

# Gorenstein-projective and semi-Gorenstein-projective modules

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**Abstract:** Let  $A$  be an artin algebra. An  $A$ -module  $M$  will be said to be semi-Gorenstein-projective provided that  $\text{Ext}^i(M, A) = 0$  for all  $i \geq 1$ . All Gorenstein-projective modules are semi-Gorenstein-projective and only few and quite complicated examples of semi-Gorenstein-projective modules which are not Gorenstein-projective have been known. The aim of the paper is to provide conditions on  $A$  such that all semi-Gorenstein-projective modules are Gorenstein-projective (we call such an algebra left weakly Gorenstein). In particular, we show that in case there are only finitely many isomorphism classes of indecomposable modules which are both semi-Gorenstein-projective and torsionless, then  $A$  is left weakly Gorenstein. On the other hand, we exhibit a 6-dimensional algebra  $\Lambda$  with a semi-Gorenstein-projective module  $M$  which is not torsionless (thus not Gorenstein-projective). Actually, also the  $\Lambda$ -dual module  $M^*$  is semi-Gorenstein-projective module and all the syzygy-modules of  $M$  and  $M^*$  are 3-dimensional, thus the example can be visualized quite easily.

## 1. Introduction.

**1.1. Notations and definitions.** Let  $A$  be an artin algebra. All modules will be finitely generated. Usually, the modules we are starting with will be left modules, but some constructions then yield right modules. Let  $\text{mod } A$  be the category of all finitely generated left  $A$ -modules and  $\text{proj}(A)$  the full subcategory of all projective modules.

If  $M$  is a module, let  $PM$  be a projective cover of  $M$ , and  $\Omega M$  the kernel of the canonical map  $PM \rightarrow M$ . The modules  $\Omega^t M$  with  $t \geq 0$  are called the *syzygy* modules of  $M$ . A module  $M$  is said to be  $\Omega$ -*periodic* provided that there is some  $t \geq 1$  with  $\Omega^t M = M$ .

The right  $A$ -module  $M^* = \text{Hom}(M, A)$  is called the  $A$ -*dual* of  $M$ . Let  $\phi_M: M \rightarrow M^{**}$  be defined by  $\phi_M(m)(f) = f(m)$  for  $m \in M$ ,  $f \in M^*$ . A module  $M$  is said to be *torsionless* provided that  $M$  is a submodule of a projective module, or, equivalently, provided that  $\phi_M$  is injective. A module  $M$  is called *reflexive* provided that  $\phi_M$  is bijective.

Let  $\text{Tr } M$  be the cokernel of  $f^*$ , where  $f$  is a minimal projective presentation of  $M$  (this is the canonical map  $P(\Omega M) \rightarrow PM$ ). Note also here, that  $\text{Tr } M$  is a right  $A$ -module, called the *transpose* of  $M$ .

Recall that a map  $f: M \rightarrow M'$  is said to be *left minimal* provided that any map  $h: M' \rightarrow M'$  with  $hf = f$  is an automorphism [AR]. A left  $\text{proj}(A)$ -approximation will be called *minimal* provided that it is left minimal. We denote by  $\mathcal{U}M$  the cokernel of a minimal left  $\text{proj}(A)$ -approximation of  $M$ .

A *complete projective resolution* is a (double infinite) exact sequence

$$P^\bullet : \quad \dots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \dots$$

of projective left  $A$ -modules, such that  $\text{Hom}_A(P^\bullet, A)$  is again exact. A module  $M$  is *Gorenstein-projective*, if there is a complete projective resolution  $P^\bullet$  with  $M$  isomorphic to the image of  $d^0$ .

A module  $M$  is said to be *semi-Gorenstein-projective* provided that  $\text{Ext}^i(M, A) = 0$  for all  $i \geq 1$ . Note that all Gorenstein-projective modules are semi-Gorenstein-projective. If  $M$  is semi-Gorenstein-projective, then also  $\Omega M$  is semi-Gorenstein-projective. We denote by  $\text{gp}(A)$  the class of all Gorenstein-projective modules and by  ${}^\perp A$  the class of all semi-Gorenstein-projective modules. As we mentioned already,  $\text{gp}(A) \subseteq {}^\perp A$ . We propose to call an artin algebra  $A$  *left weakly Gorenstein* provided that  ${}^\perp A = \text{gp}(A)$ , thus provided that any semi-Gorenstein-projective module is Gorenstein-projective.

**1.2.** It is well-known that a semi-Gorenstein-projective module  $M$  is Gorenstein-projective if and only if  $\text{Tr } M$  is semi-Gorenstein-projective, if and only if  $M$  is reflexive and  $M^*$  is semi-Gorenstein-projective (we will use our approach in order to include a proof, see 4.5). We want to give various characterizations of the left weakly Gorenstein algebras.

**Theorem.** *Let  $A$  be an artin algebra. The following statements are equivalent:*

- (1)  *$A$  is left weakly Gorenstein.*
- (2) *Any semi-Gorenstein-projective module is torsionless.*
- (3) *Any semi-Gorenstein-projective module is reflexive.*
- (4) *For any semi-Gorenstein-projective module  $M$ , the map  $\phi_M$  is surjective.*
- (5) *For any semi-Gorenstein-projective module  $M$ , the module  $M^*$  is semi-Gorenstein projective.*
- (6) *Any semi-Gorenstein-projective module  $M$  satisfies  $\text{Ext}^1(M^*, A_A) = 0$ .*
- (7) *Any semi-Gorenstein-projective module  $M$  satisfies  $\text{Ext}^1(\text{Tr } M, A_A) = 0$ .*

**1.3.** The second result concerns artin algebras with finitely many semi-Gorenstein-projective modules or with finitely many torsionless modules.

**Theorem.** *If the number of isomorphism classes of indecomposable modules which are both semi-Gorenstein-projective and torsionless is finite, then  $A$  is left weakly Gorenstein and any indecomposable non-projective semi-Gorenstein-projective module is  $\Omega$ -periodic.*

This combines two different directions of thoughts. First of all, Yoshino [Y] has shown that for certain commutative rings  $R$  (in particular all artinian commutative rings) the finiteness of the number of isomorphism classes of indecomposable semi-Gorenstein-projective  $R$ -modules implies that  $R$  is left weakly Gorenstein. Here we show the corresponding assertion for artin algebras. Second, according to Marczinzik [M1], all torsionless-finite artin algebras (these are the artin algebras with only finitely many isomorphism classes of torsionless indecomposable modules) are left weakly Gorenstein (note that a lot of interesting classes of artin algebras are known to be torsionless-finite, see 3.6).

**1.4.** Let  $\mathcal{C}$  be an extension-closed full subcategory of  $\text{mod } A$ . The embedding of  $\mathcal{C}$  into  $\text{mod } A$  provides an exact structure on  $\mathcal{C}$ ; this exact structure will be called its *canonical* exact structure (for the basic properties of exact structures, see for example the Appendix A of [K]).

An exact category  $\mathcal{F}$  is called a *Frobenius category* provided that it has enough projective and enough injective objects and that the projective objects in  $\mathcal{F}$  are just the injective

objects in  $\mathcal{F}$ . Denote by  $\mathcal{P}(\mathcal{F})$  (and by  $\mathcal{I}(\mathcal{F})$ ) the full subcategory of the projective (respectively injective) objects in  $\mathcal{F}$ .

**Theorem.** *Let  $\mathcal{F}$  be an extension-closed full subcategory of  $\text{mod } A$  such that  $\mathcal{F}$  is a Frobenius category with respect to its canonical exact structure. If  $\mathcal{P}(\mathcal{F}) \subseteq {}^\perp A \subseteq \mathcal{F}$ , then  $\mathcal{F} = \text{gp}(A) = {}^\perp A$ .*

A (full) subcategory  $\mathcal{C}$  of  $\text{mod } A$  is said to be *resolving* provided that it contains all the projective modules and is closed under extensions, direct summands and kernels of surjective maps.

**Corollary 1.** *Let  $A$  be an artin algebra and  $\mathcal{F}$  a resolving subcategory of  $\text{mod } A$  with  ${}^\perp A \subseteq \mathcal{F}$ . Assume that  $\mathcal{F}$  with its canonical exact structure is a Frobenius subcategory. Then  $\text{gp}(A) = {}^\perp A = \mathcal{F}$ .*

Note that  ${}^\perp A$  and  $\text{gp}(A)$  are resolving subcategories and that  $\text{gp}(A)$  with its canonical exact structure is always Frobenius. Thus, an immediate consequence is a further characterization of the left weakly Gorenstein algebras.

**Corollary 2.** *An artin algebra  $A$  is left weakly Gorenstein if and only if  ${}^\perp A$  with its canonical exact structure is a Frobenius subcategory.*

We should remark that the corollaries follow already from a result in [ZX] which asserts that  $\text{gp}(A)$  is the largest resolving Frobenius subcategory of  $\text{mod } A$ .

**1.5.** The first example of a semi-Gorenstein-projective module which is not Gorenstein-projective was constructed by Jorgensen and Şega [JS] in 2006, for a commutative algebra of dimension 8. Recently, Marczinik [M2] constructed some non-commutative algebras with semi-Gorenstein-projective modules which are not Gorenstein-projective (using the Liu-Schulz example [LS,R1] and gendo-symmetric algebras [FK]). In 6.1, we will exhibit a class of 6-dimensional  $k$ -algebras  $\Lambda(q)$  with parameter  $q \in k \setminus \{0\}$  and a family  $M(\alpha)$  of 3-dimensional indecomposable  $\Lambda(q)$ -modules (with  $\alpha \in k$ ) in order to find new examples:

**Theorem.** *Let  $\Lambda(q)$  be the algebra defined in 6.1. If the multiplicative order of  $q$  is infinite, then the  $\Lambda$ -modules  $M(q)$  and  $M(q)^*$  both are semi-Gorenstein-projective, but  $M(q)$  is not torsionless, thus not Gorenstein-projective; all the syzygy modules  $\Omega^t M(q)$  and  $\Omega^t(M(q)^*)$  with  $t \geq 0$  are 3-dimensional and indecomposable; the module  $M(q)^{**} \simeq \Omega M(1)$  is also 3-dimensional, but decomposable.*

**Addendum.** *For any  $q$ , the  $\Lambda(q)$ -modules  $M(\alpha)$  with  $\alpha \in k \setminus q^{\mathbb{Z}}$  are Gorenstein-projective. Thus, if  $k$  is infinite, then there are infinitely many isomorphism classes of 3-dimensional Gorenstein-projective modules.*

As a consequence, we see the independence (as pointed out in [JŞ]) of the following three conditions:

- (G1) the module  $M$  is semi-Gorenstein-projective;
- (G2) the  $A$ -dual  $M^*$  of  $M$  is semi-Gorenstein-projective;
- (G3) the module  $M$  is reflexive.

Namely, if an  $A$ -module  $M$  (such as the  $\Lambda(q)$ -module  $M(q)$ ) satisfies (G1) and (G2), but not (G3), the module  $\Omega^2 M$  satisfies (G1),(G3), but not (G2), and the  $A^{\text{op}}$ -module  $N = (\Omega^2 M)^*$  satisfies (G2), (G3), but not (G1), see 4.6. Actually, for our example

$A = \Lambda(q)$ , there is also an  $A$ -module which satisfies (G2), (G3), but not (G1), namely the module  $M' = M(1)$ , see 7.3.

**1.6. Outline of the paper.** The proofs of theorem 1.2, 1.3 and 1.4 will be given in sections 2, 3 and 5, respectively. The main tool will be what we call approximation sequences: these are the exact sequences  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  with  $Y$  projective and  $\text{Ext}^1(Z, A) = 0$ , see section 2. Also, we will stress the relevance of the cokernel  $\mathcal{U}M$  of a minimal left  $\text{proj}(A)$ -approximation of a module  $M$  (the notation  $\mathcal{U}M$ , pronounced "agemo"  $M$ , should be a reminder that the construction  $\mathcal{U}$  has to be considered as a kind of inverse of  $\Omega$ ). Of special interest are the approximation sequences with both  $X$  and  $Z$  indecomposable and non-projective; in this case, we have  $X = \Omega Z$  and  $Z = \mathcal{U}X$ , thus we call them  $\Omega\mathcal{U}$ -sequences, see section 3. In section 4 we will introduce the  $\Omega\mathcal{U}$ -quiver of  $A$ . Its vertices are the isomorphism classes  $[M]$  of the indecomposable non-projective  $A$ -modules  $M$  and there is an arrow from  $[Z]$  to  $[X]$  provided that  $X = \Omega Z$  and  $\text{Ext}^1(Z, A) = 0$ , thus provided that there exists an  $\Omega\mathcal{U}$ -sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ . The components of the  $\Omega\mathcal{U}$ -quiver are linearly oriented quivers and directed cycles (see 4.1) and they allow to identify certain types of modules, in particular the torsionless and the reflexive modules, as well as the Gorenstein projective and the semi-Gorenstein-projective modules. An essential ingredient in this setting seems to be Lemma 4.4 which shows that the kernel of the canonical map  $\mathcal{U}^{i-1}M \rightarrow (\mathcal{U}^{i-1}M)^{**}$  is equal to  $\text{Ext}^i(\text{Tr } M, A)$ .

In sections 6 and 7, we present a 6-dimensional algebra  $\Lambda = \Lambda(q)$  and analyze some 3-dimensional representations which we will denote by  $M(\alpha)$  with  $\alpha \in k$ . The essential properties of the modules  $M(\alpha)$  can be found in sections 6.3 and 6.5 and are labeled by (1) to (9). The properties (1) to (5) in 6.3 are those which are needed in order to exhibit a module (namely  $M(q)$ ) which is semi-Gorenstein-projective, but not torsionless, see 6.4. The remaining properties (6) to (9) show, in particular, that also the  $\Lambda$ -dual  $M(q)^*$  of  $M(q)$  is semi-Gorenstein-projective. Altogether, the proof of Theorem 1.5 and its Addendum is given in 6.7 and 6.8. In 7.1 and 7.2, we describe some of the components of the  $\Omega\mathcal{U}$ -quivers of the algebras  $\Lambda$  and  $\Lambda^{\text{op}}$ .

The final section 8 mentions four open questions.

**1.7. Terminology.** We end the introduction with some remarks concerning the terminology and its history. The usual reference for the introduction of the class of Gorenstein-projective modules are the Memoirs by Auslander and Bridger [AB] from 1969, where they appear under the name *modules of Gorenstein dimension zero*. Actually, Bridger, in his 1967 thesis [Br], attributes the concept of the Gorenstein dimension to Auslander: In January 1967, Auslander gave four lectures at the Séminaire Pierre Samuel [MPS] where he discussed the class of all reflexive modules  $M$  such that both  $M$  and  $M^*$  are semi-Gorenstein-projective modules and denotes it by  $G(A)$  (see [MPS], Definition 3.2.2), thus  $G(A)$  is the class of the Gorenstein-projective modules. In [AB], Proposition 3.8, it is shown that a module  $M$  belongs to  $G(A)$  if and only if both  $M$  and  $\text{Tr } M$  are semi-Gorenstein-projective. Of course, these investigations concern finitely generated modules over a **commutative** noetherian ring  $A$ , however all the essential consideration in [MPS, Br, AB] are formulated for general abelian categories with enough projectives. Enochs and Jenda [EJ1, EJ2] reformulated the definition of Gorenstein-projective modules in terms of complete projective resolutions. Several other names for the Gorenstein-projective mod-

ules are in use, they are also called "totally reflexive" modules [AM] and "maximal Cohen-Macaulay" modules [Buch, Bel].

We should apologize that we propose a new name for the modules  $M$  with  $\text{Ext}^i(M, A) = 0$  for all  $i \geq 1$ , namely "semi-Gorenstein-projective". These modules have been called for example Cohen-Macaulay modules or stable modules. However, in our opinion, the name "Cohen-Macaulay module" is in conflict with its established use for commutative rings, and the wording "balanced" may be too vague as a proper identifier. We hope that the name "semi-Gorenstein-projective" describes well what is going on: that there is something like a half of a complete projective resolution ("semi" means "half"). We also propose the name "left weakly Gorenstein" for an algebra  $A$  with  $\text{gp}(A) = {}^\perp A$  (in contrast to "nearly Gorenstein" in [M2]). Of course, a Gorenstein algebra  $A$  satisfies  $\text{gp}(A) = {}^\perp A$ , but the algebras with  $\text{gp}(A) = {}^\perp A$  seem to be quite far away from being Gorenstein.

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## 2. Approximation sequences. Proof of Theorem 1.2.

**2.1. Lemma.** *Let  $\epsilon: 0 \rightarrow X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \rightarrow 0$  be an exact sequence with  $Y$  projective.*

*Then the following conditions are equivalent:*

- (i)  $\omega$  is a left  $\text{proj}(A)$ -approximation.
- (ii)  $\text{Ext}^1(Z, A) = 0$ .
- (iii) The  $A$ -dual sequence  $\epsilon^*$  of  $\epsilon$  is exact.

A sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  with  $Y$  projective satisfying the equivalent properties will be called an *approximation sequence*.

Proof of the equivalence of the properties. Since  $Y$  is projective, applying  $\text{Hom}(-, A)$  to  $\epsilon$  we get the exact sequence  $0 \rightarrow Z^* \xrightarrow{\pi^*} Y^* \xrightarrow{\omega^*} X^* \rightarrow \text{Ext}^1(Z, A) \rightarrow 0$ . Note that  $\omega$  is a left  $\text{proj}(A)$ -approximation if and only if  $\omega^*$  is surjective. From this we get the equivalence of (i) and (ii) and the equivalence of (ii) and (iii).  $\square$

Typical examples of approximation sequences: If  $M$  is torsionless, with minimal left  $\text{proj}(A)$ -approximation  $\omega: X \rightarrow P$ , then

$$0 \rightarrow M \xrightarrow{\omega} P \rightarrow \cup M \rightarrow 0$$

is an approximation sequence (these are the approximation sequences  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  such that  $Z$  has to non-zero projective direct summand). If  $\text{Ext}^1(M, A) = 0$ , then

$$0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0$$

is an approximation sequence (these are the approximation sequences  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  such that  $X$  has to non-zero projective direct summand).

**2.2.** The following basic lemma is well-known (see, for example [R2]).

**Lemma.** *Let  $P_{-1} \xrightarrow{f} P_0 \xrightarrow{g} P_1$  be an exact sequence of projective modules and let  $g = up$  be a factorization with  $p: P_0 \rightarrow I$  epi and  $u: I \rightarrow P_1$  mono. Then  $P_{-1}^* \xleftarrow{f^*} P_0^* \xleftarrow{g^*} P_1^*$  is exact if and only if  $u$  is a left  $\text{proj}(A)$ -approximation.*

For the convenience of the reader, we insert the proof. Since  $f^*g^* = (gf)^* = 0$ , we have  $\text{Im } g^* \subseteq \text{Ker } f^*$ . Assume now that  $u$  is a left  $\text{proj}(A)$ -approximation and let  $h \in \text{Ker } f^*$ , thus  $hf = 0$ . Since  $p$  is a cokernel of  $f$ , there is  $h'$  with  $h = h'p$ . Since  $u$  is a left  $\text{add}(A)$ -approximation, there is  $h''$  with  $h' = h''u$ . Thus  $h = h'p = h''up = h''g = g^*(h'')$  belongs to the image of  $g^*$ , there also  $\text{Ker } f^* \subseteq \text{Im } g^*$ .

Conversely, we assume that  $\text{Im } g^* = \text{Ker } f^*$  and let  $h: I \rightarrow A$  be a map. Then  $hpf = 0$ , so that  $f^*(hp) = 0$ . Therefore  $hp$  belongs to  $\text{Ker } f^*$ , thus to  $\text{Im } g^*$ . There is  $h'' \in P_1^*$  with  $hp = g^*(h'') = h''g = h''up$ , and therefore  $h = h''u$ .  $\square$

This Lemma will be used in various settings, see 4.3.

### 2.3. A semi-Gorenstein-projective and $\Omega$ -periodic module is Gorenstein-projective.

Proof. Let  $M$  be semi-Gorenstein-projective and assume that  $\Omega^t M = M$  for some  $t \geq 1$ . Let  $\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$  be a minimal projective resolution of  $M$ . Then

$$(+) \quad 0 \rightarrow \Omega^t M \rightarrow P_{t-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

is the concatenation of approximation sequences. Since  $\Omega^t M = M$ , we can concatenate countably many copies of (+) in order to obtain a double infinite acyclic chain complex of projective modules. As a concatenation of approximation sequences, it is a complete projective resolution. Therefore,  $M$  is Gorenstein projective.  $\square$

### 2.4. Here are two essential observations.

(a) Let  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an approximation sequence. Then  $\phi_X$  is surjective if and only if  $Z$  is torsionless. We can also say:  $X$  is reflexive if and only if  $Z$  is torsionless.

(b) Let  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an approximation sequence. Then  $\text{Ext}^1(X^*, A) = 0$  if and only if  $\phi_Z$  is surjective.

Proof of (a) and (b). Since  $0 \rightarrow X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \rightarrow 0$  is an approximation sequence, it follows that

$$0 \longrightarrow Z^* \xrightarrow{\pi^*} Y^* \longrightarrow X^* \longrightarrow 0$$

is an exact sequence of right  $A$ -modules. This induces an exact sequence

$$0 \longrightarrow X^{**} \longrightarrow Y^{**} \xrightarrow{\pi^{**}} Z^{**} \longrightarrow \text{Ext}_A^1(X^*, A) \longrightarrow 0$$

of left  $A$ -modules, and we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \downarrow \phi_X & & \parallel & & \downarrow \phi_Z & & \\ 0 & \longrightarrow & X^{**} & \longrightarrow & Y^{**} & \xrightarrow{\pi^{**}} & Z^{**} & \longrightarrow & \text{Ext}_A^1(X^*, A) & \longrightarrow & 0. \end{array}$$

By the Snake Lemma, the kernel of  $\phi_Z$  is isomorphic to the cokernel of  $\phi_X$ , Thus  $\phi_Z$  is a monomorphism if and only if  $\phi_X$  is an epimorphism. Since  $X$  is torsionless,  $X$  is reflexive if and only if  $\phi_X$  is surjective. This is (a).

By the commutative diagram above, we see that  $\phi_Z$  is epic if and only if so is  $\pi^{**}$ , and if and only if  $\text{Ext}_A^1(X^*, A) = 0$ . This is (b).  $\square$

**Corollary.** *A module  $X$  is reflexive if and only if both  $X$  and  $\mathcal{U}X$  are torsionless.*

Proof. If  $X$  is reflexive, then it is torsionless. Thus we may assume from the beginning that  $X$  is torsionless. Any minimal left  $\text{add}(A)$ -approximation  $X \rightarrow Y$  is injective and its cokernel is  $\mathcal{U}X$ . The exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow \mathcal{U}X \rightarrow 0$  is an approximation sequence, and 2.4 (a) asserts that  $X$  is reflexive iff  $\mathcal{U}X$  is torsionless.  $\square$

**2.5. Lemma.** *Let  $M$  be an indecomposable non-projective module with  $\text{Ext}^i(M, A) = 0$  for  $i = 1, 2$ . Then  $\text{Tr } M = (\Omega^2 M)^*$ .*

Proof: Let  $\pi: PM \rightarrow M$  and  $\pi': P\Omega M \rightarrow \Omega M$  be projective covers with inclusion maps  $\omega: \Omega M \rightarrow PM$  and  $\omega': \Omega^2 M \rightarrow P\Omega M$ . Then  $\omega\pi'$  is a minimal projective presentation of  $M$ . By definition,  $\text{Tr } M$  is the cokernel of  $(\omega\pi')^*$ . Since  $\text{Ext}^i(M, A) = 0$  for  $i = 1, 2$ , the exact sequences

$$0 \rightarrow \Omega^2 M \xrightarrow{\omega'} P\Omega M \xrightarrow{\pi'} \Omega M \rightarrow 0, \quad 0 \rightarrow \Omega M \xrightarrow{\omega} PM \xrightarrow{\pi} M \rightarrow 0$$

are approximation sequences. As a consequence, the corresponding  $A$ -dual sequences

$$0 \leftarrow (\Omega^2 M)^* \xleftarrow{(\omega')^*} (P\Omega M)^* \xleftarrow{(\pi')^*} (\Omega M)^* \leftarrow 0, \quad 0 \leftarrow (\Omega M)^* \xleftarrow{\omega^*} (PM)^* \xleftarrow{\pi^*} M^* \leftarrow 0$$

are exact. The concatenation

$$0 \leftarrow (\Omega^2 M)^* \xleftarrow{(\omega')^*} (P\Omega M)^* \xleftarrow{(\omega\pi')^*} (PM)^* \xleftarrow{\pi^*} M^* \leftarrow 0$$

shows that also  $(\Omega^2 M)^*$  is a cokernel of  $(\omega\pi')^*$ , thus  $\text{Tr } M = (\Omega^2 M)^*$ .  $\square$

## 2.6. Proof of Theorem 1.

(1) implies (2) to (7): This follows directly from well-known properties of Gorenstein-projective modules. Namely, assume (1) and let  $M$  be Gorenstein-projective. Then  $M$  is reflexive, this yields (3), but, of course, also (2) and (4). Second,  $M^*$  is Gorenstein-projective, thus semi-Gorenstein-projective, therefore we get (5) and (6). Finally,  $\text{Tr } M$  is Gorenstein-projective, thus semi-Gorenstein-projective, therefore we get (7).

Both (3) and (4) imply (2): Let  $M$  be semi-Gorenstein-projective. Consider the approximation sequence  $0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0$  and note that  $\Omega M$  is again semi-Gorenstein-projective. If (3) or just (4) holds, we know that  $\phi_{\Omega M}$  is surjective, thus by 2.4 (a),  $M$  is torsionless.

Both (6) and (7) imply (2): Let  $M$  be semi-Gorenstein-projective. Consider the approximation sequences  $0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0$  and  $0 \rightarrow \Omega^2 M \rightarrow P\Omega M \rightarrow \Omega M \rightarrow 0$ . Since  $M$  is semi-Gorenstein-projective, also  $\Omega^2 M$  is semi-Gorenstein-projective. If (6) holds, we use (6) for  $\Omega^2 M$  in order to see that  $\text{Ext}^1((\Omega^2 M)^*, A) = 0$ . If (7) holds, we use (7) for  $M$  in order to see that  $\text{Ext}^1(\text{Tr } M, A) = 0$ . According to 2.5, we see that

$\text{Tr } M = (\Omega^2 M)^*$ . Thus in both cases (6) and (7), we have  $\text{Ext}^1((\Omega^2 M)^*, A) = 0$ . According to 2.4 (b), it follows from  $\text{Ext}^1((\Omega^2 M)^*, A) = 0$  that  $\phi_{\Omega M}$  is surjective. By 2.4 (a),  $M$  is torsionless.

Trivially, (5) implies (6). Altogether we have shown that any one of the assertions (3) to (7) implies (2).

It remains to show that (2) implies (1). Let  $M$  be semi-Gorenstein-projective and torsionless. We want to show that  $M$  is Gorenstein-projective. Let  $M_i = \mathcal{U}^i M$  for all  $i \geq 0$  (with  $M_0 = M$ ). Since  $M_0$  is torsionless, there is an approximation sequence  $0 \rightarrow M_0 \rightarrow P_1 \rightarrow M_1 \rightarrow 0$ , and  $M_1$  is again semi-Gorenstein-projective. By assumption,  $M_1$  is again torsionless. Inductively, starting with a torsionless module  $M_i$ , we obtain an approximation sequence  $\epsilon_i: 0 \rightarrow M_i \rightarrow P_{i+1} \rightarrow M_{i+1} \rightarrow 0$ , we conclude that with  $M_i$  also  $M_{i+1}$  is semi-Gorenstein-projective. By (2) we see that  $M_{i+1}$  is torsionless, again. Concatenating a minimal projective resolution of  $M$  with these approximation sequences  $\epsilon_i$ , for  $0 \leq i$ , we obtain a complete projective resolution of  $M$ .  $\square$

### 3. $\Omega\mathcal{U}$ -sequences. Proof of theorem 1.3.

**3.1.** An approximation sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  will be called an  $\Omega\mathcal{U}$ -sequence provided that both  $X$  and  $Z$  are indecomposable and not projective (the relevance of such sequences was stressed already in [RX]).

**Lemma.** *An approximation sequence is the direct sum of  $\Omega\mathcal{U}$ -sequences and of sequences of the form  $0 \rightarrow P \xrightarrow{1} P \rightarrow 0 \rightarrow 0$  and  $0 \rightarrow 0 \rightarrow P \xrightarrow{1} P \rightarrow 0$  with  $P$  indecomposable projective.*

Proof: Let  $0 \rightarrow X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \rightarrow 0$  be an approximation sequence. Since  $Y$  is projective and  $\pi$  is surjective, a direct decomposition  $Z = Z_1 \oplus Z_2$  yields a direct sum decomposition of the sequence.

Since  $\omega$  is a left  $\text{proj}(A)$ -approximation, there is also the corresponding assertion: If  $X = X_1 \oplus X_2$ , then  $X \xrightarrow{\omega} Y$  is the direct sum of two maps  $X_1 \rightarrow Y_1$  and  $X_2 \rightarrow Y_2$ , thus again we obtain a direct sum decomposition of the sequence. This shows that for an indecomposable approximation sequence  $0 \rightarrow X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \rightarrow 0$ , the modules  $X$  and  $Z$  are indecomposable or zero (and, of course, not both can be zero).

If  $Z$  is indecomposable and projective, then the sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  splits off  $0 \rightarrow 0 \rightarrow Z \xrightarrow{1} Z \rightarrow 0$ , thus  $X = 0$ . Similarly, if  $X$  is indecomposable and projective, then the sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  splits off  $0 \rightarrow X \xrightarrow{1} X \rightarrow 0 \rightarrow 0$ , thus  $Z = 0$ .

It remains that  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is an approximation sequence with both  $X$  and  $Z$  being indecomposable and non-projective.  $\square$

**3.2. Lemma.** *Let  $\epsilon: 0 \rightarrow X \xrightarrow{\omega} Y \xrightarrow{\pi} Z \rightarrow 0$  be an exact sequence. The following conditions are equivalent:*

- (i)  $\epsilon$  is an  $\Omega\mathcal{U}$ -sequence.
- (ii)  $X$  and  $Z$  are indecomposable and not projective,  $\omega$  is a minimal left  $\text{proj}(A)$ -approximation,  $\pi$  is a projective cover,  $X = \Omega Z$ ,  $Z = \mathcal{U}X$ .
- (iii)  $X$  is indecomposable and not projective,  $\omega$  is a minimal left  $\text{proj}(A)$ -approximation.
- (iv)  $Z$  is indecomposable and not projective,  $\pi$  is a projective cover, and  $\text{Ext}^1(Z, A) = 0$ .



- (v)  $X = \Omega Z$ ,  $Y$  is projective,  $Z = \mathcal{U}X$ , and  $X$  is indecomposable.  
(vi)  $X = \Omega Z$ ,  $Y$  is projective,  $Z = \mathcal{U}X$ , and  $Z$  is indecomposable.

Proof: In (ii) we have collected all the relevant properties of an  $\Omega\mathcal{U}$ -sequence. (i) implies (ii): Let  $\epsilon$  be an  $\Omega\mathcal{U}$ -sequence. Then  $\omega$  has to be minimal, since otherwise  $\epsilon$  would split off a non-zero sequence of the form  $0 \rightarrow 0 \rightarrow P \xrightarrow{1} P \rightarrow 0$ . Similarly,  $\pi$  has to be a projective cover, since otherwise  $\epsilon$  would split off a non-zero sequence of the form  $0 \rightarrow P \xrightarrow{1} P \rightarrow 0 \rightarrow 0$ . Since  $\omega$  is a minimal left  $\text{proj}(A)$ -approximation and  $Z$  is the cokernel of  $\omega$ , we see that  $Z = \mathcal{U}X$ . Since  $\pi$  is a projective cover of  $Z$  and  $X$  is its kernel,  $X = \Omega Z$ .

The condition (iii), (iv), (v) and (vi) single out some of these properties, thus (ii) implies these conditions.

(iii) implies (i): Since  $X$  is indecomposable and not projective,  $\epsilon$  has no direct summand  $0 \rightarrow P \xrightarrow{1} P \rightarrow 0 \rightarrow 0$ . Since  $\omega$  is left minimal,  $\epsilon$  has no direct summand  $0 \rightarrow 0 \rightarrow P \xrightarrow{1} P \rightarrow 0$ .

Similarly, (iv) implies (i).

Both (v) and (vi) imply (i): Since  $Z = \mathcal{U}X$ , we have  $\text{Ext}^1(Z, A) = 0$ . This shows that the sequence is an approximation sequence. Since  $X = \Omega Z$ , the sequence  $\epsilon$  has no direct summand of the form  $0 \rightarrow P \xrightarrow{1} P \rightarrow 0 \rightarrow 0$ . Since  $Z = \mathcal{U}X$ , the sequence  $\epsilon$  has no direct summand of the form  $0 \rightarrow 0 \rightarrow P \xrightarrow{1} P \rightarrow 0$ . Thus,  $\epsilon$  is a direct sum of  $\Omega\mathcal{U}$ -sequences. Finally, since  $X$  or  $Z$  is indecomposable,  $\epsilon$  is an  $\Omega\mathcal{U}$ -sequence.  $\square$

**3.3. Corollary.** *If  $M$  is indecomposable, non-projective, semi-Gorenstein-projective, also  $\Omega M$  is indecomposable, non-projective, semi-Gorenstein-projective and  $M = \mathcal{U}\Omega M$ .*

Proof. Since  $M$  is semi-Gorenstein-projective module, the canonical sequence  $\epsilon: 0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0$  is an approximation sequence. Since  $M$  is indecomposable and not projective, and  $PM \rightarrow M$  is a projective cover,  $\epsilon$  is an  $\Omega\mathcal{U}$ -sequence, thus  $\Omega M$  is indecomposable and non-projective, and  $M = \mathcal{U}\Omega M$ , by 3.2. Of course, with  $M$  also  $\Omega M$  is semi-Gorenstein-projective.  $\square$

**3.4. Lemma.** *If the number of isomorphism classes of indecomposable modules which are both semi-Gorenstein-projective and torsionless is finite, then any indecomposable non-projective semi-Gorenstein-projective module is  $\Omega$ -periodic.*

Proof. According to 3.3, The modules  $\Omega^t M$  with  $t \geq 1$  are indecomposable modules which are torsionless and semi-Gorenstein-projective. Since there are only finitely many isomorphism classes of indecomposable torsionless semi-Gorenstein-projective modules is finite, there are natural numbers  $1 \leq s < t$  with  $\Omega^s M = \Omega^t M$ . Then  $M = \mathcal{U}^s \Omega^s M = \mathcal{U}^s \Omega^t M = \Omega^{t-s} M$  and  $t - s \geq 1$ , thus  $M$  is  $\Omega$ -periodic.  $\square$

**3.5. Proof of Theorem 1.3.** We assume that the number of isomorphism classes of indecomposable torsionless semi-Gorenstein-projective modules is finite. According to 3.4, any indecomposable non-projective semi-Gorenstein-projective module is  $\Omega$ -periodic. 2.3 shows that any semi-Gorenstein-projective  $\Omega$ -periodic module is Gorenstein-projective.  $\square$

**3.6. Torsionless-finite algebras.** An artin algebra  $A$  is said to be *torsionless-finite* if there are only finitely many isomorphism classes of indecomposable torsionless modules.

Theorem 1.3 implies that *any torsionless-finite artin algebra is left weakly Gorenstein*, as Marczinzik [M1] has shown. Note that many interesting classes of algebras are known to be torsionless-finite. In particular, we have

*The following algebras are torsionless-finite, and hence left weakly Gorenstein.*

- (1) *Algebras  $A$  such that  $A/\text{soc}({}_A A)$  is representation-finite.*
- (2) *Algebras stably equivalent to hereditary algebras, in particular all algebras with radical square zero.*
- (3) *Minimal representation-infinite algebras.*
- (4) *Special biserial algebras without indecomposable projective-injective modules.*

See for example [R2], where also other algebras are listed which are known to be torsionless-finite.

Chen [Che] has shown that a connected algebra  $A$  with radical square zero either is self-injective, or else all the Gorenstein-projective modules are projective. The assertion that algebras with radical square zero are left weakly Gorenstein complements this result.

#### 4. The $\Omega\mathcal{U}$ -quiver.

**4.1. Definition.** *The  $\Omega\mathcal{U}$ -quiver of  $A$  has as vertices the isomorphism classes  $[X]$  of the indecomposable non-projective modules  $X$  and there is an arrow*

$$[X] \leftarrow \text{-----} [Z]$$

provided that  $X = \Omega Z$  and  $Z = \mathcal{U}X$ , thus provided that there exists an  $\Omega\mathcal{U}$ -sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ . (The arrows of an  $\Omega\mathcal{U}$ -quiver will often be drawn as dashed arrows, in order to stress that they correspond to extensions and not to maps; the direction of these arrows follows the convention used for Ext-quivers.) A component of the  $\Omega\mathcal{U}$ -quiver will be called an  $\Omega\mathcal{U}$ -*component*; a path in the  $\Omega\mathcal{U}$ -quiver will be called an  $\Omega\mathcal{U}$ -*path*.

In the  $\Omega\mathcal{U}$ -quiver, an arrow ending at  $[X]$  starts at  $[\mathcal{U}X]$ ; an arrow starting at  $[Z]$  ends at  $[\Omega Z]$ . Thus, there is the immediate consequence:

**Lemma.** *At any vertex of the  $\Omega\mathcal{U}$ -quiver at most one arrow starts and at most one arrow ends.* □

If  $I \subseteq \mathbb{Z}$  is a subset, we will consider it as a quiver with an arrow from  $z$  to  $z-1$ , provided that both  $z-1$  and  $z$  belong to  $I$ . For example, the interval  $\{1, 2, \dots, n\}$  is the quiver  $\mathbb{A}_n$  with linear orientation (with 1 being the unique sink and  $n$  the unique source).

**Corollary.** *Any  $\Omega\mathcal{U}$ -component is a linearly oriented quiver  $\mathbb{A}_n$  with  $n \geq 1$  vertices, an oriented cycle  $\tilde{\mathbb{A}}_n$  with  $n+1 \geq 1$  vertices, or of the form  $-\mathbb{N}$ , or  $\mathbb{N}$ , or  $\mathbb{Z}$ .*

As we will see in 7.1 and 7.4, all cases mentioned here can arise as  $\Omega\mathcal{U}$ -components. Note that an algebra  $A$  is left weakly Gorenstein provided that there is no  $\Omega\mathcal{U}$ -component of the form  $-\mathbb{N}$ .

An indecomposable non-projective module  $M$  will be said to be of  $\Omega\mathcal{U}$ -*type*  $\Delta$  where  $\Delta \in \{\mathbb{A}_n, \tilde{\mathbb{A}}_n, -\mathbb{N}, \mathbb{N}, \mathbb{Z}\}$  in case the  $\Omega\mathcal{U}$ -component containing  $[M]$  is of the form  $\Delta$ .

## 4.2. The $A$ -dual of an $\Omega\mathcal{U}$ -sequence.

**Lemma.** (a) *Let  $\epsilon: 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an approximation sequence and assume that  $X$  is reflexive. Then  $\text{Ext}^1(X^*, A) = 0$  if and only if  $Z$  is reflexive, if and only if the  $A$ -dual  $\epsilon^*$  of  $\epsilon$  is again an approximation sequence.*

(b) *Let  $\epsilon: 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an  $\Omega\mathcal{U}$ -sequence with  $X$  reflexive. Then  $Z$  is reflexive, if and only if the  $A$ -dual  $\epsilon^*$  of  $\epsilon$  is again an  $\Omega\mathcal{U}$ -sequence.*

*Proof.* (a) By 2.4 (a), we see that  $Z$  is always torsionless. Thus 2.4 (b) shows that  $\text{Ext}^1(X^*, A) = 0$  if and only if  $Z$  is reflexive. First, assume that  $Z$  is reflexive. Then  $\text{Ext}^1(X^*, A) = 0$ , and therefore we see that the  $A$ -dual sequence  $\epsilon^*$  is exact. We dualize a second time: the sequence  $\epsilon^{**}$  is isomorphic to the sequence  $\epsilon$ , since the three modules  $X, Y, Z$  are reflexive. This means that  $\epsilon^{**}$  is exact, and therefore  $\epsilon^*$  is an approximation sequence. Second, conversely, if  $\epsilon^*$  is an approximation sequence, then it is exact, and therefore  $\text{Ext}^1(X^*, A) = 0$ , thus  $Z$  is reflexive.

(b) Assume now that  $\epsilon$  is an  $\Omega\mathcal{U}$ -sequence. First, assume that  $Z$  is reflexive. Since  $X, Z$  both are reflexive, indecomposable and non-projective, also  $X^*$  and  $Z^*$  are indecomposable and non-projective, as we will show below. Thus  $\epsilon^*$  is an  $\Omega\mathcal{U}$ -sequence. Of course, conversely, if  $\epsilon$  is an  $\Omega\mathcal{U}$ -sequence, then it is an approximation sequence and thus  $Z$  is reflexive by (a).  $\square$

We have used some basic facts about the  $A$ -dual  $M^*$  of a module  $M$ .

- (1)  $M^*$  is always torsionless.
- (2) If  $M$  is non-zero and torsionless, then  $M^*$  is non-zero.
- (3) If  $M$  is reflexive, indecomposable and non-projective, then  $M^*$  is reflexive, indecomposable and non-projective.

Here are the proofs (or see for example [L]). (1) There is a surjective map  $u: P \rightarrow M$  with  $P$  projective. Then  $u^*: M^* \rightarrow P^*$  is an embedding of  $M^*$  into the projective module  $P^*$ . (2) is obvious.

(3) Let  $M$  be reflexive, indecomposable and non-projective. Consider a direct decomposition  $M^* = N_1 \oplus N_2$  with  $N_1 \neq 0$  and  $N_2 \neq 0$ . Since  $M^*$  is torsionless by (1), both modules  $N_1$  and  $N_2$  are torsionless, therefore  $N_1^* \neq 0, N_2^* \neq 0$ , thus there is a proper direct decomposition  $M^{**} = N_1^* \oplus N_2^*$ . Since  $M$  is reflexive and indecomposable, this is impossible. Thus  $M^*$  has to be indecomposable. If  $M^*$  is projective, then also  $M^{**}$  is projective. Again, since  $M$  is reflexive, this is impossible.

It remains to show that  $M^*$  is reflexive. But since  $M^{**}$  is isomorphic to  $M$ , we see that  $M^{***}$  is isomorphic to  $M^*$ , thus the canonical map  $M^* \rightarrow M^{***}$  has to be an isomorphism (since it is a monomorphism of modules of equal length).  $\square$

**4.3.** Lemma 2.2 outlines the importance of left  $\text{add}(A)$ -approximations when dealing with exact sequences of projective modules. We have mentioned there that this Lemma will be relevant for our considerations. Now we want to give a unified treatment of the relevance of approximation sequences and of  $\Omega\mathcal{U}$ -sequences.

(a) *An exact sequence  $\cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots$  is a complete projective resolution if and only if it is the concatenation of approximation sequences.*

(a') An indecomposable non-projective module  $M$  is Gorenstein-projective if and only if  $[M]$  is the start of an infinite  $\Omega\mathcal{U}$ -path and the end of an infinite  $\Omega\mathcal{U}$ -path.

(b) A module  $M$  is semi-Gorenstein-projective if and only if a projective resolution (or, equivalently, any projective resolution) is the concatenation of approximation sequences.

(b') An indecomposable non-projective module  $M$  is semi-Gorenstein-projective if and only if  $[M]$  is the start of an infinite  $\Omega\mathcal{U}$ -path.

(c) A module  $M$  is reflexive and  $M^*$  is semi-Gorenstein-projective if and only if there is an exact sequence  $0 \rightarrow M \rightarrow P^1 \rightarrow P^2 \rightarrow \dots$  which is the concatenation of approximation sequences.

(c') An indecomposable non-projective module  $M$  is reflexive and  $M^*$  is semi-Gorenstein-projective if and only if  $[M]$  is the end of an infinite  $\Omega\mathcal{U}$ -path.

Proof: We use that the  $A$ -dual of an approximation sequence is exact, thus the  $A$ -dual of the concatenation of approximation sequences is exact.

(a) Let  $P^\bullet$  be a double infinite exact sequence of projective modules with maps  $d^i: P^i \rightarrow P^{i+1}$ . Write  $d^i = \omega^i \pi^i$  with  $\pi^i$  epi and  $\omega^i$  mono. If  $P^\bullet$  is a complete projective resolution, then the exactness of  $(P^\bullet)^*$  at  $(P^i)^*$  implies that  $\omega^i$  is a left  $\text{proj}(A)$ -approximation, see 2.2. Thus  $P^\bullet$  is the concatenation of approximation sequences.

(b) Let  $\dots \rightarrow P_i \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  be a projective resolution of  $M$ . Write the map  $P_{i+1} \rightarrow P_i$  as  $\omega_i \pi_i$  with  $\pi_i$  epi and  $\omega_i$  mono. If the  $A$ -dual of the sequence  $\dots \rightarrow P_i \rightarrow \dots \rightarrow P_0$  is exact, then all the maps  $\omega_i$  with  $i \geq 1$  have to be left  $\text{proj}(A)$ -approximations. This shows that the projective resolution is the concatenation of approximation sequences.

(b') Let  $M$  be indecomposable, non-projective and semi-Gorenstein-projective. Since  $\text{Ext}^1(M, A) = 0$ , the sequence  $0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0$  is an  $\Omega\mathcal{U}$ -sequence and  $\Omega M$  is again indecomposable and non-projective. Also,  $\Omega M$  is semi-Gorenstein-projective. Thus, we can iterate the procedure and obtain the infinite path

$$(*) \quad \dots \longleftarrow [\Omega^2 M] \longleftarrow [\Omega M] \longleftarrow [M]$$

Conversely, assume that there is an infinite path starting with  $[M]$ , then it is of the form  $(*)$ . Thus, for all  $i \geq 1$ , we have  $\text{Ext}^i(M, A) \simeq \text{Ext}^1(\Omega^{i-1} M, A) = 0$ .

Proof of (c) and (c'). Assume that there are given approximation sequences  $\epsilon_i: 0 \rightarrow M^i \rightarrow P^{i+1} \rightarrow M^{i+1} \rightarrow 0$  for all  $i \geq 0$ , with  $M^0 = M$ . Then all the modules  $M^i$  are torsionless, thus reflexive by 2.4 (a). In particular,  $M$  itself is reflexive. The  $A$ -dual of  $\epsilon_i$  is the sequences

$$\epsilon_i^*: 0 \leftarrow (M^i)^* \leftarrow (P^{i+1})^* \leftarrow (M^{i+1})^* \leftarrow 0,$$

which again is an approximation sequence by 4.2 (a). The concatenation of the sequences  $\epsilon_i^*$  is a projective resolution of  $M^* = (M^0)^*$ . According to (b),  $M^*$  is semi-Gorenstein-projective, since all the sequences  $\epsilon_i^*$  are approximation sequences.

Conversely, assume that  $M$  is reflexive and  $M^*$  is semi-Gorenstein-projective. We want to construct a sequence  $0 \rightarrow M \rightarrow P^1 \rightarrow P^2 \rightarrow \dots$  which is the concatenation of approximation sequences. It is sufficient to consider the case where  $M$  is indecomposable

(in general, take the direct sum of the sequences). If  $M$  is projective, then  $0 \rightarrow M \rightarrow M \rightarrow 0 \rightarrow \dots$  is the concatenation of approximation sequences.

Thus, it remains to consider the case where  $M$  is indecomposable and not projective. Since  $M$  is torsionless, there is an  $\Omega\mathcal{U}$ -sequence  $\epsilon_0: 0 \rightarrow M \rightarrow P^1 \rightarrow M^1 \rightarrow 0$  (with  $M^1 = \mathcal{U}M$ ). Note that  $M^1$  is indecomposable, not projective, and that the  $A$ -dual  $\epsilon_0^*: 0 \leftarrow M^* \leftarrow (P^1)^* \leftarrow (M^1)^* \leftarrow 0$  is exact. Since  $M$  is reflexive,  $M^1$  is torsionless by 2.4 (a). Since  $M^*$  is semi-Gorenstein-projective,  $\text{Ext}^1(M^*, A) = 0$ , therefore  $\phi_{M^1}$  is surjective and  $\epsilon_0^*$  is an  $\Omega\mathcal{U}$ -sequence, by 4.2. Altogether we know now that  $M^1$  is reflexive, but also that  $(M^1)^* = \Omega(M^*)$ . With  $M^*$  also  $\Omega(M^*)$  is semi-Gorenstein-projective.

Thus  $M^1$  satisfies again the assumptions of being indecomposable, not projective, reflexive and that its  $A$ -dual  $(M^1)^*$  is semi-Gorenstein-projective. Thus we can iterate the procedure for getting the next  $\Omega\mathcal{U}$ -sequence  $\epsilon_1: 0 \rightarrow M^1 \rightarrow P^2 \rightarrow M^2 \rightarrow 0$ , with  $M^2 = \mathcal{U}^2M$ , and so on. Altogether, we obtain the infinite path:

$$[M] \dashrightarrow [\mathcal{U}M] \dashrightarrow [\mathcal{U}^2M] \dashrightarrow \dots$$

This completes the proof of (c') and thus also of (c).

(a') This follows immediately from (b') and (c'). □

**4.4. Modules at the end of a path of length  $t$ .** Modules at the end of a path of length 1 or 2 are torsionless, or reflexive, respectively. Now we look at modules which occur at the end of a path of arbitrary length  $t \geq 1$ .

**Proposition.** *Let  $M$  any module and  $t \geq 1$ . The following conditions are equivalent:*

- (i)  $\mathcal{U}^{i-1}M$  is torsionless for  $1 \leq i \leq t$ .
- (ii)  $\text{Ext}^i(\text{Tr } M, A) = 0$  for  $1 \leq i \leq t$ .

*If  $M$  is indecomposable and not projective, then these conditions are equivalent to*

- (iii)  $M$  is the end of an  $\Omega\mathcal{U}$ -path of length  $t$ .

The modules  $M$  with property (ii) have been called  $t$ -torsion free by Auslander in [MPS], Chapter 3, and then by Bridger [Br] and Auslander and Bridger [AB]. Note that a module is 1-torsion free iff it is torsionless, and 2-torsion free iff it is reflexive (see Corollary 2.4). For a discussion of these special cases  $t = 1$  and  $t = 2$ , we may refer to [ARS], Corollary IV.3.3.

Proof of the equivalence of (i) and (ii). We use the following notation: For any module  $M$ , let us denote by  $\text{K}M$  the kernel of  $\phi_M: M \rightarrow M^{**}$ , this is, of course, the kernel of any left  $\text{proj}(A)$ -approximation of  $M$ . Thus, a module  $M$  is torsionless iff  $\text{K}M = 0$ . The following Lemma shows for any  $i \geq 1$ , that  $\mathcal{U}^{i-1}M$  is torsionless iff  $\text{Ext}^i(\text{Tr } M, A) = 0$ . In this way, we see that (ii) and (iii) are equivalent. □

**Lemma.** *Let  $M$  be a module and  $i \geq 1$ . Then*

$$\text{K}(\mathcal{U}^{i-1}M) \simeq \text{Ext}^i(\text{Tr } M, A).$$

*In particular,  $\mathcal{U}^{i-1}M$  is torsionless if and only if  $\text{Ext}^i(\text{Tr } M, A) = 0$ .*

Proof. Let  $P^0 \xrightarrow{f} P^1 \xrightarrow{p^1} M \rightarrow 0$  be a minimal projective presentation of  $M$ , thus  $\text{Tr } M$  is the cokernel of  $f^*$ . We want to extend the presentation  $(P^0)^* \xleftarrow{f^*} (P^1)^*$  of  $\text{Tr } M$  to a projective resolution. We construct a complex of projective modules

$$P^\bullet : \quad P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \dots$$

as follows: Let  $d^0 = f$ , its cokernel is by definition the map  $p^1: P^1 \rightarrow M = \mathcal{U}^0 M$ . For any  $i \geq 1$ , let  $u^i q^i: \mathcal{U}^{i-1} M \rightarrow P^{i+1}$  be a minimal  $\text{proj}(A)$ -approximation with  $q^i$  epi and  $u^i$  mono, thus the kernel of  $q^i$  is just  $\text{K}(\mathcal{U}^{i-1} M)$ . Let  $p^{i+1}: P^{i+1} \rightarrow \mathcal{U}^i M$  be the cokernel of  $u^i$ , thus also of  $d^i = u^i q^i p^i$ . To repeat, the maps  $d^i$  for  $i \geq 1$  are factored as follows:

$$P^i \xrightarrow{p^i} \mathcal{U}^{i-1} M \xrightarrow{q^i} \mathcal{U}^{i-1} M / \text{K}(\mathcal{U}^{i-1} M) \xrightarrow{u^i} P^{i+1}.$$

We claim that the cohomology of  $P^\bullet$  at the position  $i \geq 1$  is

$$H^i(P^\bullet) = \text{K}(\mathcal{U}^{i-1} M).$$

Namely, for  $i = 1$ , by the definition  $d^0 = f$ , we have  $P^1 / \text{Im}(d^0) = P^1 / \text{Im}(f) = M$  and  $\text{Ker}(d^1) / \text{Im}(d^0) = \text{Ker}(q^1 p^1) / \text{Im}(f) = \text{Ker}(q^1) = \text{K} M$ . For  $i \geq 2$ , we have  $\text{Ker}(d^i) / \text{Im}(d^{i-1}) = \text{Ker}(q^i p^i) / \text{Im}(u^{i-1}) = \text{Ker}(q^i) = \text{K}(\mathcal{U}^{i-1} M)$ .

Second, we show that the complex

$$(P^\bullet)^* : \quad (P^0)^* \xleftarrow{(d^0)^*} (P^1)^* \xleftarrow{(d^1)^*} (P^2)^* \xleftarrow{(d^2)^*} \dots$$

is exact. Namely, let  $i \geq 1$  and consider an element  $h: P^i \rightarrow A$  in the kernel of  $(d^{i-1})^*$ , thus  $h d^{i-1} = 0$ . Therefore  $h$  factors through the cokernel  $p^i$  of  $d^{i-1}$ , say  $h = h' p^i$  for some  $h': \mathcal{U}^{i-1} M \rightarrow A$ . Since  $u^i q^i$  is a left  $\text{proj}(A)$ -approximation, we obtain  $h'': P^{i+1} \rightarrow A$  with  $h' = h'' u^i q^i$ . This shows that  $h = h' p^i = h'' u^i q^i p^i = h'' d^i = (d^i)^*(h'')$  is in the image of  $(d^i)^*$ .

Recall that the cokernel of  $f^* = (d^0)^*$  is  $\text{Tr } M$ , thus  $(P^\bullet)^*$  is a projective resolution of  $\text{Tr } M$  and hence  $\text{Ext}^i(\text{Tr } M, A) = H^i((P^\bullet)^{**}) = H^i(P^\bullet)$ , since  $(P^\bullet)^{**} \simeq P^\bullet$ .  $\square$

Proof of the equivalence of (i) and (iii). First, assume that  $M$  is indecomposable and not projective. If (iii) is satisfied, there is an  $\Omega\mathcal{U}$ -path of length  $t$  ending in  $M$ . This path has to be  $\mathcal{U}^t M, \mathcal{U}^{t-1} M, \dots, \mathcal{U} M, M$ . This shows that for any module  $\mathcal{U}^i M$  with  $0 \leq i < t$ , there is an arrow starting in  $\mathcal{U}^i M$ , and therefore  $\mathcal{U}^i M$  has to be torsionless.

Conversely, assume that (i) is satisfied. We show (iii) by induction on  $t$ . For any  $t \geq 1$ , there is the arrow  $\mathcal{U} M \rightarrow M$ , since  $M$  is indecomposable, non-projective and torsionless. According to 3.2, the module  $\mathcal{U} M$  is again indecomposable and non-projective. Thus, if  $t \geq 2$ , we can use induction in order to obtain a path of length  $t - 1$  ending in  $\mathcal{U} M$ , since all the modules  $\mathcal{U}^i(\mathcal{U} M)$  with  $0 \leq i < t - 1$  are torsionless.  $\square$

**4.5. Summary.** Let us collect what can be read out about an indecomposable non-projective module when looking at its position in the  $\Omega\mathcal{U}$ -quiver.

*Let  $M$  be an indecomposable non-projective module.*

- (0)  $[M]$  is an isolated vertex iff  $\text{Ext}^1(M, A) = 0$  and  $M$  is not torsionless.
- (1)  $[M]$  is the start of a path of length  $t \geq 1$  iff  $\text{Ext}^i(M, A) = 0$  for  $1 \leq i \leq t$ .  
In particular:  $[M]$  is the start of an arrow iff  $\text{Ext}^1(M, A) = 0$ .
- (1')  $[M]$  is the start of an infinite path iff  $M$  is semi-Gorenstein-projective.
- (2)  $[M]$  is the end of a path of length  $t \geq 1$  iff  $\text{Ext}^i(\text{Tr } M, A) = 0$  for  $1 \leq i \leq t$  iff  $\mathcal{U}^{i-1}M$  is torsionless for  $1 \leq i \leq t$ .  
In particular:  $[M]$  is the end of an arrow iff  $M$  is torsionless;  
and  $[M]$  is the end of a path of length 2 iff  $M$  is reflexive
- (2')  $[M]$  is the end of an infinite path iff ( $M$  is reflexive and  $M^*$  is semi-Gorenstein-projective) iff  $\text{Tr } M$  is semi-Gorenstein-projective.
- (3)  $[M]$  is the start of an infinite path and also the end of an infinite path iff  $M$  is Gorenstein-projective.  
 $M$  is of  $\Omega\mathcal{U}$ -type  $\mathbb{Z}$  iff  $M$  is Gorenstein-projective and not  $\Omega$ -periodic.  
 $M$  is of  $\Omega\mathcal{U}$ -type  $\mathbb{A}_n$  for some  $n \geq 0$  iff  $M$  is Gorenstein-projective and  $\Omega$ -periodic.
- (4)  $A$ -duality provides a bijection between the isomorphism classes of the reflexive indecomposable  $A$ -modules of type  $\mathbb{A}_n$  and the isomorphism classes of the reflexive indecomposable  $A^{\text{op}}$ -modules of type  $\mathbb{A}_n$ .  
Thus, for any  $n \geq 3$ ,  $A$  has  $\Omega\mathcal{U}$ -components of form  $\mathbb{A}_n$  iff  $A^{\text{op}}$  has  $\Omega\mathcal{U}$ -components of form  $\mathbb{A}_n$ .
- (5)  $A$ -duality provides a bijection between the isomorphism classes of the reflexive indecomposable  $A$ -modules of type  $\mathbb{N}$  and the isomorphism classes of the reflexive indecomposable  $A^{\text{op}}$ -modules of type  $-\mathbb{N}$ .  
Thus,  $A$  has  $\Omega\mathcal{U}$ -components of form  $\mathbb{N}$  iff  $A^{\text{op}}$  has  $\Omega\mathcal{U}$ -components of form  $-\mathbb{N}$ .

Proof. (1) follows from the fact that  $\text{Ext}^t(M, A) = \text{Ext}^{t-1}(\Omega M, A)$  for  $t \geq 2$ . For the special case  $t = 2$ , see Corollary 2.4. (2) is Proposition 4.4. For (1'), (2') and (3), see 4.3. For (4) and (5), we refer to 4.2 (b). Note that in an  $\Omega\mathcal{U}$ -component of the form  $\mathbb{A}_n$  with  $n \geq 3$ , as well as in those of the form  $-\mathbb{N}$ , all but precisely two vertices are the isomorphism classes of reflexive modules, whereas any vertex of an  $\Omega\mathcal{U}$ -component of the form  $\mathbb{N}$  is the isomorphism class of a reflexive module.  $\square$

**Remark. Characterizations of Gorenstein-projective modules.** The  $\Omega\mathcal{U}$ -quiver shows nicely that an indecomposable module  $M$  is Gorenstein-projective if and only if both  $M$  and  $\text{Tr } M$  are semi-Gorenstein-projective, if and only if  $M$  is reflexive and both  $M$  and  $M^*$  are semi-Gorenstein projective: See (1'), (2') and (3).  $\square$

**4.6. Independence.** As we know, a module is Gorenstein-projective if and only if it satisfies the conditions (G1), (G2), (G3) mentioned in 1.5. These three conditions are independent, as Jorgenson and Şega have shown. Section 6 of the present paper provides new (and in our opinion less technical) examples. But we should stress that only one example is essential, namely to exhibit a module  $M$  which satisfies (G1) and (G2), but not (G3):

Let  $M$  be a module which satisfies (G1), (G2) and not (G3). Then  $\Omega^2 M$  satisfies (G1) and (G3), but not (G2), whereas  $N = (\Omega^2 M)^*$  is a right  $A$ -module which satisfies (G2) and (G3), but not (G1).

Proof. Assume that  $M$  satisfies (G1), (G2) and not (G3). Then  $\Omega^2 M$  is reflexive and semi-Gorenstein-projective. By Lemma 2.5,  $N = (\Omega^2 M)^* = \text{Tr } M$ , thus  $N$  is not semi-Gorenstein-projective (otherwise,  $M$  would be Gorenstein-projective). Using 4.5, we see that  $(\Omega^2 M)^*$  is reflexive and  $N^* = (\Omega^2 M)^{**} = \Omega M$  is semi-Gorenstein-projective.  $\square$

**4.7. The constructions  $\Omega$  and  $\mathcal{U}$ .** For any module  $M$ ,  $\Omega M$  is torsionless and  $\mathcal{U}M$  satisfies  $\text{Ext}^1(\mathcal{U}M, A) = 0$ ; in addition,  $\mathcal{U}M$  has no non-zero projective direct summands. If  $Z$  satisfies  $\text{Ext}^1(Z, A) = 0$  and has no non-zero projective direct summand, then  $\mathcal{U}\Omega Z \simeq Z$ . If  $X$  is torsionless and has no non-zero projective direct summand, then  $\Omega\mathcal{U}X \simeq X$ . Thus,  $\Omega$  and  $\mathcal{U}$  are inverse bijections between isomorphism classes as follows:

$$\left\{ \begin{array}{l} \text{indecomposable} \\ \text{non-projective modules } X \\ \text{which are torsionless} \end{array} \right\} \begin{array}{c} \xrightarrow{\mathcal{U}} \\ \xleftarrow{\Omega} \end{array} \left\{ \begin{array}{l} \text{indecomposable} \\ \text{non-projective modules } Z \\ \text{with } \text{Ext}^1(Z, A) = 0 \end{array} \right\}$$

Actually, this is not only a set-theoretical bijection. There is the following categorical version. Let  $\mathcal{L}(A)$  be the full subcategory of all torsionless modules, and  $\mathcal{Z}(A)$  the full subcategory of all modules  $Z$  with  $\text{Ext}^1(Z, A) = 0$ . If  $\mathcal{C}' \subseteq \mathcal{C}$  are full subcategories of  $\text{mod } A$ , let  $\mathcal{C}/\mathcal{C}'$  be the category with the same objects as  $\mathcal{C}$  such that  $\text{Hom}_{\mathcal{C}/\mathcal{C}'}(X, Y)$  is the factor group of  $\text{Hom}_{\mathcal{C}}(X, Y)$  modulo the subspace of all maps  $X \rightarrow Y$  which factor through a direct sum of modules in  $\mathcal{C}'$ .

**Proposition.** *The constructions  $\Omega$  and  $\mathcal{U}$  provide inverse categorical equivalences*

$$\mathcal{L}(A)/\text{proj}(A) \begin{array}{c} \xrightarrow{\mathcal{U}} \\ \xleftarrow{\Omega} \end{array} \mathcal{Z}(A)/\text{proj}(A)$$

The proof is left to the reader.

## 5. Proof of Theorem 1.4.

Since  $\text{add}(A) \subseteq {}^\perp A \subseteq \mathcal{F}$ , we see that  $\text{add}(A) \subseteq \mathcal{P}(\mathcal{F}) = \mathcal{I}(\mathcal{F})$ . Thus  $\text{Ext}_A^1(X, A) = 0$ , for all  $X \in \mathcal{F}$ .

For  $X \in \mathcal{F}$ , there is an exact sequence  $0 \rightarrow K \rightarrow Q \rightarrow X \rightarrow 0$  with  $Q \in \mathcal{P}(\mathcal{F})$  and  $K \in \mathcal{F}$ . By  $\mathcal{P}(\mathcal{F}) \subseteq {}^\perp A$  we have  $Q \in {}^\perp A$ . Thus  $\text{Ext}_A^1(X, A) = 0$  and  $\text{Ext}_A^{m+1}(X, A) = \text{Ext}_A^m(K, A)$  for  $m \geq 1$ . So  $\text{Ext}_A^2(X, A) = 0$ , and in particular  $\text{Ext}_A^2(K, A) = 0$ . Repeating this process we see that  $X \in {}^\perp A$ . Thus  $\mathcal{F} \subseteq {}^\perp A$ , and hence  ${}^\perp A = \mathcal{F}$  is Frobenius.

For  $L \in \mathcal{P}({}^\perp A)$ , consider an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow L \rightarrow 0$  with  $P \in \text{add}(A)$ . Since  $L$  and  $P$  are in  ${}^\perp A$ ,  $K \in {}^\perp A$ . So  $\text{Ext}_A^1(L, K) = 0$ , thus the exact sequence splits and  $L \in \text{add}(A)$ . This shows  $\mathcal{P}({}^\perp A) \subseteq \text{add}(A) \subseteq \mathcal{P}({}^\perp A)$ , and hence  $\mathcal{P}({}^\perp A) = \text{add}(A)$ .

For  $X \in {}^\perp A$ , since  ${}^\perp A$  is Frobenius, there is an exact sequence  $0 \rightarrow X \rightarrow I \rightarrow C \rightarrow 0$  with  $I \in \mathcal{I}({}^\perp A) = \mathcal{P}({}^\perp A) = \text{add}(A)$  and  $C \in {}^\perp A$ . So  $X$  is torsionless. Taking a left  $\text{add}(A)$ -approximation  $X \xrightarrow{f} P^0$  of  $X$ , we get an exact sequence  $0 \rightarrow X \xrightarrow{f}$





**Lemma.** *The intersection of the kernels of all the homomorphisms  $M(q) \rightarrow {}_{\Lambda}\Lambda$  is  $zM(q) = kv''$  and  $M(q)/zM(q) \simeq \Lambda m_1$ . In particular,  $M(q)$  is not torsionless and  $M(q)^* = (\Lambda m_1)^*$ .*

Proof. Let  $f: M(q) = \Lambda v \rightarrow {}_{\Lambda}\Lambda$  be a homomorphism. Let  $f(v) = c_1x + c_2y + c_3z + c_4yx + c_5xz$  with  $c_i \in k$ . By  $qf(v') = f(xv) = xf(v) = -c_2qyx + c_3xz$  and  $f(v') = f(yv) = yf(v) = c_1yx$ , we get  $c_2 = -c_1$  and  $c_3 = 0$ . Thus,  $f(v) = c_1(x - y) + c_4yx + c_5xz$ . It follows that  $f(v'') = f(zv) = zf(v) = 0$ . This shows that  $v''$  is contained in the kernel of any map  $f: M(q) = \Lambda v \rightarrow {}_{\Lambda}\Lambda$ . On the other hand, the homomorphism  $g: M(q) = \Lambda v \rightarrow \Lambda$  given by  $g(v) = x - y = m_1$  has kernel  $kv''$ . This completes the proof of the first assertion.

The map  $g$  provides a surjective map  $p: M(q) \rightarrow \Lambda m_1$  and  $p^*: M(q)^* \rightarrow (\Lambda m_1)^*$  is bijective, thus an isomorphism of right  $\Lambda$ -modules.  $\square$

**6.3. The modules  $M(\alpha)$  with  $\alpha \in k$ .** We consider now the modules  $M(\alpha)$  in general, and relate them to the left ideals  $\Lambda m_\alpha$ , and to the right ideals  $m_\alpha \Lambda$ . Let us denote by  $U_\alpha$  the twosided ideal generated by  $m_\alpha$ , it is 3-dimensional with basis  $m_\alpha, yx, xz$ . Actually, for any  $\alpha \in k$ , the right ideal  $m_\alpha \Lambda$  is equal to  $U_\alpha$  (but we prefer to write  $U_\alpha$  instead of  $m_\alpha \Lambda$  when we consider it as a left module). For  $\alpha \neq 1$ , the left ideal  $\Lambda m_\alpha$  is equal to  $U_\alpha$ .

If  $M$  is a module and  $m \in M$ , we denote by  $r(m): {}_{\Lambda}\Lambda \rightarrow M$  the right multiplication by  $m$  (defined by  $r(m)(\lambda) = \lambda m$ ). Similarly, if  $N$  is a right  $\Lambda$ -module and  $a \in N$ , let  $l(a): {}_{\Lambda}\Lambda \rightarrow N$  be the left multiplication by  $a$ .

We denote by  $u_\alpha: \Lambda m_\alpha \rightarrow \Lambda$  and  $u'_\alpha: m_\alpha \Lambda \rightarrow \Lambda$  the canonical embeddings.

(1) *The right ideal  $m_\alpha \Lambda$  is 3-dimensional (and equal to  $U_\alpha$ ), for all  $\alpha \in k$ .*

(2) *The left ideal  $\Lambda m_\alpha$  is 3-dimensional (and equal to  $U_\alpha$ ), for  $\alpha \in k \setminus \{1\}$ , whereas  $\Lambda m_1$  is 2-dimensional.*

(3) *We have  $M(\alpha) \simeq \Lambda/U_\alpha$  for all  $\alpha \in k$ .*

Proof. The map  $r(v): \Lambda \rightarrow M(\alpha)$  is surjective (thus a projective cover) and

$$r(v)(m_\alpha) = m_\alpha v = (x - \alpha y)v = xv - \alpha yv = \alpha v' - \alpha v' = 0.$$

Thus,  $\Lambda m_\alpha \subseteq \text{Ker}(r(v))$ . Also,  $xz \in \text{Ker}(r(v))$ , thus  $\text{Ker}(r(v)) = U_\alpha$ . This shows that  $M(\alpha)$  is isomorphic to  $\Lambda/U_\alpha$ .  $\square$

(4) *For  $\alpha \in k \setminus \{1\}$ , we have  $M(q\alpha) \simeq \Lambda m_\alpha$ .*

Proof. Consider the map  $r(m_\alpha): \Lambda \rightarrow \Lambda m_\alpha$ . Since  $r(m_\alpha)(m_{q\alpha}) = m_{q\alpha}m_\alpha = 0$ , we see that  $U_{q\alpha} \subseteq \text{Ker}(r(m_\alpha))$ . For  $\alpha \neq 1$ , the module  $\Lambda m_\alpha$  is 3-dimensional, therefore  $r(m_\alpha)$  yields an isomorphism  $\Lambda/U_{q\alpha} \rightarrow \Lambda m_\alpha$ . Using (3) for  $M(q\alpha)$ , we see that  $M(q\alpha) \simeq \Lambda/U_{q\alpha} \simeq \Lambda m_\alpha$ .  $\square$

(5) *For any map  $f: \Lambda m_\alpha \rightarrow \Lambda$ , there is  $\lambda \in \Lambda$  with  $f = r(\lambda)u_\alpha$ , for all  $\alpha \in k$ . Thus  $u_\alpha$  is a left  $\text{add}(\Lambda)$ -approximation.*

Proof. Let  $f: \Lambda m_\alpha \rightarrow \Lambda$  be any map. Let  $f(m_\alpha) = c_1x + c_2y + c_3z + c_4yx + c_5xz$  with  $c_i \in k$ . Since  $f(y m_\alpha) = f(yx)$  and  $y f(m_\alpha) = c_1yx$ , we see that  $f(yx) = c_1yx$ . Since

$f(xm_\alpha) = f(-\alpha xy) = q\alpha f(yx) = q\alpha c_1 yx$  and  $xf(m_\alpha) = c_2 xy + c_3 xz = -qc_2 yx + c_3 xz$ , we see that  $q\alpha c_1 yx = -qc_2 yx + c_3 xz$ , therefore  $c_2 = -\alpha c_1$  and  $c_3 = 0$ . Thus,  $f(m_\alpha) = c_1(x - \alpha y) + c_4 yx + c_5 xz$  belongs to  $U_\alpha = m_\alpha \Lambda$ , say  $f(m_\alpha) = m_\alpha \lambda$  with  $\lambda \in \Lambda$ . Therefore  $f(m_\alpha) = m_\alpha \lambda = r(\lambda)u_\alpha(m_\alpha)$ , but this means that  $f = r(\lambda)u_\alpha$ .  $\square$

**6.4. Lemma.** *Let  $\alpha \in k \setminus \{1\}$ . Then there is an  $\Omega\mathcal{U}$ -sequence*

$$0 \rightarrow M(q\alpha) \rightarrow \Lambda \rightarrow M(\alpha) \rightarrow 0.$$

Proof. According to (3),  $M(\alpha) \simeq \Lambda/U_\alpha$ . Since  $\alpha \neq 1$ , we have  $U_\alpha = \Lambda m_\alpha$  by (2). Thus, we have the following exact sequence

$$0 \rightarrow \Lambda m_\alpha \xrightarrow{u_\alpha} \Lambda \rightarrow M(\alpha) \rightarrow 0$$

According to (5) the embedding  $u_\alpha: \Lambda m_\alpha \rightarrow \Lambda$  is a left  $\text{add}(\Lambda)$ -approximation. Thus, the sequence is an  $\Omega\mathcal{U}$ -sequence. Finally, (4) shows that  $\Lambda m_\alpha \simeq M(q\alpha)$ .  $\square$

**Corollary.** *If  $o(q) = \infty$ , then the module  $M(q)$  is semi-Gorenstein-projective.*

Proof. We assume that  $o(q) = \infty$ . Then  $q^t \neq 1$  for all  $t \geq 1$ . By 6.4, all the sequences

$$0 \rightarrow M(q^{t+1}) \rightarrow \Lambda \rightarrow M(q^t) \rightarrow 0.$$

with  $t \geq 1$  are  $\Omega\mathcal{U}$ -sequences. They can be concatenated in order to obtain a minimal projective resolution of  $M(q)$ . This shows that  $M(q)$  is semi-Gorenstein-projective.  $\square$

**6.5. The right  $\Lambda$ -modules  $m_\alpha \Lambda$  and  $M(\alpha)^*$ .** We have started in 6.3 to present essential properties of the modules  $M(\alpha)$ . We look now also at the modules  $m_\alpha \Lambda$  and  $M(\alpha)^*$ . We hope that the use of consecutive numbers will be helpful.

(6)  $\Omega(m_{q\alpha} \Lambda) = m_\alpha \Lambda$  for all  $\alpha \in k$ .

Proof. We consider the composition of the following right  $\Lambda$ -module maps

$$\Lambda_\Lambda \xrightarrow{l(m_\alpha)} \Lambda_\Lambda \xrightarrow{l(m_{q\alpha})} \Lambda_\Lambda$$

Since  $m_{q\alpha} m_\alpha = 0$ , the composition is zero. The image of  $l(m_\alpha)$  is the right ideal  $m_\alpha \Lambda$ , the image of  $l(m_{q\alpha})$  is the right ideal  $m_{q\alpha} \Lambda$ . Both right ideals are 3-dimensional, thus the sequence is exact. Thus  $m_\alpha \Lambda = \text{Ker}(p)$ , for a surjective map  $p: \Lambda_\Lambda \rightarrow m_{q\alpha} \Lambda$ . Now  $p$  is a projective cover, thus  $\text{Ker}(p) = \Omega(m_{q\alpha} \Lambda)$ , and therefore  $\Omega(m_{q\alpha} \Lambda) \simeq m_\alpha \Lambda$ .  $\square$

(7)  $(\Lambda m_\alpha)^* = m_\alpha \Lambda$  for all  $\alpha \in k$ .

Proof. First, let us show that  $(\Lambda m_\alpha)^*$  is 3-dimensional. On the one hand, besides  $u_\alpha$ , there are homomorphisms  $\Lambda m_\alpha \rightarrow \Lambda$  with image  $kyx$  and with image  $kxz$ , which shows that  $(\Lambda m_\alpha)^*$  is at least 3-dimensional. According to (5), any homomorphism  $\Lambda m_\alpha \rightarrow \Lambda$  maps into  $\Lambda m_\alpha \Lambda = U_\alpha$ . Since  $U_\alpha$  is 3-dimensional, we have  $\dim \text{Hom}(\Lambda_\Lambda, U_\alpha) = 3$ , therefore  $\dim(\Lambda m_\alpha)^* = \dim \text{Hom}(\Lambda m_\alpha, \Lambda) = \dim \text{Hom}(\Lambda m_\alpha, U_\alpha) \leq \dim \text{Hom}(\Lambda_\Lambda, U_\alpha) = 3$ .

Second, using again (5), we see that  $(\Lambda m_\alpha)^*$  is, as a right  $\Lambda$ -module, generated by  $u_\alpha$ . Thus, there is a surjective homomorphism  $\theta_\alpha: \Lambda_\Lambda \rightarrow (\Lambda m_\alpha)^*$  of right  $\Lambda$ -modules defined by  $\theta_\alpha(1) = u_\alpha$ . We have

$$(\theta_\alpha(m_{q^{-1}\alpha}))(m_\alpha) = (\theta_\alpha(1)m_{q^{-1}\alpha})(m_\alpha) = (u_\alpha m_{q^{-1}\alpha})(m_\alpha) = m_\alpha m_{q^{-1}\alpha} = 0,$$

therefore  $\theta_\alpha(m_{q^{-1}\alpha}) = 0$ . It follows that  $\theta_\alpha$  yields a surjective map  $\Lambda_\Lambda/m_{q^{-1}\alpha}\Lambda \rightarrow (\Lambda m_\alpha)^*$ . Actually, this map has to be an isomorphism, since  $m_{q^{-1}\alpha}\Lambda$  is 3-dimensional. Therefore  $\Lambda_\Lambda/m_{q^{-1}\alpha}\Lambda \simeq (\Lambda m_\alpha)^*$ . By (6),  $\Lambda_\Lambda/m_{q^{-1}\alpha}\Lambda \simeq m_\alpha\Lambda$ . This completes the proof.  $\square$

(8)  $M(q\alpha)^* = m_\alpha\Lambda$  for all  $\alpha \in k$ .

Proof. For  $\alpha \neq 1$ , we have  $M(q\alpha) \simeq \Lambda m_\alpha$  by (4), thus we use (7). For  $\alpha = 1$ , we use 6.2 and then (7).  $\square$

Let us stress that (7) and (8) show that  $M(q)^*$  and  $(\Lambda m_1)^*$  are isomorphic, namely isomorphic to  $m_1\Lambda$ , whereas  $M(q)$  and  $\Lambda m_1$  themselves are not isomorphic.

(9) Let  $\alpha \in k \setminus \{1, q\}$ . For any homomorphism  $g: m_\alpha\Lambda \rightarrow \Lambda$  there is  $\lambda \in \Lambda$  with  $g = l(\lambda)u'_\alpha$ . Thus,  $u'_\alpha$  is a left  $\text{add}(\Lambda)$ -approximation.

Proof: Let  $g: m_\alpha\Lambda \rightarrow \Lambda_\Lambda$  be a homomorphism. We claim that  $g(m_\alpha) \in \Lambda m_\alpha$ . Let  $g(m_\alpha) = c_1x + c_2y + c_3z + c_4yx + c_5xz$  with  $c_i \in k$ . Now,  $g(m_\alpha x) = g(-\alpha yx) = -\alpha g(yx)$  and  $g(m_\alpha x) = c_2xy + c_3xz$ . Also,  $g(m_\alpha y) = g(xy) = -qg(yx)$ , and  $g(m_\alpha y) = c_1yx + c_3xz = -c_1qyx + c_3xz$ , thus  $g(yx) = -q^{-1}g(m_\alpha y) = -q^{-1}(-c_1qyx + c_3xz) = c_1yx - q^{-1}c_3xz$ . It follows that  $c_2yx + c_3xz = -\alpha g(yx) = -\alpha(c_1yx - q^{-1}c_3xz) = -\alpha c_1yx + \alpha q^{-1}c_3xz$ . Therefore  $c_2 = -\alpha c_1$  and  $c_3 = \alpha q^{-1}c_3$ . Since we assume that  $\alpha \neq q$ , it follows that  $c_3 = 0$ . Therefore  $g(m_\alpha) = c_1x - \alpha c_1y + c_3z + c_4yx + c_5xz = c_1(x - \alpha y) + c_4yx + c_5xz$  belongs to  $U_\alpha$ . Since we also assume that  $\alpha \neq 1$ , we have  $U_\alpha = \Lambda m_\alpha$ . Thus  $g(m_\alpha) \in \Lambda m_\alpha$ .

As a consequence, there is  $\lambda \in \Lambda$  with  $g(m_\alpha) = \lambda m_\alpha$ , therefore  $g(m_\alpha) = \lambda m_\alpha = l(\lambda)u'_\alpha(m_\alpha)$ . It follows that  $g = l(\lambda)u'_\alpha$ .  $\square$

**6.6. Lemma.** Let  $\alpha \in k \setminus \{1, q\}$ . Then there is an  $\Omega\mathcal{U}$ -sequence of right  $\Lambda$ -modules

$$0 \rightarrow m_\alpha\Lambda \xrightarrow{u'_\alpha} \Lambda_\Lambda \rightarrow m_{q\alpha}\Lambda \rightarrow 0.$$

Proof. This is 6.5 (6) and (9).  $\square$

**6.7. Proof of Theorem 1.5.** According to 6.5 (8), we have  $M(q)^* = m_1\Lambda$ . As we know from 6.3,  $M(q)$  is not torsionless.

We assume now that  $o(q) = \infty$ . The Corollary in 6.4 shows that  $M(q)$  is semi-Gorenstein-projective. Since  $q^{-t} \neq 1$  for all  $t \geq 1$ , the sequences

$$0 \rightarrow m_{q^{-t}}\Lambda \xrightarrow{u'_\alpha} \Lambda_\Lambda \rightarrow m_{q^{-t+1}}\Lambda \rightarrow 0.$$

with  $t \geq 1$  are  $\Omega\mathcal{U}$ -sequences, by 6.5. They can be concatenated in order to obtain a minimal projective resolution of  $m_1\Lambda$  and show that  $m_1\Lambda$  is semi-Gorenstein-projective.

Finally, we want to show that  $M(q)^{**} = \Omega M(1)$ . According to 6.3 (5), the map  $u_1: \Lambda m_1 \rightarrow \Lambda$  is a minimal left  $\text{add}(\Lambda)$ -approximation, thus we may consider as in 2.4 (a) the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda m_1 & \xrightarrow{u_1} & \Lambda & \xrightarrow{\pi_1} & \Lambda/\Lambda m_1 & \longrightarrow & 0 \\
& & \downarrow & & \parallel & & \downarrow \phi & & \\
0 & \longrightarrow & (\Lambda m_1)^{**} & \longrightarrow & \Lambda & \xrightarrow{\pi_1^{**}} & (\Lambda/\Lambda m_1)^{**} & \longrightarrow & \text{Ext}^1(M'(q)^*, \Lambda)
\end{array}$$

where  $\phi = \phi_{\Lambda/\Lambda m_1}$ . The submodule  $xz(\Lambda/\Lambda m_1)$  belongs to the kernel of any map  $\Lambda/\Lambda m_1 \rightarrow \Lambda$ , and it is the kernel of the map  $p: \Lambda/\Lambda m_1 \rightarrow M(1)$  defined by  $p(\bar{1}) = v$ . This shows that  $xz(\Lambda/\Lambda m_1)$  is the kernel of  $\phi$ , thus the image of  $\phi$  is just  $M(1)$ . But the image of  $\phi$  coincides with the image of  $\pi_1^{**}$ . In this way, we see that  $(\Lambda m_1)^{**}$  is the kernel of a projective cover of  $M(1)$ , thus equal to  $\Omega M(1)$ .

Of course,  $\Omega M(1)$  is decomposable, namely isomorphic to  $\Lambda m_1 \oplus kxz$ .  $\square$

**6.8. Proof of Addendum 1.5.** We denote by  $q^{\mathbb{Z}}$  the set of elements of  $k$  which are of the form  $q^i$  with  $i \in \mathbb{Z}$ . Assume that  $\alpha \in k \setminus q^{\mathbb{Z}}$ , thus  $q^t \alpha \neq 1$  for all  $t \in \mathbb{Z}$ . According to 6.4, all the sequences

$$0 \rightarrow M(q^{t+1}\alpha) \rightarrow \Lambda \rightarrow M(q^t\alpha) \rightarrow 0.$$

with  $t \in \mathbb{Z}$  are  $\Omega\mathcal{U}$ -sequences. They can be concatenated in order to obtain a complete projective resolution for  $M(\alpha)$ , thus  $M(\alpha)$  is Gorenstein-projective.

The following lemma shows that there are infinitely many elements  $\alpha \in k \setminus q^{\mathbb{Z}}$ .

**Lemma.** *Assume that  $k$  is an infinite field and  $q \in k$ . Then  $k \setminus q^{\mathbb{Z}}$  is an infinite set.*

We include a proof. The assertion is clear if  $o(q)$  is finite. Thus, let  $o(q)$  be infinite (in particular,  $q \neq 0$ ). Assume that the multiplicative group  $k^* = k \setminus \{0\}$  is cyclic, say  $k^* = w^{\mathbb{Z}}$ . Then  $o(w) = \infty$ , and each element in  $k^*$  different from 1 has infinite multiplicative order. Since  $(-1)^2 = 1$ , we see that  $k$  is of characteristic 2. Now  $w+1 \neq 0$  shows that  $w+1 = w^n$  for some  $n > 1$ , thus  $w$  is algebraic over the prime field  $\mathbb{Z}_2$ . Thus  $k = \mathbb{Z}_2(w)$  is a finite field, a contradiction. Since  $k^*$  is not cyclic, there is  $a \in k^* \setminus q^{\mathbb{Z}}$ . Then  $a \cdot q^{\mathbb{Z}}$  is an infinite subset of  $k^* \setminus q^{\mathbb{Z}}$ .  $\square$

## 7. Further details for $\Lambda = \Lambda(q)$ .

**7.1. The  $\Omega\mathcal{U}$ -components involving modules  $M(\alpha)$ .** *The only  $\Omega\mathcal{U}$ -sequences which involve a module of the form  $M(\alpha)$  with  $\alpha \in k$  are those exhibited in 6.4.*

Proof. We show that there is no  $\Omega\mathcal{U}$ -sequence ending in  $M(1)$  and no  $\Omega\mathcal{U}$ -sequence starting in  $M(q)$ . Since  $\Omega M(1)$  is decomposable, there is no  $\Omega\mathcal{U}$ -sequence ending in  $M(1)$ . By 6.2, the module  $M(q)$  is not torsionless, thus there is no  $\Omega\mathcal{U}$ -sequence starting in  $M(q)$ .

Let us assume that  $o(q) = \infty$  (for the case of  $o(q)$  being finite, see 7.4). *There are three kinds of  $\Omega\mathcal{U}$ -components which involve modules of the form  $M(\alpha)$  with  $\alpha \in k$ . There*

is one component of the form  $-\mathbb{N}$ , it has  $M(q)$  as its source, and there is one component of the form  $\mathbb{N}$ , it has  $M(1)$  as its sink:

$$\begin{array}{ccccccc} \leftarrow \leftarrow M(q^4) \leftarrow \leftarrow M(q^3) \leftarrow \leftarrow M(q^2) \leftarrow \leftarrow M(q) & & M(1) \leftarrow \leftarrow M(q^{-1}) \leftarrow \leftarrow M(q^{-2}) \leftarrow \leftarrow \end{array}$$

The remaining ones (containing the modules  $M(\alpha)$  with  $\alpha \notin q^{\mathbb{Z}}$ ) are of the form  $\mathbb{Z}$ :

$$\leftarrow \leftarrow M(q^4\alpha) \leftarrow \leftarrow M(q^3\alpha) \leftarrow \leftarrow M(q^2\alpha) \leftarrow \leftarrow M(q\alpha) \leftarrow \leftarrow M(\alpha) \leftarrow \leftarrow M(q^{-1}\alpha) \leftarrow \leftarrow M(q^{-2}\alpha) \leftarrow \leftarrow$$

For laziness, we have labeled the vertices just  $M$ , and not  $[M]$ . The positions of the reflexive modules are shaded.

According to 4.5, there are the following observations concerning the behavior of the modules  $M(\alpha)$  with  $\alpha \in k$ .

*The module  $M(\alpha)$  is Gorenstein-projective iff  $\alpha \notin q^{\mathbb{Z}}$ .*

*The module  $M(\alpha)$  is not Gorenstein-projective, but semi-Gorenstein-projective iff  $\alpha = q^t$  for some  $t \geq 1$ .*

*The module  $M(\alpha)$  is torsionless iff  $\alpha \neq q$ .*

*The module  $M(\alpha)$  is reflexive iff  $\alpha \notin \{q, q^2\}$ .*

It seems worthwhile to know the canonical maps  $\phi_X: X \rightarrow X^{**}$  for the non-reflexive modules  $X = M(q)$  and  $X = M(q^2)$ . For  $M(q)$  we refer to 6.7: there it is shown that  $M(q)^{**} = \Omega M(1)$  and that the image of  $\phi_{M(q)}$  is  $\Lambda m_1$ .

It remains to look at  $X = M(q^2)$ . The module  $M(q^2)^{**}$  is the the submodule  $\Lambda m_q + \Lambda z$  of  $\Lambda$  and  $\phi_{M(q^2)}$  is the inclusion map

$$M(q^2) = \Lambda m_q \longrightarrow \Lambda m_q + \Lambda z = M(q^2)^{**}.$$

Proof. Since  $M(q^2)$  is torsionless, the map  $\phi_{M(q^2)}$  is injective. There is the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M(q^2) & \xrightarrow{u_q} & \Lambda & \xrightarrow{\pi_q} & M(q) & \longrightarrow & 0 \\ & & \downarrow \phi_{M(q^2)} & & \parallel & & \downarrow \phi_{M(q)} & & \\ 0 & \longrightarrow & M(q^2)^{**} & \xrightarrow{u_q^{**}} & \Lambda & \xrightarrow{\pi_q^{**}} & M(q)^{**} & \longrightarrow & \text{Ext}^1(M(q^2)^*, \Lambda) \end{array}$$

As we know already, the image of  $\phi_{M(q)}$  and therefore of  $\pi_q^{**}$ , is  $\Lambda m_1$ . Thus the kernel of  $\pi_q^{**}$  is the submodule  $\Lambda m_q + \Lambda z$  of  $\Lambda$ . Therefore  $M(q^2)^{**} = \Lambda m_q + \Lambda z$  and  $\phi_{M(q^2)}$  is the inclusion map  $M(q^2) = \Lambda m_q \longrightarrow \Lambda m_q + \Lambda z = M(q^2)^{**}$ .  $\square$

**7.2. The  $\Omega\mathcal{U}$ -components involving the right  $\Lambda$ -modules  $m_\alpha\Lambda$ .** We assume again that  $o(q) = \infty$ . The  $\Omega\mathcal{U}$ -sequences which involve a right  $\Lambda$ -module of the form  $m_\alpha\Lambda$  with  $\alpha \in k$  are those exhibited in 6.6 as well as

$$0 \rightarrow m_1\Lambda \xrightarrow{\begin{bmatrix} u'_1 \\ h \end{bmatrix}} \Lambda_\Lambda \oplus \Lambda_\Lambda \rightarrow Z \rightarrow 0, \quad 0 \rightarrow m_q\Lambda \xrightarrow{\begin{bmatrix} u'_q \\ h' \end{bmatrix}} \Lambda_\Lambda \oplus \Lambda_\Lambda \rightarrow Z' \rightarrow 0,$$

where  $h: m_1\Lambda \rightarrow \Lambda_\Lambda$  is defined by  $h(m_1) = xz$  and  $h': m_q\Lambda \rightarrow \Lambda_\Lambda$  is defined by  $h'(m_q) = z$ .

Proof. It is easy to check that the maps  $\begin{bmatrix} u_1 \\ h \end{bmatrix}$  and  $\begin{bmatrix} u_q \\ h' \end{bmatrix}$  are minimal left  $\text{add}(\Lambda_\Lambda)$ -approximations. Clearly, the corresponding cokernels  $Z$  and  $Z'$  are not torsionless.

In addition, we have to show that there is no  $\Omega\mathcal{U}$ -sequence ending in  $m_{q^2}\Lambda$  or in  $m_q\Lambda$ . But this follows from the fact that the inclusion maps  $u'_q: m_q\Lambda = \Omega(m_{q^2}\Lambda) \rightarrow P(m_{q^2}\Lambda)$  and  $u'_1: m_1\Lambda = \Omega(m_q\Lambda) \rightarrow P(m_q\Lambda)$  are not  $\text{add}(\Lambda_\Lambda)$ -approximations.

There are four kinds of  $\Omega\mathcal{U}$ -components involving right  $\Lambda$ -modules of the form  $m_\alpha\Lambda$  with  $\alpha \in k$ , namely a component of the form  $\mathbb{N}$  with  $m_{q^2}\Lambda$  as a sink, a component of the form  $-\mathbb{N}$  with  $Z$  as a source, and a component of the form  $\mathbb{A}_2$  with sink  $m_q\Lambda$  and source  $Z'$ :

$$\begin{array}{ccccccc} \cdots & \rightarrow & m_{q^3}\Lambda & \rightarrow & m_{q^2}\Lambda & & \\ & & & & & \nearrow & \\ & & & & m_q\Lambda & & \\ & & & & & \nearrow & \\ & & & & m_1\Lambda & \rightarrow & m_{q^{-1}}\Lambda \rightarrow m_{q^{-2}}\Lambda \rightarrow m_{q^{-3}}\Lambda \rightarrow \cdots \end{array}$$

The remaining  $\Omega\mathcal{U}$ -components (containing the right  $\Lambda$ -modules  $m_\alpha\Lambda$  with  $\alpha \in k \setminus q^{\mathbb{Z}}$ ) are of the form  $\mathbb{Z}$ :

$$\cdots \rightarrow m_{q^3\alpha}\Lambda \rightarrow m_{q^2\alpha}\Lambda \rightarrow m_{q\alpha}\Lambda \rightarrow m_\alpha\Lambda \rightarrow m_{q^{-1}\alpha}\Lambda \rightarrow m_{q^{-2}\alpha}\Lambda \rightarrow m_{q^{-3}\alpha}\Lambda \rightarrow \cdots$$

For the convenience of the reader, the pictures in 7.1 and 7.2 have been arranged so that the  $A$ -duality is respected. Thus, in 7.1, the arrows are drawn from right to left, in 7.2 from left to right. Also we recall from 6.3 (8) that the  $A$ -dual of  $M(q\alpha)$  is  $m_\alpha\Lambda$ , therefore the position of  $m_\alpha\Lambda$  in the pictures 7.2 is the same as the position of  $M(q\alpha)$  in 7.1.

We complete the description of the behavior of the modules  $M(\alpha)$  started in 7.1.

*The module  $M(\alpha)$  is not Gorenstein-projective, but  $M(\alpha)^*$  is semi-Gorenstein-projective, iff  $\alpha = q^t$  for some  $t \leq 1$ .*

Proof. According to 7.1, the module  $M(\alpha)$  is Gorenstein-projective iff  $\alpha \notin q^{\mathbb{Z}}$ . Thus, we can assume that  $\alpha = q^t$  for some  $t \in \mathbb{Z}$ . According to 6.3 (8), the module  $M(q^t)$  is isomorphic to  $m_{q^{t-1}}\Lambda$ . The display of the  $\Omega\mathcal{U}$ -components shows that  $m_{q^{t-1}}\Lambda$  is semi-Gorenstein-projective iff  $t - 1 \leq 0$ , thus iff  $t \leq 1$ , see 4.5.  $\square$

**7.3.** We have mentioned already that one may use the algebra  $\Lambda = \Lambda(q)$  with  $o(q) = \infty$  in order to exhibit examples of modules  $M$  which satisfy precisely two of the three properties (G1), (G2) and (G3).

- (1)  $M = M(q)$  satisfies (G1), (G2), but not (G3).
- (2)  $M = M(q^3)$  satisfies (G1), (G3), but not (G2).
- (3)  $M = M(1)$  satisfies (G2), (G3), but not (G1).

Proof: For (1): this is the main assertion of Theorem 1.5. For (2): see 7.1 and 7.2. For (3): according to 7.1,  $M(1)$  is reflexive, but not Gorenstein-projective. According to 6.3 (8), we have  $M(1)^* = m_{q^{-1}}\Lambda$  and  $m_{q^{-1}}\Lambda$  is semi-Gorenstein-projective, see 7.2.  $\square$

Let us look for similar examples for  $\Lambda^{\text{op}}$ , thus, for right  $\Lambda$ -modules  $N$ .

- (1\*) *There is **no** right  $\Lambda$ -module of the form  $N = m_\alpha \Lambda$  satisfying (G1), (G2), but not (G3).*
- (2\*) *The right  $\Lambda$ -module  $N = m_{q^{-2}} \Lambda$  satisfies (G1), (G3), but not (G2).*
- (3\*) *The right  $\Lambda$ -module  $N = m_{q^2} \Lambda$  satisfies (G2), (G3), but not (G1).*

Proof: (2\*) There starts an infinite  $\Omega\mathcal{U}$ -path at  $N = m_{q^{-2}} \Lambda$ , thus  $N$  satisfies (G1). There ends an  $\Omega\mathcal{U}$ -path of length 2 at  $N$ , thus  $N$  satisfies (G3). Of course,  $N^*$  cannot be semi-Gorenstein-projective, since otherwise  $N$  would be Gorenstein-projective.

(3\*) Let  $N = m_{q^2} \Lambda$ . According to 6.5 (8),  $N = M(q^3)^*$ . As we know from 7.1,  $M(q^3)$  is reflexive, thus  $N$  is reflexive and  $N^* = M(q^3)^{**} = M(q^3)$  is semi-Gorenstein-projective.

(1\*) Assume that  $N = m_\alpha \Lambda$  and  $N^*$  are both semi-Gorenstein-projective. Since  $N$  cannot be Gorenstein-projective, it is not reflexive. Thus  $\alpha \in \{1, q\}$ . Since  $[m_q \Lambda]$  is the sink of an  $\Omega\mathcal{U}$ -component,  $m_\alpha \Lambda$  is not semi-Gorenstein-projective. Thus  $\alpha = 1$ . But  $(m_1 \Lambda)^* = M(q)^{**} = \Omega M(1)$ , according to 6.5 (8) and Theorem 1.5. As we have mentioned already in the proof 6.7,  $\Omega M(1) \simeq \Lambda m_1 \oplus k$ , where  $k$  is the simple  $\Lambda$ -module. But we claim that  $k$  is not semi-Gorenstein-projective, thus  $\Omega M(1)$  is not semi-Gorenstein-projective.

**Lemma.** *The simple  $\Lambda$ -module  $k$  is not semi-Gorenstein-projective.*

Proof. Assume that  $k$  is semi-Gorenstein-projective. Since  ${}^\perp \Lambda$  is closed under extensions, it follows that  ${}^\perp \Lambda = \text{mod } \Lambda$ . However, as we have seen in 7.1, the modules  $M(q^t)$  with  $t \leq 0$  are not semi-Gorenstein-projective.  $\square$

Actually, we also can show directly that  $\text{Ext}^1(k, \Lambda) \neq 0$ . We show that the inclusion map  $u: \text{rad } \Lambda \rightarrow \Lambda$  is not a left  $\text{proj}(\Lambda)$ -approximation. Let  $f: \text{rad } \Lambda \rightarrow \Lambda$  be a homomorphism with  $f(z) = yx$  (such a map exists, since  $z$  is in the top of  $\text{rad } \Lambda$  and  $yx$  in its socle). Assume that  $f$  factors through  $u$ , say  $f = r(a)u$  for some  $a \in \Lambda$ , where  $r(a): \Lambda \rightarrow \Lambda$  is the right multiplication by  $a$ . Thus  $r(a)u(z) = za$ , but  $f(z) = yx$  does not belong to  $z\Lambda$ .  $\square$

**7.4.** Let us look also at the case when  $o(q) = n < \infty$ . *There are two kinds of  $\Omega\mathcal{U}$ -components which involve modules of the form  $M(\alpha)$  with  $\alpha \in k$ . There is one component of the form  $\mathbb{A}_n$ , it has  $M(q)$  as its source, and  $M(1)$  as its sink:*

$$M(1) \leftarrow \cdots \leftarrow M(q^{n-1}) \leftarrow \cdots \leftarrow M(q^3) \leftarrow \cdots \leftarrow M(q^2) \leftarrow \cdots \leftarrow M(q)$$

*The remaining ones (containing the modules  $M(\alpha)$  with  $\alpha \notin q^{\mathbb{Z}}$ ) are directed cycles of cardinality  $n$ :*

$$M(\alpha) \leftarrow \cdots \leftarrow M(q^{n-1}\alpha) \leftarrow \cdots \leftarrow M(q^3\alpha) \leftarrow \cdots \leftarrow M(q^2\alpha) \leftarrow \cdots \leftarrow M(q\alpha) \leftarrow \cdots \leftarrow M(\alpha)$$

All modules in the cycles are reflexive. In the component of type  $\mathbb{A}_n$ , the modules  $M(q)$  and  $M(q^2)$  are not reflexive (they coincide for  $o(q) = 1$ ); for  $o(q) \geq 3$ , there are  $n - 2$  additional modules  $M(1) = M(q^n)$ ,  $M(q^{n-2})$ ,  $\dots$ ,  $M(q^4)$ ,  $M(q^3)$  in the component, and these modules are reflexive.

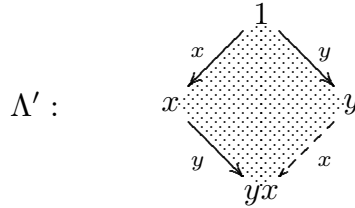
Proof: According to 7.1, the  $\Omega\mathcal{U}$ -sequences presented here are the only ones involving modules of the form  $M(\alpha)$ . Thus,  $[M(q)]$  is a source in the  $\Omega\mathcal{U}$ -quiver and  $[M(1)]$  is a sink.



This holds true also for  $o(q) = 1$ : here  $q = 1$  and  $[M(1)]$  is both a sink and a source, thus a singleton component (without any arrow). Finally, for any  $n$ , the elements  $1, q, \dots, q^{n-1}$  are pairwise different, as are the elements  $\alpha, q\alpha, \dots, q^{n-1}\alpha$  for  $\alpha \in k \setminus q^{\mathbb{Z}}$ .  $\square$

We have shown in section 4.1 that any  $\Omega\mathcal{U}$ -component is a linearly oriented quiver of type  $\mathbb{A}_n$  (with  $n \geq 1$  vertices), a directed cycle  $\tilde{\mathbb{A}}_n$  (with  $n + 1 \geq 1$  vertices), or of the form  $-\mathbb{N}$ , or  $\mathbb{N}$ , or  $\mathbb{Z}$ . Now we see in 7.1 and 7.4 that all these cases do arise.

**7.5. The quantum exterior algebra  $\Lambda' = \Lambda'(q)$  in two variables.** Let  $\Lambda'$  be the  $k$ -algebra generated by  $x, y$  with the relations  $x^2, y^2, xy + qyx$ . It has a basis  $1, x, y,$  and  $yx$ . We may use the following picture as an illustration:



If we factor out the socle of  $\Lambda'$ , we obtain the 3-dimensional local algebra  $\Lambda''$  with radical square zero (it is generated by  $x, y$  with relations  $x^2, y^2, xy, yx$ ).

Note that  $\Lambda'(q)$  is a subalgebra of  $\Lambda(q)$ , and that  $\Lambda z \Lambda = \Lambda z = \text{span}\{z, xz\}$ . The composition  $\Lambda' \hookrightarrow \Lambda \twoheadrightarrow \Lambda/\Lambda z \Lambda$  of the canonical maps is an isomorphism of algebras. In this way, the  $\Lambda'$ -modules may be considered as the  $\Lambda$ -modules which are annihilated by  $z$ . We should stress that the elements  $m_\alpha = x - \alpha y$  (which play a decisive role in our investigation) belong to  $\Lambda'$ .

For  $\alpha \in k$ , let  $M'(\alpha)$  be the  $\Lambda'$ -module with basis  $v, v'$ , such that  $xv = \alpha v', yv = v'$ , and  $xv' = 0 = yv'$ . In addition, we define  $M'(\infty)$  as the  $\Lambda'$ -module with basis  $v, v'$ , such that  $xv = v', yv = xv' = yv' = 0$ . Here are the corresponding illustrations:



The modules  $M'(\alpha)$  with  $\alpha \in k \cup \{\infty\}$  are pairwise non-isomorphic and indecomposable, and any two-dimensional indecomposable  $\Lambda'$ -module is of this form. In particular, the left ideal  $\Lambda' m_\alpha$  is isomorphic to  $M'(q\alpha)$ , for any  $\alpha \in k \cup \{\infty\}$ . The essential property of the modules  $M'(\alpha)$  is the following:  $\Omega_{\Lambda'} M'(\alpha) = M'(q\alpha)$ . This follows from the fact that  $m_{q\alpha} m_\alpha = 0$  and it is this equality which has been used frequently in sections 6 and 7.

For all  $\alpha \in k$ ,  $M(\alpha)$  considered as a  $\Lambda'$ -module, is equal to  $M'(\alpha) \oplus k$ , where  $k$  is the simple  $\Lambda'$ -module. Also, we should stress that  $\text{rad } \Lambda$  considered as a left  $\Lambda'$ -module is the direct sum of  $I$  and  $M'(\infty)$ , where  $I$  is the indecomposable injective  $\Lambda''$ -module.

**7.6. A variation.** Let  $\tilde{\Lambda}$  be the algebra defined by a quiver with two vertices, say labeled by 1 and 2, with three arrows  $1 \rightarrow 2$  labeled by  $x, y, z$  and with three arrows  $2 \rightarrow 1$ , also labeled by  $x, y, z$ , satisfying the "same" relations as  $\Lambda$  (of course, now we have 14 relations: seven concerning paths  $1 \rightarrow 2 \rightarrow 1$  and seven concerning paths  $2 \rightarrow 1 \rightarrow 2$ ).

Whereas  $\Lambda$  is a local algebra, the algebra  $\tilde{\Lambda}$  is a connected algebra with two simple modules  $S(1)$  and  $S(2)$ .

For all the previous considerations in sections 6 and 7, there are corresponding ones for  $\tilde{\Lambda}$ , but always we have to take into account that now we deal with two simple modules  $S(1)$  and  $S(2)$ : Corresponding to the module  $M(\alpha)$ , there are two different modules, namely  $M^1(\alpha)$  with top  $S(1)$  and  $M^2(\alpha)$  with top  $S(2)$ . The modules  $M^1(\alpha)$  and  $M^2(\alpha)$  have similar properties as  $M(\alpha)$ , in particular,  $M^1(q)$  and  $M^2(q)$  are semi-Gorenstein-projective and not Gorenstein-projective provided that  $o(q) = \infty$ . There is one decisive difference between the  $\Lambda$ -modules and the  $\tilde{\Lambda}$ -modules: The endomorphism ring of  $M^1(\alpha)$  and  $M^2(\alpha)$  is equal to  $k$ , whereas the endomorphism ring of any  $M(\alpha)$  is 3-dimensional.

## 8. Questions.

**1.** One may ask whether or not the finiteness of  $\text{gp } A$  implies that  $A$  is left weakly Gorenstein, There is a weaker question: is  $A$  left weakly Gorenstein, in case all the Gorenstein-projective  $A$ -modules are projective?

**2.** An artin  $A$  will be said to be *right weakly Gorenstein* provided its opposite algebra  $A^{\text{op}}$  is left weakly Gorenstein. Following Marczinzik [M1, question 1], one may ask whether an artin algebra is left weakly Gorenstein algebra if and only if it is right weakly Gorenstein. In terms of the  $\Omega\mathcal{U}$ -quiver, the question may be reformulated as follows: Does the existence of an  $\Omega\mathcal{U}$ -component of the form  $-\mathbb{N}$  imply that also an  $\Omega\mathcal{U}$ -component of the form  $\mathbb{N}$  exists?

Let us show: *If any right weakly Gorenstein algebra is left weakly Gorenstein, then the Gorenstein symmetry conjecture holds true* (the Gorenstein symmetry conjecture asserts that if the injective dimension of  ${}_A A$  is finite, then the projective dimension of an injective cogenerator  $Q$  is finite).

Thus, assume that the injective dimension of  ${}_A A$  is at most  $n$ . It is well-known that this implies that for any module  $M$  the syzygy module  $\Omega^n M$  is semi-Gorenstein-projective. [Namely, for all  $i \geq 1$ , we have  $\text{Ext}^i(\Omega^n M, A) = \text{Ext}^{n+i}(M, A) = \text{Ext}^i(M, \Sigma^n A) = 0$ ; here,  $\Sigma N$  denotes the cokernel of an injective envelope of a module  $N$ .] In particular,  $A$  cannot have any  $\Omega\mathcal{U}$ -component of type  $\mathbb{N}$  (namely, if  $[M]$  is the end of an infinite  $\Omega\mathcal{U}$ -path, then  $M = \Omega^n(\mathcal{U}^n M)$  implies that  $M$  is semi-Gorenstein-projective, thus  $M$  is in an  $\Omega\mathcal{U}$ -component of type  $\mathbb{Z}$ ). Therefore  $A$  is right weakly Gorenstein and thus  $A$  is left weakly Gorenstein. So any semi-Gorenstein-projective module is Gorenstein-projective, thus all the modules  $\Omega^n M$  are Gorenstein-projective. In particular,  $Q' = \Omega^n Q$  is Gorenstein-projective. A well-known argument shows that if  $Q'$  is Gorenstein-projective, then  $Q'$  is even projective (and therefore the projective dimension of  $Q$  is at most  $n$ ). [Namely, assume that  $Q' = \Omega^n Q$  is Gorenstein-projective. Then there is a Gorenstein-projective module  $Q''$  such that  $Q' = P' \oplus \Omega^{n+1} Q''$  with  $P'$  projective. Now  $\text{Ext}^1(\Omega^n Q'', Q') \simeq \text{Ext}^{n+1}(Q'', Q') \simeq \text{Ext}^1(Q'', Q) = 0$ , here the first isomorphism is the usual index shift, whereas the second isomorphism comes from the fact that  $Q''$  is (semi-)Gorenstein-projective and  $Q' = \Omega^n Q$  (for a semi-Gorenstein-projective module  $N$ , and any module  $Z$ , we have  $\text{Ext}^{i+1}(N, \Omega Z) \simeq \text{Ext}^i(N, Z)$  for all  $i \geq 1$ ). But  $\text{Ext}^1(\Omega^n Q'', P' \oplus \Omega^{n+1} Q'') = 0$  implies that  $\text{Ext}^1(\Omega^n Q'', \Omega^{n+1} Q'') = 0$ , thus the canonical exact sequence  $0 \rightarrow \Omega^{n+1} Q'' \rightarrow P(\Omega^n Q'') \rightarrow \Omega^n Q'' \rightarrow 0$  splits and  $\Omega^{n+1} Q''$  has to be projective (even zero). It follows that  $Q' = P' \oplus \Omega^{n+1} Q''$  is projective.]  $\square$

**3.** Assume that there exists a non-reflexive  $A$ -module  $M$  such that both  $M$  and  $M^*$  are semi-Gorenstein-projective. Is then the same true for  $A^{\text{op}}$ ? Even for  $A = \Lambda(q)$  with  $o(q) = \infty$ , we do not know the answer. According to 7.3 (1\*), a right  $A$ -module  $N$  of the form  $N = m_\alpha \Lambda(q)$  is reflexive, if both  $N$  and  $N^*$  are semi-Gorenstein-projective. But, there could exist some other right  $A$ -module  $N$  satisfying (G1), (G2) and not (G3).

**4.** Does there exist a semi-Gorenstein-projective module  $M \neq 0$  with  $M^* = 0$ ? Such a module would be an extreme example of a module satisfying (G1), (G2) and not (G3). Marczinik has pointed out that this question concerns the Nunke condition [H] which asks for any non-zero module  $M$  for an integer  $i \geq 0$  such that  $\text{Ext}^i(M, A) \neq 0$ ; Colby and Fuller [CF] have conjectured that the Nunke condition should hold for any module  $M$ , they called this the *strong Nakayama conjecture*. The strong Nakayama conjecture obviously implies the generalized Nakayama conjecture which asserts that *for any simple module  $S$  there should exist some  $i \geq 0$  such that  $\text{Ext}^i(S, A) \neq 0$* . It is known that the Nunke condition is satisfied in case the finitistic dimension conjecture holds true.

Note that if  $M$  is indecomposable semi-Gorenstein-projective, then  $M^*$  may be decomposable, as Theorem 1.5 shows: the  $\Lambda(q)^{\text{op}}$  module  $M(q)^*$  is indecomposable and semi-Gorenstein-projective, but  $M(q)^{**}$  is decomposable.

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