}^1(Z, A) = 0$.

(v) *$X = \Omega Z$, Y is projective, $Z = \mho X$, and X is indecomposable.*

(vi) *$X = \Omega Z$, Y is projective, $Z = \mho X$, and Z is indecomposable.*

Proof: In (ii) we have collected all the relevant properties of an $\Omega\mho$-sequence. (i) implies (ii): Let ϵ be an $\Omega\mho$-sequence. Then ω has to be minimal, since otherwise ϵ would split off a non-zero sequence of the form $0 \to 0 \to P \xrightarrow{1} P \to 0$. Similarly, π has to be a projective cover, since otherwise ϵ would split off a non-zero sequence of the form $0 \to P \xrightarrow{1} P \to 0 \to 0$. Since ω is a minimal left $\mathrm{proj}(A)$-approximation and Z is the cokernel of ω, we see that $Z = \mho X$. Since π is a projective cover of Z and X is its kernel, $X = \Omega Z$.

The condition (iii), (iv), (v) and (vi) single out some of these properties, thus (ii) implies these conditions.

(iii) implies (i): Since X is indecomposable and not projective, ϵ has no direct summand $0 \to P \xrightarrow{1} P \to 0 \to 0$. Since ω is left minimal, ϵ has no direct summand $0 \to 0 \to P \xrightarrow{1} P \to 0$.

Similarly, (iv) implies (i).

Both (v) and (vi) imply (i): Since $Z = \mho X$, we have $\mathrm{Ext}^1(Z, A) = 0$. This shows that the sequence is an approximation sequence. Since $X = \Omega Z$, the sequence ϵ has no direct summand of the form $0 \to P \xrightarrow{1} P \to 0 \to 0$. Since $Z = \mho X$, the sequence ϵ has no direct summand of the form $0 \to 0 \to P \xrightarrow{1} P \to 0$. Thus, ϵ is a direct sum of $\Omega\mho$-sequences. Finally, since X or Z is indecomposable, ϵ is an $\Omega\mho$-sequence. □

3.3. Corollary. *If M is indecomposable, non-projective, semi-Gorenstein-projective, also ΩM is indecomposable, non-projective, semi-Gorenstein-projective and $M = \mho\Omega M$.*

Proof. Since M is semi-Gorenstein-projective module, the canonical sequence $\epsilon\colon 0 \to \Omega M \to PM \to M \to 0$ is an approximation sequence. Since M is indecomposable and not projective, and $PM \to M$ is a projective cover, ϵ is an $\Omega\mho$-sequence, thus ΩM is indecomposable and non-projective, and $M = \mho\Omega M$, by 3.2. Of course, with M also ΩM is semi-Gorenstein-projective. □

3.4. Lemma. *If the number of isomorphism classes of indecomposable modules which are both semi-Gorenstein-projective and torsionless is finite, then any indecomposable non-projective semi-Gorenstein-projective module is Ω-periodic.*

Proof. According to 3.3, The modules $\Omega^t M$ with $t \geq 1$ are indecomposable modules which are torsionless and semi-Gorenstein-projective. Since there are only finitely many isomorphism classes of indecomposable torsionless semi-Gorenstein-projective modules is finite, there are natural numbers $1 \leq s < t$ with $\Omega^s M = \Omega^t M$. Then $M = \mho^s\Omega^s M = \mho^s\Omega^t M = \Omega^{t-s} M$ and $t - s \geq 1$, thus M is Ω-periodic. □

3.5. Proof of Theorem 1.3. We assume that the number of isomorphism classes of indecomposable torsionless semi-Gorenstein-projective modules is finite. According to 3.4, any indecomposable non-projective semi-Gorenstein-projective module is Ω-periodic. 2.3 shows that any semi-Gorenstein-projective Ω-periodic module is Gorenstein-projective. □

3.6. Torsionless-finite algebras. An artin algebra A is said to be *torsionless-finite* if there are only finitely many isomorphism classes of indecomposable torsionless modules. Theorem 1.3 implies that *any torsionless-finite artin algebra is weakly Gorenstein,* as Marczinzik [M1] has shown. Note that many interesting classes of algebras are known to be torsionless-finite. In particular, we have

The following algebras are torsionless-finite, and hence weakly Gorenstein.

(1) *Algebras A such that $A/\operatorname{soc}({}_AA)$ is representation-finite.*
(2) *Algebras stably equivalent to hereditary algebras, in particular all algebras with radical square zero.*
(3) *Minimal representation-infinite algebras.*
(4) *Special biserial algebras without indecomposable projective-injective modules.*

See for example [R2], where also other algebras are listed which are known to be torsionless-finite.

Chen [Chen] has shown that a connected algebra A with radical square zero either is self-injective, or else all the Gorenstein-projective modules are projective. The assertion that algebras with radical square zero are weakly Gorenstein complements this result.

4. The $\Omega\mho$-quiver.

4.1. Definition. *The $\Omega\mho$-quiver* of A has as vertices the isomorphism classes $[X]$ of the indecomposable non-projective modules X and there is an arrow

$$[X] \dashleftarrow [Z]$$

provided that $X = \Omega Z$ and $Z = \mho X$, thus provided that there exists an $\Omega\mho$-sequence $0 \to X \to Y \to Z \to 0$. (The arrows of an $\Omega\mho$-quiver will usually be drawn as dashed arrows, in order to stress that they correspond to extensions and not to maps; the direction of these arrows follows the convention used for Ext-quivers.) An arrow ending at $[X]$ starts at $[\mho X]$; an arrow starting at $[Z]$ ends at $[\Omega Z]$. Thus, there is the immediate consequence:

Lemma. *At any vertex of the $\Omega\mho$-quiver at most one arrow starts and at most one arrow ends.* □

The components of the $\Omega\mho$-quiver will be called $\Omega\mho$-components. Any interval $I \subseteq \mathbb{Z}$ will be considered as a quiver with an arrow from z to $z-1$, provided that both $z-1$ and z belong to I.

Corollary. *Any $\Omega\mho$-component is a linearly directed quiver of type $\mathbb{A}_n$ with $n \geq 1$ vertices, an oriented cycle $\widetilde{\mathbb{A}}_n$ with $n+1 \geq 1$ vertices, or of the form $-\mathbb{N}$, or $\mathbb{N}$, or $\mathbb{Z}$.*

As we will see in 7.1 and 7.4, all cases mentioned here can arise as $\Omega\mho$-components. Note that an algebra A is weakly Gorenstein provided there is no $\Omega\mho$-component of the form $-\mathbb{N}$.

4.2. The A-dual of an $\Omega\mho$-sequence.

Lemma. (a) *Let $\epsilon\colon 0 \to X \to Y \to Z \to 0$ be an approximation sequence and assume that X is reflexive. Then* $\operatorname{Ext}^1(X^*, A) = 0$ *if and only if Z is reflexive, if and only if the A-dual ϵ^* of ϵ is again an approximation sequence.*

(b) *Let $\epsilon\colon 0 \to X \to Y \to Z \to 0$ be an $\Omega\mho$-sequence with X reflexive. Then Z is reflexive, if and only if the A-dual ϵ^* of ϵ is again an $\Omega\mho$-sequence.*

Proof. (a) By 2.4 (a), we see that Z is always torsionless. Thus 2.4 (b) shows that $\operatorname{Ext}^1(X^*, A) = 0$ if and only if Z is reflexive. First, assume assume that Z is reflexive. Then $\operatorname{Ext}^1(X^*, A) = 0$, and therefore we see that the A-dual sequence ϵ^* is exact. We dualize a second time: the sequence ϵ^{**} is isomorphic to the sequence ϵ, since the three modules X, Y, Z are reflexive. This means that ϵ^{**} is exact, and therefore ϵ^* is an approximation sequence. Second, conversely, if ϵ^* is an approximation sequence, then it is exact, and therefore $\operatorname{Ext}^1(X^*, A) = 0$, thus Z is reflexive.

(b) Assume now that ϵ is an $\Omega\mho$-sequence. First, assume that Z is reflexive. Since X, Z both are reflexive, indecomposable and non-projective, also X^* and Z^* are indecomposable and non-projective, as we will show below. Thus ϵ^* is an $\Omega\mho$-sequence. Of course, conversely, if ϵ is an $\Omega\mho$-sequence, then it is an approximation sequence and thus Z is reflexive by (a). □.

We have used some basic facts about the A-dual M^* of a module M.

(1) *M^* is always torsionless.*
(2) *If M is non-zero and torsionless, then M^* is non-zero.*
(3) *If M is reflexive, indecomposable and non-projective, then M^* is reflexive, indecomposable and non-projective.*

Here are the proofs (or see for example [L]). (1) There is a surjective map $u\colon P \to M$ with P projective. Then $u^*\colon M^* \to P^*$ is an embedding of M^* into the projective module P^*. (2) is obvious.

(3) Let M be reflexive, indecomposable and non-projective. Consider a direct decomposition $M^* = N_1 \oplus N_2$ with $N_1 \neq 0$ and $N_2 \neq 0$. Since M^* is torsionless by (1), both modules N_1 and N_2 are torsionless, therefore $N_1^* \neq 0, N_2^* \neq 0$, thus there is a proper direct decomposition $M^{**} = N_1^* \oplus N_2^*$. Since M is reflexive and indecomposable, this is impossible. Thus M^* has to be indecomposable. If M^* is projective, then also M^{**} is projective. Again, since M is reflexive, this is impossible.

It remains to show that M^* is reflexive. But since M^{**} is isomorphic to M, we see that $(M^{***}$ is isomorphic to M^*, thus the canonical map $M^* \to M^{***}$ has to be an isomorphism (since it is a monomorphism of modules of equal length). □

4.3. Lemma 2.2 outlines the importance of left add(A)-approximations when dealing with exact sequences of projective modules. We have mention there that this Lemma will be relevant for our considerations. Now we want to give a unified treatment of the relevance of approximation sequences and of $\Omega\mho$-sequences.

(a) *An exact sequence* $\cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$ *is a complete projective resolution if and only if it is the concatenation of approximation sequences.*

(a′) *An indecomposable non-projective module M is Gorenstein-projective if and only if $[M]$ is the start of an infinite path and the end of an infinite path in the $\Omega\mho$-quiver.*

(b) *A module M is semi-Gorenstein-projective if and only if a projective resolution (or, equivalently, any projective resolution) is the concatenation of approximation sequences.*

(b′) *An indecomposable non-projective module M is semi-Gorenstein-projective if and only if $[M]$ is the start of an infinite path in the $\Omega\mho$-quiver.*

(c) *A module M is reflexive and M^* is semi-Gorenstein-projective if and only if there is an exact sequence $0 \to M \to P^1 \to P^2 \to \cdots$ which is the concatenation of approximation sequences.*

(c′) *An indecomposable non-projective module M is reflexive and M^* is semi-Gorenstein-projective if and only if $[M]$ is the end of an infinite path in the $\Omega\mho$-quiver.*

Proof: We use that the A-dual of an approximation sequence is exact, thus the A-dual of the concatenation of approximation sequences is exact.

(a) Let $P^\bullet$ be a double infinite exact sequence of projective modules with maps $d^i\colon P^i \to P^{i+1}$ Write $d^i = \omega^i\pi^i$ with π^i epi and ω^i mono. If $P^\bullet$ is a complete projective resolution, then the exactness of $(P^\bullet)^*$ at $(P^i)^*$ implies that ω^i is a left proj(A)-approximation, see 2.2. Thus $P^\bullet$ is the concatenation of approximation sequences.

(b) Let $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. write the map $P_{i+1} \to P_i$ as $\omega_i\pi_i$ with π_i epi and ω_i mono. If the A-dual of the sequence $\cdots \to P_i \to \cdots \to P_0$ is exact, then all the maps ω_i with $i \geq 1$ have to be left proj(A)-approximations. This shows that the projective resolution is the concatenation of approximation sequences.

(b′) Let M be indecomposable, non-projective and semi-Gorenstein-projective. Since $\operatorname{Ext}^1(M, A) = 0$, the sequence $0 \to \Omega M \to PM \to M \to 0$ is an $\Omega\mho$-sequence and ΩM is again indecomposable and non-projective. Also, ΩM is semi-Gorenstein-projective. Thus, we can iterate the procedure and obtain the infinite path

$$(*) \qquad \cdots \dashleftarrow [\Omega^2 M] \dashleftarrow [\Omega M] \dashleftarrow [M]$$

Conversely, assume that there is an infinite path starting with $[M]$, then it is of the form $(*)$. Thus, for all $i \geq 1$, we have $\operatorname{Ext}^i(M, A) \simeq \operatorname{Ext}^1(\Omega^{i-1}M, A) = 0$.

Proof of (c) and (c′). Assume that there are given approximation sequences $\epsilon_i \colon 0 \to M^i \to P^{i+1} \to M^{i+1} \to 0$ for all $i \geq 0$, with $M^0 = M$. Then all the modules M^i are torsionless, thus reflexive by 2.4 (a). In particular, M itself is reflexive. The A-dual of ϵ_i is the sequences

$$\epsilon_i^* \colon 0 \leftarrow (M^i)^* \leftarrow (P^{i+1})^* \leftarrow (M^{i+1})^* \leftarrow 0,$$

which again is an approximation sequence by 4.2 (a). The concatenation of the sequences ϵ_i^* is a projective resolution of $M^* = (M^0)^*$. According to (b), M^* is semi-Gorenstein-projective, since all the sequences ϵ_i^* are approximation sequences.

Conversely, assume that M is reflexive and M^* is semi-Gorenstein-projective. We want to construct a sequence $0 \to M \to P^1 \to P^2 \to \cdots$ which is the concatenation of approximation sequences. It is sufficient to consider the case where M is indecomposable (in general, take the direct sum of the sequences). If M is projective, then $0 \to M \to M \to 0 \to \cdots$ is the concatenation of approximation sequences.

Thus, it remains to consider the case where M is indecomposable and not projective. Since M is torsionless, there is an $\Omega\mho$-sequence $\epsilon_0 \colon 0 \to M \to P^1 \to M^1 \to 0$ (with $M^1 = \mho M$). Note that M^1 is indecomposable, not projective, and that the A-dual $\epsilon_0^* \colon 0 \leftarrow M^* \leftarrow (P^1)^* \leftarrow (M^1)^* \leftarrow 0$ is exact. Since M is reflexive, M^1 is torsionless by 2.4 (a). Since M^* is semi-Gorenstein-projective, $\operatorname{Ext}^1(M^*, A) = 0$, therefore ϕ_{M^1} is surjective and ϵ_0^* is an $\Omega\mho$-sequence, by 4.2. Altogether we know now that M^1 is reflexive, but also that $(M^1)^* = \Omega(M^*)$. With M^* also $\Omega(M^*)$ is semi-Gorenstein-projective.

Thus M^1 satisfies again the assumptions of being indecomposable, not projective, reflexive and that its A-dual $(M^1)^*$ is semi-Gorenstein-projective. Thus we can iterate the procedure for getting the next $\Omega\mho$-sequence $\epsilon_1 \colon 0 \to M^1 \to P^2 \to M^2 \to 0$, with $M^2 = \mho^2 M$, and so on. Altogether, we obtain the infinite path:

$$[M] \dashrightarrow [\mho M] \dashrightarrow [\mho^2 M] \dashrightarrow \cdots$$

This completes the proof of (c′) and thus also of (c).

(a′) This follows immediately from (b′) and (c′). □

4.4. Modules at the end of a path of length t. Modules at the end of a path of length 1 or 2 are torsionless, or reflexive, respectively. Now we look at modules which occur at the end of a path of arbitrary length $t \geq 1$.

Proposition. *Let M be an indecomposable module and $t \geq 1$. The following conditions are equivalent:*
(i) *M is the end of a path of length t in the $\Omega\mho$-quiver.*
(ii) *$\mho^{i-1}M$ is torsionless for $1 \leq 1 \leq t$.*
(iii) $\operatorname{Ext}^i(\operatorname{Tr} M, A) = 0$ *for $1 \leq i \leq t$.*

The modules M with property (iii) have been called *t-torsion free* by Auslander in [MPS], Chapter 3, and then by Bridger [Br] and Auslander and Bridger [AB]. Note that a module is 1-torsion free iff it is torsionless, and 2-torsion free iff it is reflexive. These special cases are discussed already in [ARS], Corollary IV.3.3.

Proof. The equivalence of (i) and (ii) follows directly from the definition of the $\Omega\mho$-quiver. Thus, we only have to show the equivalence of (ii) and (iii).

For the proof, we use the following notation: For any module M, let us denote by $\operatorname{K} M$ the kernel of $\phi_M \colon M \to M^{**}$, this is, of course, the kernel of any left $\operatorname{proj}(A)$-approximation of M. Thus, a module M is torsionless iff $\operatorname{K} M = 0$. The following Lemma shows for any $i \geq 1$, that $\mho^{i-1}M$ is torsionless iff $\operatorname{Ext}^i(\operatorname{Tr} M, A) = 0$ In this way, we see that (ii) and (iii) are equivalent. $\square$

Lemma. *Let M be a module and $i \geq 1$. Then*

$$\operatorname{K}(\mho^{i-1}M) \simeq \operatorname{Ext}^i(\operatorname{Tr} M, A).$$

In particular, $\mho^{i-1}M$ is torsionless if and only if $\operatorname{Ext}^i(\operatorname{Tr} M, A) = 0$.

Proof. Let $P^0 \xrightarrow{f} P^1 \xrightarrow{p^1} M \to 0$ be a minimal projective presentation of M, thus $\operatorname{Tr} M$ is the cokernel of f^*. We want to extend the presentation $(P^0)^* \xleftarrow{f^*} (P^1)^*$ of $\operatorname{Tr} M$ to a projective resolution. We construct a complex of projective modules

$$P^\bullet: \qquad P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \cdots$$

as follows: Let $d^0 = f$, its cokernel is by definition the map $p^1 \colon P^1 \to M = \mho^0 M$. For any $i \geq 1$, let $u^i q^i \colon \mho^{i-1}M \to P^{i+1}$ be a minimal $\operatorname{proj}(A)$-approximation with q^i epi and u^i mono, thus the kernel of q^i is just $\operatorname{K}(\mho^{i-1}M)$. Let $p^{i+1} \colon P^{i+1} \to \mho^i M$ be the cokernel of u^i, thus also of $d^i = u^i q^i p^i$. To repeat, the maps d^i for $i \geq 1$ are factored as follows:

$$P^i \xrightarrow{\;p^i\;} \mho^{i-1}M \xrightarrow{\;q^i\;} \mho^{i-1}M/\operatorname{K}(\mho^{i-1}M) \xrightarrow{\;u^i\;} P^{i+1}.$$

We claim that the cohomology of $P^\bullet$ at the position $i \geq 1$ is

$$H^i(P^\bullet) = \operatorname{K}(\mho^{i-1}M).$$

Namely, for $i = 1$, by the definition $d^0 = f$, we have $P^1/\operatorname{Im}(d^0) = P^1/\operatorname{Im}(f) = M$ and $\operatorname{Ker}(d^1)/\operatorname{Im}(d^0) = \operatorname{Ker}(q^1p^1)/\operatorname{Im}(f) = \operatorname{Ker}(q^1) = \operatorname{K} M$. For $i \geq 2$, we have $\operatorname{Ker}(d^i)/\operatorname{Im}(d^{i-1}) = \operatorname{Ker}(q^ip^i)/\operatorname{Im}(u^{i-1}) = \operatorname{Ker}(q^i) = \operatorname{K}(\mho^{i-1}M)$.

Second, we show that the complex

$$(P^\bullet)^*: \qquad (P^0)^* \xleftarrow{(d^0)^*} (P^1)^* \xleftarrow{(d^1)^*} (P^2)^* \xleftarrow{(d^2)^*} \cdots$$

is exact. Namely, let $i \geq 1$ and consider an element $h\colon P^i \to A$ in the kernel of $(d^{i-1})^*$, thus $hd^{i-1} = 0$. Therefore h factors through the cokernel p^i of d^{i-1}, say $h = h'p^i$ for some $h'\colon \mho^{i-1}M$. Since $u^i q^i$ is a left $\operatorname{proj}(A)$-approximation, we obtain $h''\colon P^{i+1} \to A$ with $h' = h''u^i q^i$. This shows that $h = h'p^i = h''u^i q^i p^i = h''d^i = (d^i)^*(h'')$ is in the image of $(d^i)^*$.

Recall that the cokernel of $f^* = (d^0)^*$ is $\operatorname{Tr} M$, thus $(P^\bullet)^*$ is a projective resolution of $\operatorname{Tr} M$ and hence $\operatorname{Ext}^i(\operatorname{Tr} M, A) = H^i((P^\bullet)^{**}) = H^i(P^\bullet)$, since $(P^\bullet)^{**} \simeq P^\bullet$. □

4.5. Summary. Let us collect what can be read out about an indecomposable non-projective module when looking at its position in the $\Omega\mho$-quiver.

Let M be an indecomposable non-projective module.

(1) $[M]$ *is the start of a path of length* $t \geq 1$ *iff* $\operatorname{Ext}^i(M, A) = 0$ *for* $1 \leq i \leq t$.
In particular: $[M]$ *is the start of an arrow iff* $\operatorname{Ext}^1(M, A) = 0$.

(1′) $[M]$ *is the start of an infinite path iff M is semi-Gorenstein-projective.*

(2) $[M]$ *is the end of a path of length* $t \geq 1$ *iff* $\operatorname{Ext}^i(\operatorname{Tr} M, A) = 0$ *for* $1 \leq i \leq t$ *iff* $\mho^{i-1}M$ *is torsionless for* $1 \leq i \leq t$.
In particular: $[M]$ *is the end of an arrow iff M is torsionless;*
and $[M]$ *is the end of a path of length 2 iff M is reflexive* (see Corollary 2.4).

(2′) $[M]$ *is the end of an infinite path iff (M is reflexive and M^* is semi-Gorenstein-projective) iff* $\operatorname{Tr} M$ *is semi-Gorenstein-projective.*

(3) $[M]$ *is the start of an infinite path and also the end of an infinite path iff M is Gorenstein-projective.*
$[M]$ *belongs to a component of the form* $\mathbb{Z}$ *iff M is Gorenstein-projective and not Ω-periodic.*
$[M]$ *belongs to a component of the form* $\widetilde{\mathbb{A}}_n$ *iff M is Gorenstein-projective and Ω-periodic.*

(4) $[M]$ *is an isolated vertex iff* $\operatorname{Ext}^1(M, A) = 0$ *and M is not torsionless.*

Proof. (1) follows from the fact that $\operatorname{Ext}^t(M, A) = \operatorname{Ext}^{t-1}(\Omega M, A)$ for $t \geq 2$. (2) is Proposition 4.4. For (1′), (2′) and (3), see 4.3. □

Remark. Characterizations of Gorenstein-projective modules. The $\Omega\mho$-quiver shows nicely that an indecomposable module M is Gorenstein-projective if and only if both M and $\operatorname{Tr} M$ are semi-Gorenstein-projective, if and only if M is reflexive and both M and M^* are semi-Gorenstein projective: See (1′), (2′) and (3). □

4.6. Independence. As we know, a module is Gorenstein-projective if and only if (G1) the module is semi-Gorenstein-projective; (G2) the A-dual of the module is semi-Gorenstein-projective; and (G3) the modules is reflexive. These three conditions are independent, as Jorgenson and Şega have shown. Section 6 of the present paper provides new (and in our opinion less technical) examples. But we should stress that only one example is essential, namely to exhibit a module M which satisfies (G1) and (G2), but not (G3).

Let M be a module which satisfies (G1), (G2) *and not* (G3). *Then $\Omega^2 M$ satisfies* (G1) *and* (G3), *but not* (G2), *whereas $N = (\Omega^2 M)^*$ is a right A-module which satisfies* (G2) *and* (G3), *but not* (G1).

Proof. Assume that M satisfies (G1), (G2) and not (G3). Then $\Omega^2 M$ is reflexive and semi-Gorenstein-projective. By Lemma 2.5, $N = (\Omega^2 M)^* = \operatorname{Tr} M$, thus N is not semi-Gorenstein-projective (otherwise, M would be Gorenstein-projective). Using 4.5, we see that $(\Omega^2 M)^*$ is reflexive and $N^* = (\Omega^2 M)^{**} = \Omega M$ is semi-Gorenstein-projective. $\square$

4.7. The constructions Ω and $\mho$. *For any module M, ΩM is torsionless and $\mho M$ satisfies $\operatorname{Ext}^1(\mho M, A) = 0$; in addition, $\mho M$ has no non-zero projective direct summands. If Z satisfies $\operatorname{Ext}^1(Z, A) = 0$ and has no non-zero projective direct summand, then $\mho\Omega Z \simeq Z$. If X is torsionless and has no non-zero projective direct summand, then $\Omega\mho X \simeq X$. Thus, Ω and $\mho$ are inverse bijections between isomorphism classes as follows:*

$$\left\{\begin{matrix}\text{indecomposable}\\ \text{non-projective modules } X\\ \text{which are torsionless}\end{matrix}\right\} \underset{\Omega}{\overset{\mho}{\rightleftarrows}} \left\{\begin{matrix}\text{indecomposable}\\ \text{non-projective modules } Z\\ \text{with } \operatorname{Ext}^1(Z,A)=0\end{matrix}\right\}$$

Actually, this is not only a set-theoretical bijection. There is the following categorical version. Let $\mathcal{L}(A)$ be the full subcategory of all torsionless modules, and $\mathcal{Z}(A)$ the full subcategory of all modules Z with $\operatorname{Ext}^1(Z, A) = 0$. If $\mathcal{C}' \subseteq \mathcal{C}$ are full subcategories of $\operatorname{mod} A$, let $\mathcal{C}/\mathcal{C}'$ be the category with the same objects as $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}/\mathcal{C}'}(X, Y)$ is the factor group of $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ modulo the subspace of all maps $X \to Y$ which factor through a direct sum of modules in $\mathcal{C}'$.

Proposition. *The constructions Ω and $\mho$ provide inverse categorical equivalences*

$$\mathcal{L}(A)/\operatorname{proj}(A) \underset{\Omega}{\overset{\mho}{\rightleftarrows}} \mathcal{Z}(A)/\operatorname{proj}(A)$$

The proof is left to the reader.

5. Resolving subcategories which are Frobenius (proof of Theorem 1.4).

Proof of Theorem 1.4. Let $\mathcal{F}$ be a resolving subcategory of $\operatorname{mod} A$ with ${}^{\perp}A \subseteq \mathcal{F}$ and assume that $\mathcal{F}$ with respect to its canonical exact structure is Frobenius.

First, let us show that that the projective objects in $\mathcal{F}$ are just the projective modules. By assumption, the projective modules belong to $\mathcal{F}$ and since the deflations in $\mathcal{F}$ are epimorphisms in $\operatorname{mod} A$, the projective modules are relative projective objects in $\mathcal{F}$. Conversely, assume that M in $\mathcal{F}$ is relative projective in $\mathcal{F}$. Let $\pi\colon PM \to M$ be a projective cover. Since $\mathcal{F}$ is resolving, this is a deflation in $\mathcal{F}$, thus it splits, therefore M is a projective module.

Second, we show that $\operatorname{Ext}^1(M, A) = 0$ for $M \in \mathcal{F}$. Let $\epsilon\colon 0 \to A \to M' \to M \to 0$ be an exact sequence in $\operatorname{mod} A$. Since $\mathcal{F}$ is resolving, this sequence belongs to $\mathcal{F}$ and $A \to M'$ is an inflation. Since A is relative injective in $\mathcal{F}$, the sequence ϵ splits.

It follows that $\mathcal{F} \subseteq {}^{\perp}A$. Namely, if M belongs to $\mathcal{F}$, then for all $t \geq 0$, also $\Omega^t M$ belongs to $\mathcal{F}$, since $\mathcal{F}$ is resolving. Since $\operatorname{Ext}^t(M, A) = \operatorname{Ext}^1(\Omega^{t-1}M, A)$ for all $t \geq 2$, we see by induction that $\operatorname{Ext}^t(M, A) = 0$ for all $t \geq 1$. Thus, we see that $\mathcal{F} = {}^{\perp}A$.

Finally, we show that any module M in ${}^{\perp}A$ is torsionless. Let M belong to ${}^{\perp}A$, thus to $\mathcal{F}$. Since $\mathcal{F}$ is Frobenius, there is an admissible monomorphism $u\colon M \to M'$ with M'

relative projective in $\mathcal{F}$. As we have seen, M' is a projective module. Since the exact structure of $\mathcal{F}$ is the canonical one, u is injective. Thus M is torsionless. According to Theorem 1.2, A is weakly Gorenstein, thus $\mathrm{gp}(A) = {}^{\perp}A = \mathcal{F}$. $\square$

6. An example.

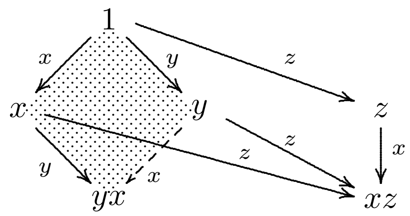
Let k be a field k and $q \in k \setminus \{0\}$. We consider a 6-dimensional local algebra $\Lambda = \Lambda(q)$. If k is infinite, there are infinitely many Gorenstein-projective Λ-modules of dimension 3. Let $o(q) = |q^{\mathbb{Z}}|$ be the multiplicative order of q. If $o(q)$ is infinite, we show that there is also a semi-Gorenstein-projective Λ-module of dimension 3 which is not Gorenstein-projective.

6.1. The algebra $\Lambda = \Lambda(q)$. The algebra Λ is generated by x, y, z, subject to the relations:

$$x^2,\ y^2,\ z^2,\ yz,\ xy + qyx,\ zx - xz,\ zy - xz.$$

The algebra Λ has a basis $1,\ x,\ y,\ z,\ yx$, and xz and may be visualized as follows:

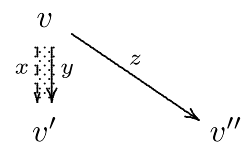
$\Lambda :$



Here, we use the following convention: a solid arrow $x\colon v \to v'$ means that $xv = v'$, a dashed arrow $x\colon v \dashrightarrow v'$ means that xv is a non-zero multiply of v' (in our case, $xy = -qyx$).

We study the following modules $M(\alpha)$ with $\alpha \in k$. The module $M(\alpha)$ has a basis v, v', v'', such that $xv = \alpha v',\ yv = v',\ zv = v''$, and such that v' and v'' are annihilated by x, y, z. That is,

$M(\alpha) :$



The modules $M(\alpha)$ with $\alpha \in k$ are pairwise non-isomorphic indecomposable Λ-modules.

For $\alpha \in k$, we define $m_\alpha = x - \alpha y \in \Lambda$. In order to provide a proof of Theorem 1.5, we now collect some general results for the modules $M(\alpha)$, Λm_α, and the right ideals $m_\alpha \Lambda$ which are needed.

6.2. The module $M(q)$.

Lemma. *The intersection of the kernels of all the homomorphisms $M(q) \to {}_\Lambda\Lambda$ is $zM(q) = kv''$ and $M(q)/zM(q) \simeq \Lambda m_1$. In particular, $M(q)$ is not torsionless and $M(q)^* = (\Lambda m_1)^*$.*

Proof. Let $f\colon M(q) = \Lambda v \to {}_\Lambda\Lambda$ be a homomorphism. Let $f(v) = c_1 x + c_2 y + c_3 z + c_4 yx + c_5 xz$ with $c_i \in k$. By $qf(v') = f(xv) = xf(v) = -c_2 qyx + c_3 xz$ and $f(v') = f(yv) = yf(v) = c_1 yx$, we get $c_2 = -c_1$ and $c_3 = 0$. Thus, $f(v) = c_1(x - y) + c_4 yx + c_5 xz$. It follows that $f(v'') = f(zv) = zf(v) = 0$. This shows that v'' is contained in the kernel of any map

$f\colon M(q) = \Lambda v \to {}_\Lambda\Lambda$. On the other hand, the homomorphism $g\colon M(q) = \Lambda v \to \Lambda$ given by $g(v) = x - y = m_1$ has kernel kv''. This completes the proof of the first assertion.

The map g provides a surjective map $p\colon M(q) \to \Lambda m_1$ and $p^*\colon M(q)^* \to (\Lambda m_1)^*$ is bijective, thus an isomorphism of right Λ-modules. □

6.3. The modules $M(\alpha)$ with $\alpha \in k$. We consider now the modules $M(\alpha)$ in general, and relate them to the left ideals Λm_α, and to the right ideals $m_\alpha\Lambda$. Let us denote by U_α the twosided ideal generated by m_α, it is 3-dimensional with basis $m_\alpha,\ yx,\ xz$. Actually, for any $\alpha \in k$, the right ideal $m_\alpha\Lambda$ is equal to U_α (but we prefer to write U_α instead of $m_\alpha\Lambda$ when we consider it as a left module). For $\alpha \neq 1$, the left ideal Λm_α is equal to U_α.

If M is a module and $m \in M$, we denote by $r(m)\colon {}_\Lambda\Lambda \to M$ the right multiplication by m (defined by $r(m)(\lambda) = \lambda m$. Similarly, if N is a right Λ-module and $a \in N$, let $l(a)\colon \Lambda_\Lambda \to N$ be the left multiplication by a.

We denote by $u_\alpha\colon \Lambda m_\alpha \to \Lambda$ and $u'_\alpha\colon m_\alpha\Lambda \to \Lambda$ the canonical embeddings.

(1) *The right ideal $m_\alpha\Lambda$ is 3-dimensional (and equal to U_α), for all $\alpha \in k$.*

(2) *The left ideal Λm_α is 3-dimensional (and equal to U_α), for $\alpha \in k \setminus \{1\}$, whereas Λm_1 is 2-dimensional.*

(3) *We have $M(\alpha) \simeq \Lambda/U_\alpha$ for all $\alpha \in k$.*

Proof. The map $r(v) : \Lambda \to M(\alpha)$ is surjective (thus a projective cover) and

$$r(v)(m_\alpha) = m_\alpha v = (x - \alpha y)v = xv - \alpha yv = \alpha v' - \alpha v' = 0.$$

Thus, $\Lambda m_\alpha \subseteq \operatorname{Ker}(r(v))$. Also, $xz \in \operatorname{Ker}(r(v))$, thus $\operatorname{Ker}(r(v)) = U_\alpha$. This shows that $M(\alpha)$ is isomorphic to Λ/U_α. □

(4) *For $\alpha \in k \setminus \{1\}$, we have $M(q\alpha) \simeq \Lambda m_\alpha$.*

Proof. Consider the map $r(m_\alpha) : \Lambda \to \Lambda m_\alpha$. Since $r(m_\alpha)(m_{q\alpha}) = m_{q\alpha}m_\alpha = 0$, we see that $U_{q\alpha} \subseteq \operatorname{Ker}(r(m_\alpha))$. For $\alpha \neq 1$, the module Λm_α is 3-dimensional, therefore $r(m_\alpha)$ yields an isomorphism $\Lambda/U_{q\alpha} \to \Lambda m_\alpha$. Using (3) for $M(q\alpha)$, we see that $M(q\alpha) \simeq \Lambda/U_{q\alpha} \simeq \Lambda m_\alpha$. □

(5) *For any map $f\colon \Lambda m_\alpha \to \Lambda$, there is $\lambda \in \Lambda$ with $f = r(\lambda)u_\alpha$, for all $\alpha \in k$. Thus u_α is a left* $\operatorname{add}(\Lambda)$*-approximation.*

Proof. Let $f\colon \Lambda m_\alpha \to \Lambda$ be any map. Let $f(m_\alpha) = c_1x + c_2y + c_3z + c_4yx + c_5xz$ with $c_i \in k$. Since $f(ym_\alpha) = f(yx)$ and $yf(m_\alpha) = c_1yx$, we see that $f(yx) = c_1yx$. Since $f(xm_\alpha) = f(-\alpha xy) = q\alpha f(yx) = q\alpha c_1yx$ and $xf(m_\alpha) = c_2xy + c_3xz = -qc_2yx + c_3xz$, we see that $q\alpha c_1yx = -qc_2yx + c_3xz$, therefore $c_2 = -\alpha c_1$ and $c_3 = 0$. Thus, $f(m_\alpha) = c_1(x - \alpha y) + c_4yx + c_5xz$ belongs to $U_\alpha = m_\alpha\Lambda$, say $f(m_\alpha) = m_\alpha\lambda$ with $\lambda \in \Lambda$. Therefore $f(m_\alpha) = m_\alpha\lambda = r(\lambda)u_\alpha(m_\alpha)$, but this means that $f = r(\lambda)u_\alpha$. □

6.4. Lemma. *Let $\alpha \in k \setminus \{1\}$. Then there is an $\Omega\mho$-sequence*

$$0 \to M(q\alpha) \to \Lambda \to M(\alpha) \to 0.$$

Proof. According to (3), $M(\alpha) \simeq \Lambda/U_\alpha$. Since $\alpha \neq 1$, we have $U_\alpha = \Lambda m_\alpha$ by (2). Thus, we have the following exact sequence

$$0 \to \Lambda m_\alpha \xrightarrow{u_\alpha} \Lambda \to M(\alpha) \to 0$$

According to (5) the embedding $u_\alpha \colon \Lambda m_\alpha \to \Lambda$ is a left add(Λ)-approximation. Thus, the sequence is an $\Omega\mho$-sequence. Finally, (4) shows that $\Lambda m_\alpha \simeq M(q\alpha)$. □

Corollary. *If $o(q) = \infty$, then the module $M(q)$ is semi-Gorenstein-projective.*

Proof. We assume that $o(q) = \infty$. Then $q^t \neq 1$ for all $t \geq 1$. By 6.4, all the sequences

$$0 \to M(q^{t+1}) \to \Lambda \to M(q^t) \to 0.$$

with $t \geq 1$ are $\Omega\mho$-sequences. They can be concatenated in order to obtain a minimal projective resolution of $M(q)$. This shows that $M(q)$ is semi-Gorenstein-projective. □

6.5. The right modules $m_\alpha\Lambda$ and $M(\alpha)^*$. We have started in 6.3 to present essential properties of the modules $M(\alpha)$. We look now also at the modules $m_\alpha\Lambda$ and $M(\alpha)^*$. We hope that the use of consecutive numbers will be helpful.

(6) $\Omega(m_{q\alpha}\Lambda) = m_\alpha\Lambda$ *for all $\alpha \in k$.*

Proof. We consider the composition of the following right Λ-module maps

$$\Lambda_\Lambda \xrightarrow{l(m_\alpha)} \Lambda_\Lambda \xrightarrow{l(m_{q\alpha})} \Lambda_\Lambda$$

Since $m_{q\alpha}m_\alpha = 0$, the composition is zero. The image of $l(m_\alpha)$ is the right ideal $m_\alpha\Lambda$, the image of $l(m_{q\alpha})$ is the right ideal $m_{q\alpha}\Lambda$. Both right ideals are 3-dimensional, thus the sequence is exact. Thus $m_\alpha\Lambda = \operatorname{Ker}(p)$, for a surjective map $p \colon \Lambda_\Lambda \to m_{q\alpha}\Lambda$. Now p is a projective cover, thus $\operatorname{Ker}(p) = \Omega(m_{q\alpha}\Lambda)$, and therefore $\Omega(m_{q\alpha}\Lambda) \simeq m_\alpha\Lambda$. □

(7) $(\Lambda m_\alpha)^* = m_\alpha\Lambda$ *for all $\alpha \in k$.*

Proof. First, let us show that $(\Lambda m_\alpha)^*$ is 3-dimensional. On the one hand, besides u_α, there are homomorphisms $\Lambda m_\alpha \to \Lambda$ with image kyx and with image kxz, which shows that $(\Lambda m_\alpha)^*$ is at least 3-dimensional. According to (5), any homomorphism $\Lambda m_\alpha \to \Lambda$ maps into $\Lambda m_\alpha \Lambda = U_\alpha$. Since U_α is 3-dimensional, we have $\dim \operatorname{Hom}({}_\Lambda\Lambda, U_\alpha) = 3$, therefore $\dim(\Lambda m_\alpha)^* = \dim \operatorname{Hom}(\Lambda m_\alpha, \Lambda) = \dim \operatorname{Hom}(\Lambda m_\alpha, U_\alpha) \leq \dim \operatorname{Hom}({}_\Lambda\Lambda, U_\alpha) = 3$.

Second, using again (5), we see that $(\Lambda m_\alpha)^*$ is, as a right Λ-module, generated by u_α. Thus, there is a surjective right-module homomorphism $\theta_\alpha \colon \Lambda_\Lambda \to (\Lambda m_\alpha)^*$ defined by $\theta_\alpha(1) = u_\alpha$. We have

$$(\theta_\alpha(m_{q^{-1}\alpha}))(m_\alpha) = (\theta_\alpha(1)m_{q^{-1}\alpha})(m_\alpha) = (u_\alpha m_{q^{-1}\alpha})(m_\alpha) = m_\alpha m_{q^{-1}\alpha} = 0,$$

therefore $\theta_\alpha(m_{q^{-1}\alpha}) = 0$. It follows that θ_α yields a surjective map $\Lambda_\Lambda/m_{q^{-1}\alpha}\Lambda \to (\Lambda m_\alpha)^*$. Actually, this map has to be an isomorphism, since $m_{q^{-1}\alpha}\Lambda$ is 3-dimensional. Therefore $\Lambda_\Lambda/m_{q^{-1}\alpha}\Lambda \simeq (\Lambda m_\alpha)^*$. By (6), $\Lambda_\Lambda/m_{q^{-1}\alpha}\Lambda \simeq m_\alpha\Lambda$. This completes the proof. □

(8) *$M(q\alpha)^* = m_\alpha\Lambda$ for all $\alpha \in k$.*

Proof. For $\alpha \neq 1$, we have $M(q\alpha) \simeq \Lambda m_\alpha$ by (4), thus we use (7). For $\alpha = 1$, we use 6.2 and then (7). □

Let us stress that (7) and (8) show that $M(q)^*$ and $(\Lambda m_1)^*$ are isomorphic, namely isomorphic to $m_1\Lambda$, whereas $M(q)$ and Λm_1 themselves are not isomorphic.

(9) *Let $\alpha \in k \setminus \{1, q\}$. For any homomorphism $g\colon m_\alpha\Lambda \to \Lambda$ there is $\lambda \in \Lambda$ with $g = l(\lambda)u'_\alpha$. Thus, u'_α is a left* $\operatorname{add}(\Lambda)$*-approximation.*

Proof: Let $g\colon m_\alpha\Lambda \to \Lambda_\Lambda$ be a homomorphism. We claim that $g(m_\alpha) \in \Lambda m_\alpha$. Let $g(m_\alpha) = c_1x+c_2y+c_3z+c_4yx+c_5xz$ with $c_i \in k$. Now, $g(m_\alpha x) = g(-\alpha yx) = -\alpha g(yx)$ and $g(m_\alpha)x = c_2xy + c_3xz$. Also, $g(m_\alpha y) = g(xy) = -qg(yx)$, and $g(m_\alpha)y = c_1xy + c_3xz = -c_1qyx + c_3xz$, thus $g(yx) = -q^{-1}g(m_\alpha y) = -q^{-1}(-c_1qyx + c_3xz) = c_1yx - q^{-1}c_3xz$. It follows that $c_2yx + c_3xz = -\alpha g(yx) = -\alpha(c_1yx - q^{-1}c_3xz) = -\alpha c_1yx + \alpha q^{-1}c_3xz$. Therefore $c_2 = -\alpha c_1$ and $c_3 = \alpha q^{-1}c_3$. Since we assume that $\alpha \neq q$, it follows that $c_3 = 0$. Therefore $g(m_\alpha) = c_1x - \alpha c_1y + c_3z + c_4yx + c_5xz = c_1(x - \alpha y) + c_4yx + c_5xz$ belongs to U_α. Since we also assume that $\alpha \neq 1$, we have $U_\alpha = \Lambda m_\alpha$. Thus $g(m_\alpha) \in \Lambda m_\alpha$.

As a consequence, there is $\lambda \in \Lambda$ with $g(m_\alpha) = \lambda m_\alpha$, therefore $g(m_\alpha) = \lambda m_\alpha = l(\lambda)u'_\alpha(m_\alpha)$. It follows that $g = l(\lambda)u'_\alpha$. □

6.6. Lemma. *Let $\alpha \in k \setminus \{1, q\}$. Then there is an $\Omega\mho$-sequence of right Λ-modules*

$$0 \to m_\alpha\Lambda \xrightarrow{u'_\alpha} \Lambda_\Lambda \to m_{q\alpha}\Lambda \to 0.$$

Proof. This is 6.5 (6) and (9). □

6.7. Proof of Theorem 1.5. According to 6.5 (8), we have $M(q)^* = m_1\Lambda$. As we know from 6.3, $M(q)$ is not torsionless.

We assume now that $o(q) = \infty$. The Corollary in 6.4 shows that $M(q)$ is semi-Gorenstein-projective. Since $q^{-t} \neq 1$ for all $t \geq 1$, the sequences

$$0 \to m_{q^{-t}}\Lambda \xrightarrow{u'_\alpha} \Lambda_\Lambda \to m_{q^{-t+1}}\Lambda \to 0.$$

with $t \geq 1$ are $\Omega\mho$-sequences, by 6.5. They can be concatenated in order to obtain a minimal projective resolution of $m_1\Lambda$ and show that $m_1\Lambda$ is semi-Gorenstein-projective.

Finally, we want to show that $M(q)^{**} = \Omega M(1)$. According to 6.3 (5), the map $u_1\colon \Lambda m_1 \to \Lambda$ is a minimal left $\operatorname{add}(\Lambda)$-approximation, thus we may consider as in 2.4 (a) the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda m_1 & \xrightarrow{u_1} & \Lambda & \xrightarrow{\pi_1} & \Lambda/\Lambda m_1 & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle\phi} & & \\
0 & \longrightarrow & (\Lambda m_1)^{**} & \longrightarrow & \Lambda & \xrightarrow{\pi_1^{**}} & (\Lambda/\Lambda m_1)^{**} & \longrightarrow & \operatorname{Ext}^1(M'(q)^*, \Lambda)
\end{array}$$

where $\phi = \phi_{\Lambda/\Lambda m_1}$. The submodule $xz(\Lambda/\Lambda m_1)$ belongs to the kernel of any map $\Lambda/\Lambda m_1) \to \Lambda$, and it is the kernel of the map $p\colon \Lambda/\Lambda m_1 \to M(1)$ defined by $p(\overline{1}) = v$. This shows that $xz(\Lambda/\Lambda m_1)$ is the kernel of ϕ, thus the image of ϕ is just $M(1)$. But the image of ϕ coincides with the image of π_1^{**}. In this way, we see that $(\Lambda m_1)^{**}$ is the kernel of a projective cover of $M(1)$, thus equal to $\Omega M(1)$.

Of course, $\Omega M(1)$ is decomposable, namely isomorphic to $\Lambda m_1 \oplus kxz$. □

6.8. Proof of Addendum 1.5. We denote by $q^{\mathbb{Z}}$ the set of elements of k which are of the form q^i with $i \in \mathbb{Z}$. Assume that $\alpha \in k \setminus q^{\mathbb{Z}}$, thus $q^t\alpha \neq 1$ for all $t \in \mathbb{Z}$. According to 6.4, all the sequences

$$0 \to M(q^{t+1}\alpha) \to \Lambda \to M(q^t\alpha) \to 0.$$

with $t \in \mathbb{Z}$ are $\Omega\mho$-sequences. They can be concatenated in order to obtain a complete projective resolution for $M(\alpha)$, thus $M(\alpha)$ is Gorenstein-projective.

The following lemma shows that there are infinitely many elements $\alpha \in k \setminus q^{\mathbb{Z}}$.

Lemma. *Assume that k is an infinite field and $q \in k$. Then $k \setminus q^{\mathbb{Z}}$ is an infinite set.*

We include a proof. The assertion is clear if $o(q)$ is finite. Thus, let $o(q)$ be infinite (in particular, $q \neq 0$). Assume that the multiplicative group $k^* = k\setminus\{0\}$ is cyclic, say $k^* = w^{\mathbb{Z}}$. Then $o(w) = \infty$, and each element in k^* different from 1 has infinite multiplicative order. Since $(-1)^2 = 1$, we see that k is of characteristic 2. Now $w+1 \neq 0$ shows that $w+1 = w^n$ for some $n > 1$, thus w is algebraic over the prime field $\mathbb{Z}_2$. Thus $k = \mathbb{Z}_2(w)$ is a finite field, a contradiction. Since k^* is not cyclic, there is $a \in k^* \setminus q^{\mathbb{Z}}$. Then $a \cdot q^{\mathbb{Z}}$ is an infinite subset of $k^* \setminus q^{\mathbb{Z}}$. □

7. Further details for $\Lambda = \Lambda(q)$.

7.1. The $\Omega\mho$-components involving modules $M(\alpha)$. *The only $\Omega\mho$-sequences which involve a module of the form $M(\alpha)$ with $\alpha \in k$ are those exhibited in 6.4.*

Proof. We show that there is no $\Omega\mho$-sequence ending in $M(1)$ and no $\Omega\mho$-sequence starting in $M(q)$. Since $\Omega M(1)$ is decomposable, there is no $\Omega\mho$-sequence ending in $M(1)$. By 6.2, the module $M(q)$ is not torsionless, thus there is no $\Omega\mho$-sequence starting in $M(q)$.

Let us assume that $o(q) = \infty$ (for the case of $o(q)$ being finite, see 7.4). *There are three kinds of $\Omega\mho$-components which involve modules of the form $M(\alpha)$ with $\alpha \in k$. There is one component of the form $-\mathbb{N}$, it has $M(q)$ as its source, and there is one component of the form $\mathbb{N}$, it has $M(1)$ as its sink:*

$$\leftarrow\text{-}\text{-}\, M(q^4) \leftarrow\text{-}\text{-}\, M(q^3) \leftarrow\text{-}\text{-}\, M(q^2) \leftarrow\text{-}\text{-}\, M(q) \qquad M(1) \leftarrow\text{-}\text{-}\, M(q^{-1}) \leftarrow\text{-}\text{-}\, M(q^{-2}) \leftarrow\text{-}\text{-}$$

The remaining ones (containing the modules $M(\alpha)$ with $\alpha \notin q^{\mathbb{Z}}$) are of the form $\mathbb{Z}$:

$$\leftarrow\text{-}\text{-}\, M(q^4\alpha) \leftarrow\text{-}\text{-}\, M(q^3\alpha) \leftarrow\text{-}\text{-}\, M(q^2\alpha) \leftarrow\text{-}\text{-}\, M(q\alpha) \leftarrow\text{-}\text{-}\, M(\alpha) \leftarrow\text{-}\text{-}\, M(q^{-1}\alpha) \leftarrow\text{-}\text{-}\, M(q^{-2}\alpha) \leftarrow\text{-}\text{-}$$

For laziness, we have labeled the vertices just M, and not $[M]$. The positions of the reflexive modules are shaded.

According to 4.5, there are the following observations concerning the behavior of the modules $M(\alpha)$ with $\alpha \in k$.

The module $M(\alpha)$ is Gorenstein-projective iff $\alpha \notin q^{\mathbb{Z}}$.

The module $M(\alpha)$ is not Gorenstein-projective, but semi-Gorenstein-projective iff $\alpha = q^t$ for some $t \geq 1$.

The module $M(\alpha)$ is torsionless iff $\alpha \neq q$.

The module $M(\alpha)$ is reflexive iff $\alpha \notin \{q, q^2\}$.

It seems worthwhile to know the canonical maps $\phi_X \colon X \to X^{**}$ for the non-reflexive modules $X = M(q)$ and $X = M(q^2)$. For $M(q)$ we refer to 6.7: there it is shown that *$M(q)^{**} = \Omega M(1)$ and that the image of $\phi_{M(q)}$ is Λm_1.*

It remains to look at $X = M(q^2)$. *The module $M(q^2)^{**}$ is the the submodule $\Lambda m_q + \Lambda z$ of Λ and $\phi_{M(q^2)}$ is the inclusion map*

$$M(q^2) = \Lambda m_q \longrightarrow \Lambda m_q + \Lambda z = M(q^2)^{**}.$$

Proof. Since $M(q^2)$ is torsionless, the map $\phi_{M(q^2)}$ is injective. There is the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(q^2) & \xrightarrow{u_q} & \Lambda & \xrightarrow{\pi_q} & M(q) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{M(q^2)}} & & \| & & \downarrow{\scriptstyle \phi_{M(q)}} & & \\
0 & \longrightarrow & M(q^2)^{**} & \xrightarrow{u_q^{**}} & \Lambda & \xrightarrow{\pi_q^{**}} & M(q)^{**} & \longrightarrow & \operatorname{Ext}^1(M(q^2)^*, \Lambda)
\end{array}$$

As we know already, the image of $\phi_{M(q)}$ and therefore of π_q^{**}, is Λm_1. Thus the kernel of π_q^{**} is the submodule $\Lambda m_q + \Lambda z$ of Λ. Therefore $M(q^2)^{**} = \Lambda m_q + \Lambda z$ and $\phi_{M(q^2)}$ is the inclusion map $M(q^2) = \Lambda m_q \longrightarrow \Lambda m_q + \Lambda z = M(q^2)^{**}$. $\square$

7.2. The $\Omega\mho$-components involving the right modules $m_\alpha\Lambda$. We assume again that $o(q) = \infty$. *The $\Omega\mho$-sequences which involve a right module of the form $m_\alpha\Lambda$ with $\alpha \in k$ are those exhibited in 6.6 as well as*

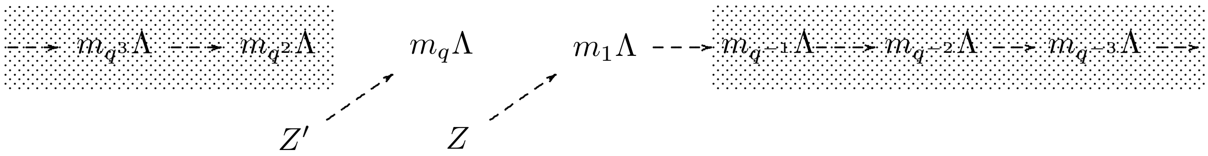
$$0 \to m_1\Lambda \xrightarrow{\begin{bmatrix} u_1' \\ h \end{bmatrix}} \Lambda_\Lambda \oplus \Lambda_\Lambda \to Z \to 0, \qquad 0 \to m_q\Lambda \xrightarrow{\begin{bmatrix} u_q' \\ h' \end{bmatrix}} \Lambda_\Lambda \oplus \Lambda_\Lambda \to Z' \to 0,$$

where $h \colon m_1\Lambda \to \Lambda_\Lambda$ is defined by $h(m_1) = xz$ and $h' \colon m_q\Lambda \to \Lambda_\Lambda$ is defined by $h'(m_q) = z$.

Proof. It is easy to check that the maps $\begin{bmatrix} u_1 \\ h \end{bmatrix}$ and $\begin{bmatrix} u_q \\ h' \end{bmatrix}$ are minimal left $\operatorname{add}(\Lambda_\Lambda)$-approximations. Clearly, the corresponding cokernels Z and Z' are not torsionless.

In addition, we have to show that there is no $\Omega\mho$-sequence ending in $m_{q^2}\Lambda$ or in $m_q\Lambda$. But this follows from the fact that the inclusion maps $u_q' \colon m_q\Lambda = \Omega(m_{q^2}\Lambda) \to P(m_{q^2}\Lambda)$ and $u_1' \colon m_1\Lambda = \Omega(m_q\Lambda) \to P(m_q\Lambda)$ are not $\operatorname{add}(\Lambda_\Lambda)$-approximations.

There are four kinds of $\Omega\mho$-components involving right modules of the form $m_\alpha\Lambda$ with $\alpha \in k$, namely a component of the form $\mathbb{N}$ with $m_{q^2}\Lambda$ as a sink, a component of the form $-\mathbb{N}$ with Z as a source, and a component of the form $\mathbb{A}_2$ with sink $m_q\Lambda$ and source Z':



The remaining $\Omega\mho$-components (containing the right modules $m_\alpha\Lambda$ with $\alpha \in k \setminus q^{\mathbb{Z}}$) are of the form $\mathbb{Z}$:

For the convenience of the reader, the pictures in 7.1 and 7.2 have been arranged so that the A-duality is respected. Thus, in 7.1, the arrows are drawn from right to left, in 7.2 from left to right. Also we recall from 6.3 (8) that the A-dual of $M(q\alpha)$ is $m_\alpha\Lambda$, therefore the position of $m_\alpha\Lambda$ in the pictures 7.2 is the same as the position of $M(q\alpha)$ in 7.1.

We complete the description of the behavior of the modules $M(\alpha)$ started in 7.1.

The module $M(\alpha)$ is not Gorenstein-projective, but $M(\alpha)^*$ is semi-Gorenstein-projective, iff $\alpha = q^t$ for some $t \leq 1$.

Proof. According to 7.1, the module $M(\alpha)$ is Gorenstein-projective iff $\alpha \notin q^{\mathbb{Z}}$. Thus, we can assume that $\alpha = q^t$ for some $t \in \mathbb{Z}$. According to 6.3 (8), the module $M(q^t)$ is isomorphic to $m_{q^{t-1}}\Lambda$. The display of the $\Omega\mho$-components shows that $m_{q^{t-1}}\Lambda$ is semi-Gorenstein-projective iff $t-1 \leq 0$, thus iff $t \leq 1$, see 4.5. □

7.3. We have mentioned already that one may use the algebra $\Lambda = \Lambda(q)$ with $o(q) = \infty$ in order to exhibit examples of modules M which satisfy precisely two of the following three properties (G1), (G2) and (G3).

(1) *$M = M(q)$ satisfies* (G1), (G2), *but not* (G3).
(2) *$M = M(q^3)$ satisfies* (G1), (G3), *but not* (G2).
(3) *$M = M(1)$ satisfies* (G2), (G3), *but not* (G1).

Proof: For (1): this is the main assertion of Theorem 1.5. For (2): see 7.1 and 7.2. For (3): according to 7.1, $M(1)$ is reflexive, but not Gorenstein-projective. According to 6.3 (8), we have $M(1)^* = m_{q^{-1}}\Lambda$ and 7.2 shows that $m_{q^{-1}}\Lambda$ is semi-Gorenstein-projective. □

Let us look for similar examples for Λ^{op}, thus, for right Λ-modules N.

(1*) *There is* **no** *right Λ-module of the form $N = m_\alpha\Lambda$ satisfying* (G1), (G2), *but not* (G3).
(2*) *The right Λ-module $N = m_{q^{-2}}\Lambda$ satisfies* (G1), (G3), *but not* (G2).
(3*) *The right module $N = m_{q^2}\Lambda$ satisfies* (G2), (G3), *but not* (G1).

Proof: (2*) In the $\Omega\mho$-quiver, there starts an infinite path at $N = m_{q^{-2}}\Lambda$, thus N satisfies (G1). There ends a path of length 2 at N, thus N satisfies (G3). Of course, N^* cannot be semi-Gorenstein-projective, since otherwise N would be Gorenstein-projective.

(3*) Let $N = m_{q^2}\Lambda$. According to 6.5 (8), $N = M(q^3)^*$. As we know from 7.1, $M(q^3)$ is reflexive, thus N is reflexive and $N^* = M(q^3)^{**} = M(q^3)$ is semi-Gorenstein-projective.

(1*) Assume that $N = m_\alpha\Lambda$ and N^* are both semi-Gorenstein-projective. Since N cannot be Gorenstein-projective, it is not reflexive. Thus $\alpha \in \{1, q\}$. Since $[m_q\Lambda]$ is the sink of an $\Omega\mho$-component, $m_\alpha\Lambda$ is not semi-Gorenstein-projective. Thus $\alpha = 1$. But $(m_1\Lambda)^* = M(q)^{**} = \Omega M(1)$, according to 6.5 (8) and Theorem 1.5. As we have mentioned already in the proof 6.7, $\Omega M(1) \simeq \Lambda m_1 \oplus k$, where k is the simple Λ-module. But we claim that k is not semi-Gorenstein-projective, thus $\Omega M(1)$ is not semi-Gorenstein-projective.

Lemma. *The simple Λ-module k is not semi-Gorenstein-projective.*

Proof. Assume that k is semi-Gorenstein-projective. Since ${}^\perp\Lambda$ is closed under extensions, it follows that ${}^\perp\Lambda = \operatorname{mod}\Lambda$. However, as we have seen in 7.1, the modules $M(q^t)$ with $t \leq 0$ are not semi-Gorenstein-projective. $\square$

Actually, we also can show directly that $\operatorname{Ext}^1(k, \Lambda) \neq 0$. We show that the inclusion map $u\colon \operatorname{rad}\Lambda \to \Lambda$ is not a left $\operatorname{proj}(\Lambda)$-approximation. Let $f\colon \operatorname{rad}\Lambda \to \Lambda$ be a homomorphism with $f(z) = yx$ (such a map exists, since z is in the top of $\operatorname{rad}\Lambda$ and yx in its socle. Assume that f factors through u, say $f = r(a)u$ for some $a \in \Lambda$, where $r(a)\colon \Lambda \to \Lambda$ is the right multiplication by a. Thus $r(a)u(z) = za$, but $f(z) = yx$ does not belong to $z\Lambda$. $\square$

7.4. Let us look also at the case when $o(q) = n < \infty$. *There are two kinds of $\Omega\mho$-components which involve modules of the form $M(\alpha)$ with $\alpha \in k$. There is one component of the form $\mathbb{A}_n$, it has $M(q)$ as its source, and $M(1)$ as its sink:*

$$M(1) \dashleftarrow M(q^{n-1}) \dashleftarrow \cdots \dashleftarrow M(q^3) \dashleftarrow M(q^2) \dashleftarrow M(q)$$

The remaining ones (containing the modules $M(\alpha)$ with $\alpha \notin q^{\mathbb{Z}}$) are directed cycles of cardinality n:

$$M(\alpha) \dashleftarrow M(q^{n-1}\alpha) \dashleftarrow \cdots \dashleftarrow M(q^3\alpha) \dashleftarrow M(q^2\alpha) \dashleftarrow M(q\alpha) \dashleftarrow M(\alpha)$$

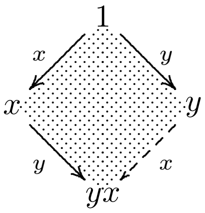
All modules in the cycles are reflexive. In the component of type $\mathbb{A}_n$, the modules $M(q)$ and $M(q^2)$ are not reflexive (they coincide for $o(q) = 1$); for $o(q) \geq 3$, there are $n-2$ additional modules $M(1) = M(q^n),\ M(q^{n-2}),\ \cdots, M(q^4), M(q^3)$ in the component, and these modules are reflexive.

Proof: According to 7.1, the $\Omega\mho$-sequences presented here are the only ones involving modules of the form $M(\alpha)$. Thus, $[M(q)]$ is a sink in the $\Omega\mho$-quiver and $[M(1)]$ is a source. This holds true also for $o(q) = 1$: here $q = 1$ and $[M(1)]$ is both a sink and a source, thus a singleton component (without any arrow). Finally, for $o(q) = n$, the elements $1, q, \dots, q^{n-1}$ are pairwise different, as are the elements $\alpha, q\alpha, \dots, q^{n-1}\alpha$ for any $\alpha \in k \setminus q^{\mathbb{Z}}$. $\square$

We have shown in section 4.1 that any $\Omega\mho$-component is a linearly directed quiver of type $\mathbb{A}_n$ (with $n \geq 1$ vertices), a directed cycle $\widetilde{\mathbb{A}}_n$ (with $n+1 \geq 1$ vertices), or of the form $-\mathbb{N}$, or $\mathbb{N}$, or $\mathbb{Z}$. Now we see in 7.1 and 7.4 that all these cases do arise.

7.5. The quantum exterior algebra $\Lambda' = \Lambda'(q)$ in two variables. Let Λ' be the k-algebra generated by x, y with the relations $x^2,\ y^2,\ xy + qyx$. It has a basis $1,\ x,\ y,$ and yx. We may use the following picture as an illustration:

Λ' :

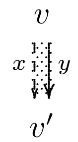


If we factor out the socle of Λ', we obtain the 3-dimensional local algebra Λ'' with radical square zero (it is generated by x, y with relations x^2, y^2, xy, yx).

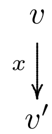
Note that $\Lambda'(q)$ is a subalgebra of $\Lambda(q)$, and that $\Lambda z \Lambda = \Lambda z = \operatorname{span}\{z, xz\}$. The composition $\Lambda' \hookrightarrow \Lambda \twoheadrightarrow \Lambda/\Lambda z \Lambda$ of the canonical maps is an isomorphism of algebras. In this way, the Λ'-modules may be considered as the Λ-modules which are annihilated by z. We should stress that the elements $m_\alpha = x - \alpha y$ (which play a decisive role in our investigation) belong to Λ'.

For $\alpha \in k$, let $M'(\alpha)$ be the Λ'-module with basis v, v', such that $xv = \alpha v'$, $yv = v'$, and $xv' = 0 = yv'$. In addition, we define $M'(\infty)$ as the Λ'-module with basis v, v', such that $xv = v', yv = xv' = yv' = 0$. Here are the corresponding illustrations:

$M'(\alpha)$:



$M'(\infty)$:



The modules $M'(\alpha)$ with $\alpha \in k \cup \{\infty\}$ are pairwise non-isomorphic and indecomposable, and any two-dimensional indecomposable Λ'-module is of this form. In particular, the left ideal $\Lambda' m_\alpha$ is isomorphic to $M'(q\alpha)$, for any $\alpha \in k \cup \{\infty\}$. The essential property of the modules $M'(\alpha)$ is the following: $\Omega_{\Lambda'} M'(\alpha) = M'(q\alpha)$. This follows from the fact that $m_{q\alpha} m_\alpha = 0$ and it is this equality which has been used frequently in sections 6 and 7.

For all $\alpha \in k$, $M(\alpha)$ considered as a Λ'-module, is equal to $M'(\alpha) \oplus k$, where k is the simple Λ'-module. Also, we should stress that $\operatorname{rad} \Lambda$ considered as a left Λ'-module is the direct sum of I and $M'(\infty)$, where I is the indecomposable injective Λ''-module.

7.6. A variation. Let $\widetilde{\Lambda}$ be the algebra defined by the quiver with two vertices, say labeled 1 and 2, with three arrows $1 \to 2$ labeled x, y, z and with three arrows $2 \to 1$, also labeled x, y, z, satisfying the "same" relations as Λ (of course, now we have 14 relations: seven concerning paths $1 \to 2 \to 1$ and seven concerning paths $2 \to 1 \to 2$). Whereas Λ is a local algebra, the algebra $\widetilde{\Lambda}$ is a connected algebra with two simple modules $S(1)$ and $S(2)$ (concentrated at the vertices 1 and 2, respectively).

For all the previous considerations in sections 6 and 7, there are corresponding ones for $\widetilde{\Lambda}$, but always we have to take into account that now we deal with two simple modules $S(1)$ and $S(2)$: Corresponding to the module $M(\alpha)$, there are two different modules, namely $M^1(\alpha)$ with top $S(1)$ and $M^2(\alpha)$ with top $S(2)$. The modules $M^1(\alpha)$ and $M^2(\alpha)$ have similar properties as the Λ-modules $M(\alpha)$, in particular, for $o(q) = \infty$, the modules

$M^1(q)$ and $M^2(q)$ are semi-Gorenstein-projective and not Gorenstein-projective. There is one decisive difference between the Λ-modules and the $\widetilde{\Lambda}$-modules: The endomorphism ring of $M^1(\alpha)$ and $M^2(\alpha)$ is equal to k, whereas the endomorphism ring of any $M(\alpha)$ is 3-dimensional.

8. Questions.

1. One may ask whether or not the finiteness of $\operatorname{gp} A$ implies that A is weakly Gorenstein, There is a weaker question: is A weakly Gorenstein, in case all the Gorenstein-projective A-modules are projective?

2. Is the opposite of a weakly Gorenstein algebra weakly Gorenstein? This is question 1 of Marczinzik [M1]. In terms of the $\Omega\mho$-quiver, the question may be reformulated as follows: Does the existence of an $\Omega\mho$-component of the form $\mathbb{N}$ imply that also an $\Omega\mho$-component of the form $-\mathbb{N}$ exists?

3. Assume that there exists a non-reflexive A-module M such that both M and M^* are semi-Gorenstein-projective. Is then the same true for $A^{\rm op}$? Even for $A = \Lambda(q)$ with $o(q) = \infty$, we do not know the answer. According to 7.3 (1*), a right module N of the form $N = m_\alpha\Lambda$ is reflexive, if both N and N^* are semi-Gorenstein-projective. But, there could exist some other right module N satisfying (G1), (G2) and not (G3).

4. Does there exist a semi-Gorenstein-projective module $M \neq 0$ with $M^* = 0$? Such a module would be an extreme example of a module satisfying (G1), (G2) and not (G3). Marczinzik has pointed out that this question concerns the Nunke condition [H] which asserts that for any non-zero module M there should exist some $i \geq 0$ such that $\operatorname{Ext}^i(M, A) \neq 0$. It is known that the Nunke condition is satisfied in case the finitistic dimension conjecture holds true.

9. References.

[A] M. Auslander. Coherent functors. In: Proceedings of the Conference on Categorical Algebra. La Jolla 1965. Springer, 189–231.

[AB] M. Auslander, M. Bridger. Stable module theory, Mem. Amer. Math. Soc. 94., Amer. Math. Soc., Providence, R.I., 1969.

[AR] M. Auslander, I. Reiten. Applications of contravariantly finite subcategories, Adv. Math. 86(1991), 111-152.

[AR] M. Auslander, I. Reiten, S. O. Smalø. Representation Theory of Artin Algebras. Cambridge studies in advanced mathematics 36. Cambridge University Press. (1995)

[AM] L. L. Avramov, A. Martsinkovsky. Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. 85(3)(2002), 393-440.

[Bel] A. Beligiannis. Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, J. Algebra 288(1)(2005), 137-211.

[Br] M. Bridger. The $\operatorname{Ext}^i_R(M, R)$ and other invariants of M. Brandeis University, Ph.D. 1967, Mathematics.

[Buch] R.-O. Buchweitz. Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Unpublished manuscript, Hamburg (1987), 155pp.

[Chr] L. W. Christensen. Gorenstein Dimensions, Lecture Notes in Math. 1747, Springer-Verlag, 2000.

[Chen] X. W. Chen. Algebras with radical square zero are either self-injective or CM-free, Proc. Amer. Math. Soc. 140(1)(2012), 93-98.

[H] D. Happel. Homological conjectures in representation theory of finite-dimensional algebras. Unpublished. See: https://www.math.uni-bielefeld.de/~sek/dim2/happel2.pdf (retrieved Aug 6, 2018).

[EJ1] E. E. Enochs. O. M. G. Jenda. Gorenstein injective and projective modules, Math. Z. 220(4)(1995), 611-633.

[EJ2] E. E. Enochs. O. M. G. Jenda. Relative homological algebra, De Gruyter Exp. Math. 30. Walter De Gruyter Co., 2000.

[FK] M. Fang, S. König. Endomorphism algebras of generators over symmetric algebras, J. Algebra 332(2011), 428-433.

[JS] D. A. Jorgensen, L. M. Şega. Independence of the total reflexivity conditions for modules, Algebras and Representation Theory 9(2)(2006), 217-226.

[K] B. Keller. Chain complexes and stable categories. Manuscripta Mathematica. 67 (1990), 379417.

[L] T. S. Lam. Lectures on Modules and Rings. Springer (1999)

[LS] S. P. Liu, R. Schulz. The existence of bounded infinite DTr-orbits, Proc. Amer. Math. Soc. 122(1994), 1003-1005.

[MPS] M. Mangeney, C. Peskine, L. Szpiro: Anneaux de Gorenstein, et torsion en algèbre commutative. Séminaire Samuel. Algèbre commutative, tome 1 (1966-1967), p. 2– 69

[M1] R. Marczinzik. Gendo-symmetric algebras, dominant dimensions and Gorenstein homological algebra. arXiv: [math.RT] 1608.04212.

[M2] R. Marczinzik. On stable modules that are not Gorenstein projective, arXiv: [math.RT] 1709.01132v3.

[R1] C. M. Ringel. The Liu-Schulz example, in: Representation Theory of Algebras, CMS Conf. Proc. 18. Providence (1996), 587-600.

[R2] C. M. Ringel. On the representation dimension of artin algebras, Bull. the Institute of Math., Academia Sinica, Vol. 7(1)(2012), 33-70.

[RX] C. M. Ringel, B. L. Xiong. Finite dimensional algebras with Gorenstein-projective nodes. In preparation.

[Y] Y. Yoshino. A functorial approach to modules of G-dimension zero, Illinois J. Math. 49(3)(2005), 345-367.

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