Nakayama algebras of Loewy length 3.

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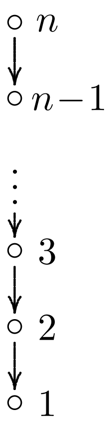
$A(n) = A(n,3)$, the algebra with quiver

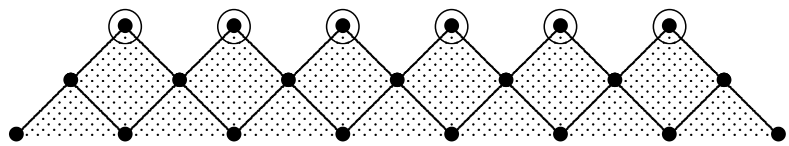


and with relations: $\alpha^3 = 0$
(all arrows are labelled α)

Indecomposable modules are serial and of length at most 3, those of length 3 are projective-injective.

$S(t)$ denotes the simple module corresponding to the vertex t.
$P(t)$ or just t will denote the projective module with top $S(t)$.

The Auslander-Reiten quiver for $n = 8$:



$\mathcal{C}(\Lambda)$ the category of bounded complexes $M_\bullet$
$\mathcal{P}(\Lambda)$ the category of perfect complexes (i.e.: all M_i projective)
$\underline{\mathcal{P}}(\Lambda)$ the homotopy category of perfect complexes

$$\begin{array}{ccc} & & \mathcal{C}(\Lambda) \\ & & \cup \\ & & \mathcal{P}(\Lambda) \\ & & \downarrow \\ D^b(\operatorname{mod}\Lambda) & \simeq & \underline{\mathcal{P}}(\Lambda) \end{array}$$

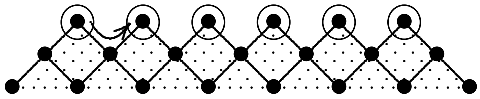
Seidel, Happel.
Lenzing, Geigle, Kussin, Meltzer, de la Peña.

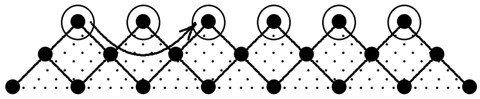
Typical maps between indecomposable projective modules:

$$P(i+1) \longleftarrow P(i), \qquad P(i+2) \longleftarrow P(i)$$

step jump

For example, for $i = 3$:





Note: Composing two steps is a jump, thus non-zero.
composing jumps yields zero.
composing a jump and a step yields zero.

$M_\bullet$ indecomposable with all M_i indecomposable projective or zero:

$$P(i_0) \longleftarrow P(i_1) \longleftarrow P(i_2) \longleftarrow \cdots \longleftarrow P(i_{t-1}) \longleftarrow P(i_t)$$

consists of jumps and steps, without consecutive steps.

Λ	Type of $\mathcal{P}(\Lambda)$		
$A(1)$	A_1		
$A(2)$	A_2		
$A(3)$	A_3		
$A(4)$	D_4		
$A(5)$	D_5		
$A(6)$	E_6		
$A(7)$	E_7		
$A(8)$	E_8		
$A(9)$	$\widetilde{E}_8$	$C(5,3,2)$	
$A(10)$	$\textcircled{E_8}$	$C(6,3,2)$	
$A(11)$	wild	$C(7,3,2)$	
$A(12)$	not piecewise hereditary	$C(7,3,2)^+$	?
$A(13)$	$\cdots$		
$A(14)$	$\cdots$		

$C(p,q,r)$ ist the canonical algebra of type (p,q,r)
$C(p,q,r)^+$ is its one-point extension using the simple projective module.

Seidel: A_n piecewise hereditary $\iff$ $n \leq 11$.
Happel-Zacharia: piecewise hered. $\iff$ bounded strong global dimension.

Aims:

1.

For $n = 9$: a one-parameter family. For $n = 11$: wildness.

2.

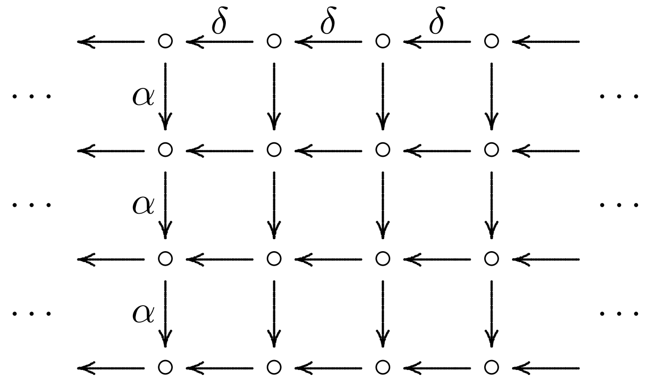
For any n: Search for shift-periodic complexes $M_\bullet$ (Calabi-Yau complexes): $\tau^s M_\bullet = M_\bullet[t]$ for some $s \geq 1$ and some t.

Important: The basic wings (jumping between the boundaries).
The neighbors of the basic wings.
Further wings.

3.

Indecomposable perfect complexes of large width.

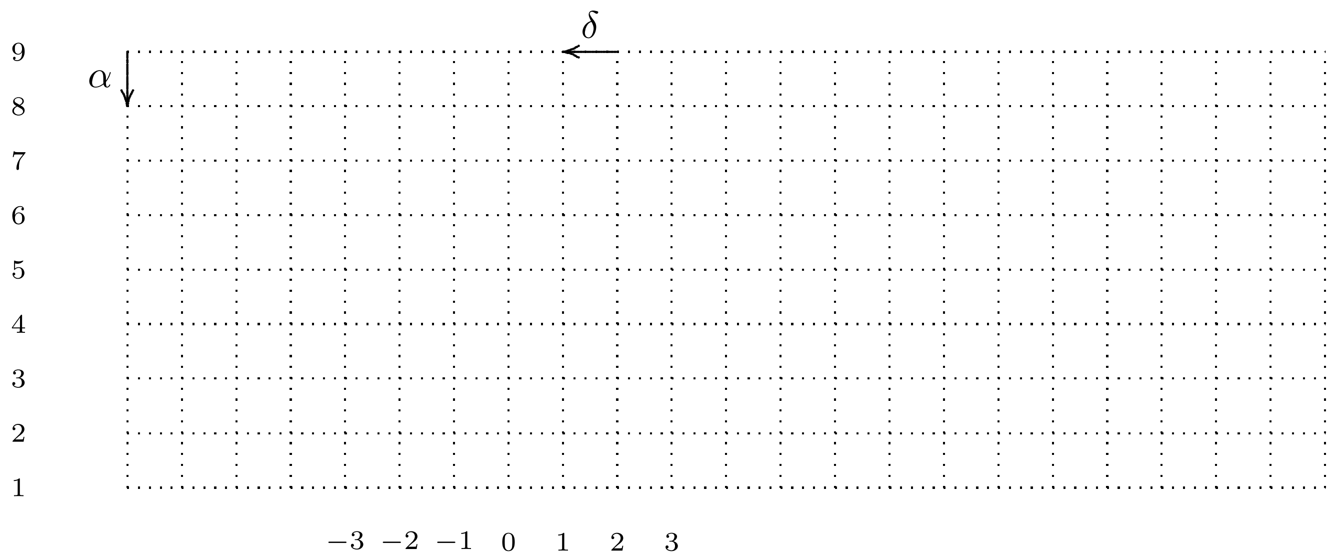
The quiver of $\mathcal{C}(A(n))$: a horicontal stripe with arrows pointing downwards and to the left. Here $n = 4$:



with $\alpha^3 = 0,\ \delta^2 = 0,\ \alpha\delta = \delta\alpha$
(here, δ is the differential, pointing to the left).

Such a complex is perfect iff always $\mathrm{Ker}(\alpha) = \mathrm{Im}(\alpha^2)$.

For $n = 9$:



with relations: $\alpha^3 = 0, \quad \delta^2 = 0, \quad \alpha\delta = \delta\alpha.$

The vertices of the quiver are pairs (i, t) of integers, with $i \in \mathbb{Z},\ 1 \le t \le n$. The corresponding simple module will be denoted by $S(t)_i$.

$\mathcal{P}(A(n))$: the objects with vertical restrictions being projective $A(n)$-modules. The indecomposable projective $A(n)$ module with top $S(t)_i$ is denoted by t_i.

1. One-parameter families for $n = 9$ and consequences.

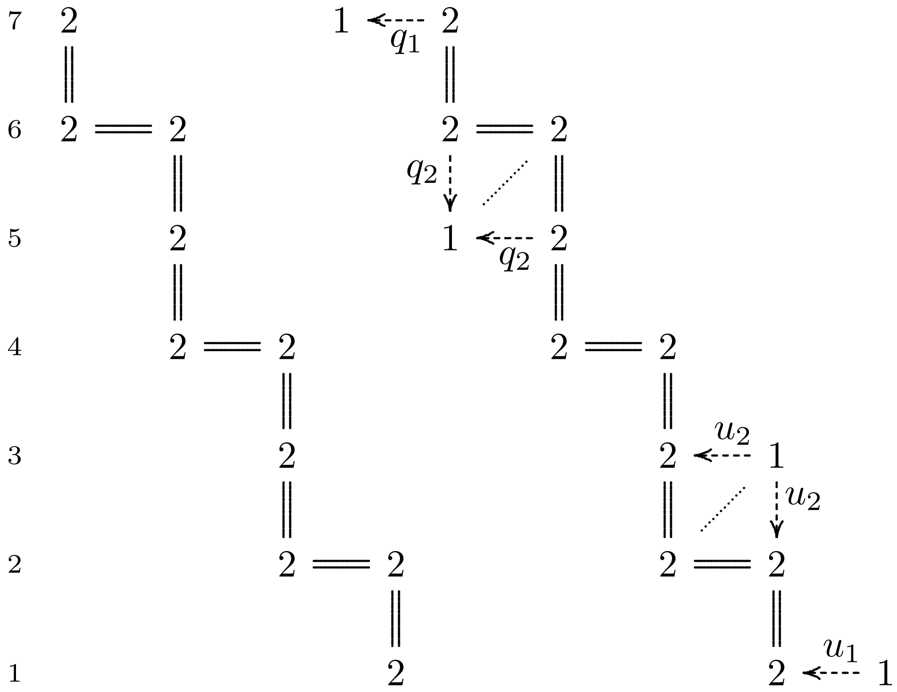
$A(1)$	A_1	
$A(2)$	A_2	
$A(3)$	A_3	
$A(4)$	D_4	
$A(5)$	D_5	
$A(6)$	E_6	
$A(7)$	E_7	
$A(8)$	E_8	
$A(9)$	$\widetilde{E}_8$	$C(5,3,2)$
$A(10)$	$\textcircled{E_8}$	$C(6,3,2)$
$A(11)$	wild	$C(7,3,2)$
$A(12)$	not piecewise hereditary	$C(7,3,2)^+$
$A(13)$	$\cdots$	
$A(14)$	$\cdots$	

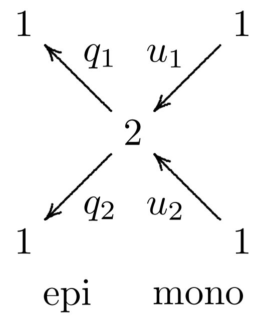
First, we will present a direct construction of a one-parameter family for $n = 9$, thus the $n \geq 9$: infinite type.

On the other hand, it is easy to see: for $n \leq 8$ locally finite type.

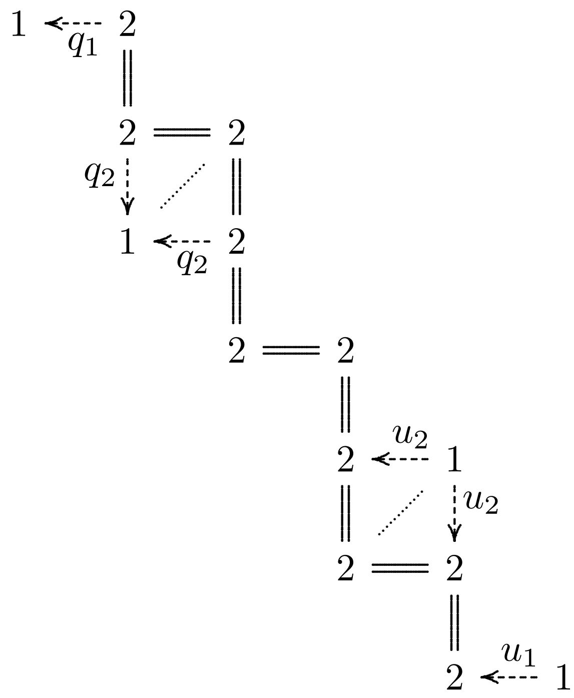
1. One-parameter families for $n = 9$.

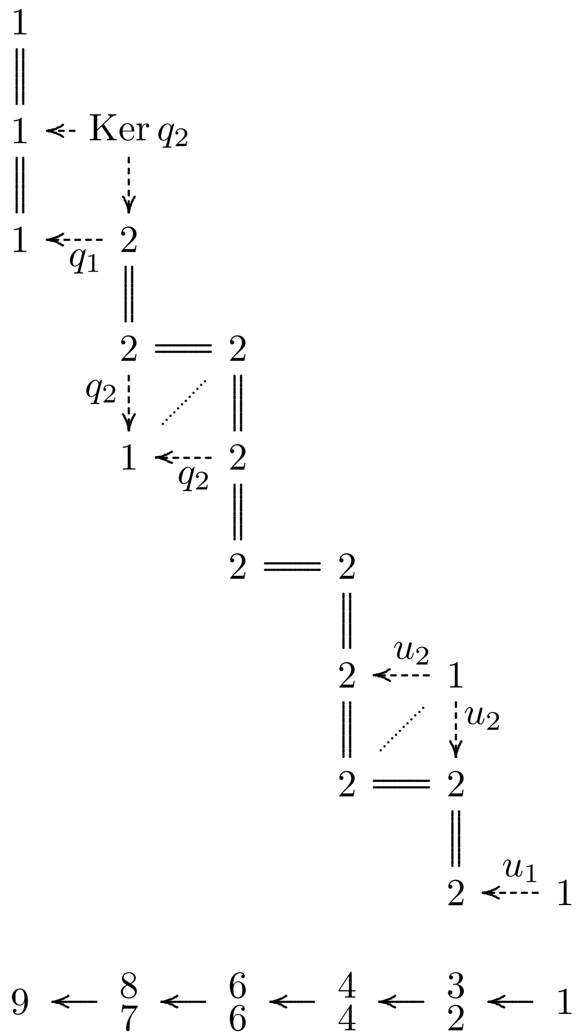
We want to construct one-parameter families of indecomposable objects.



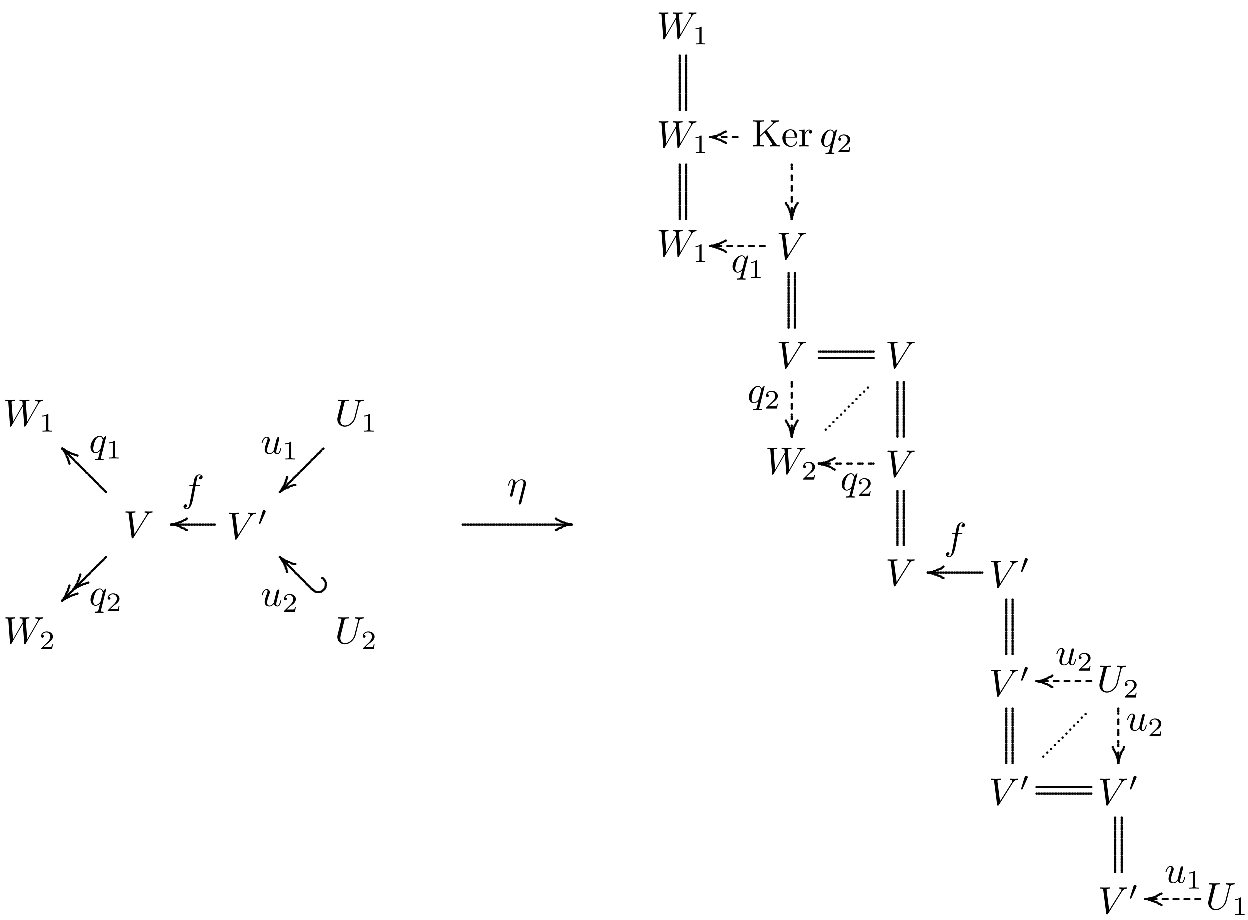


These are complexes, however not perfect ones: the problem arises at the top of the stairs: We have to prolong the stairs as follows:


q_1
q_2
u_2
u_1


$\mathrm{Ker}\, q_2$
q_1
q_2
u_2
u_1

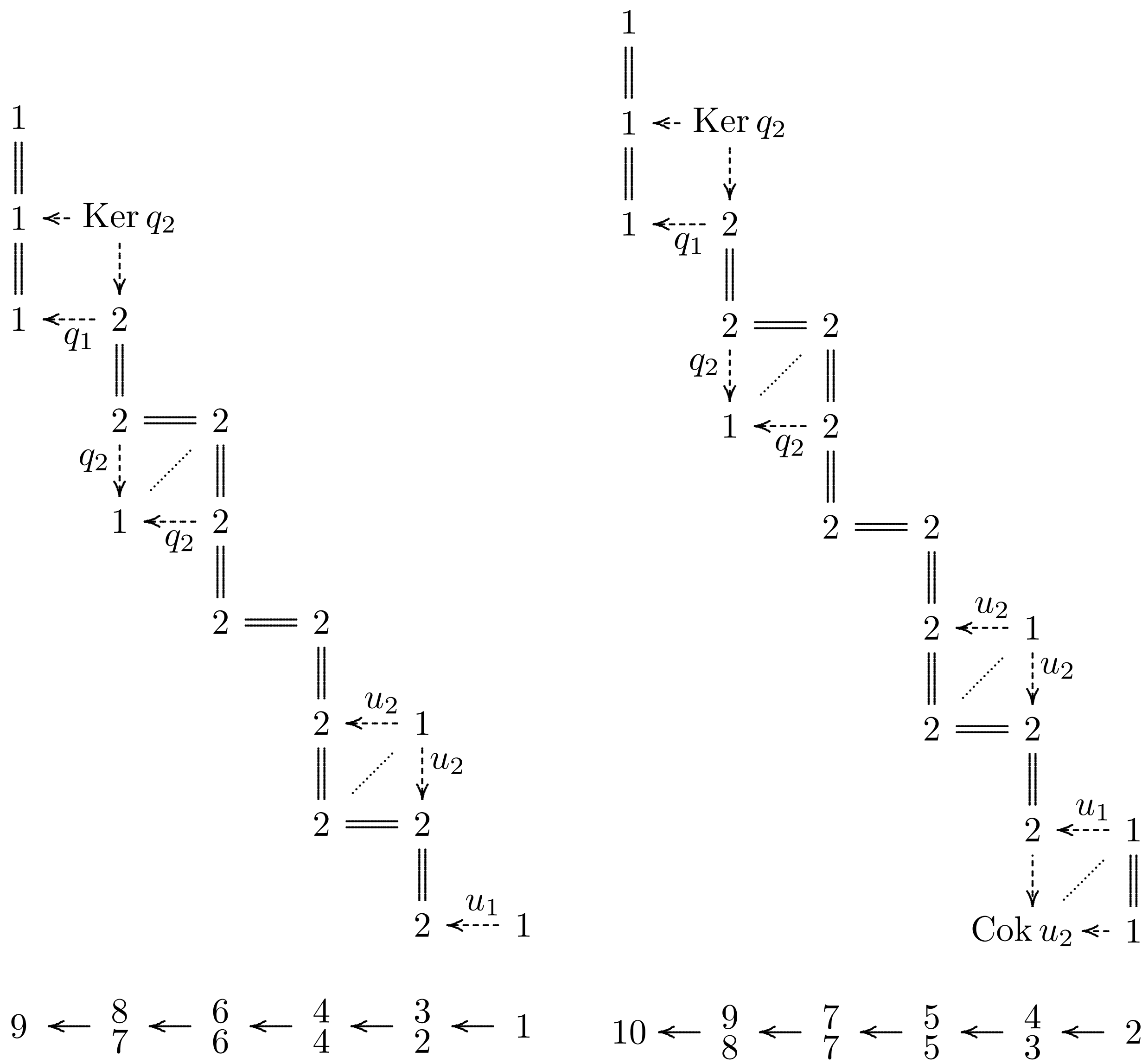
Let us repeat: There is a full exact embedding functor η



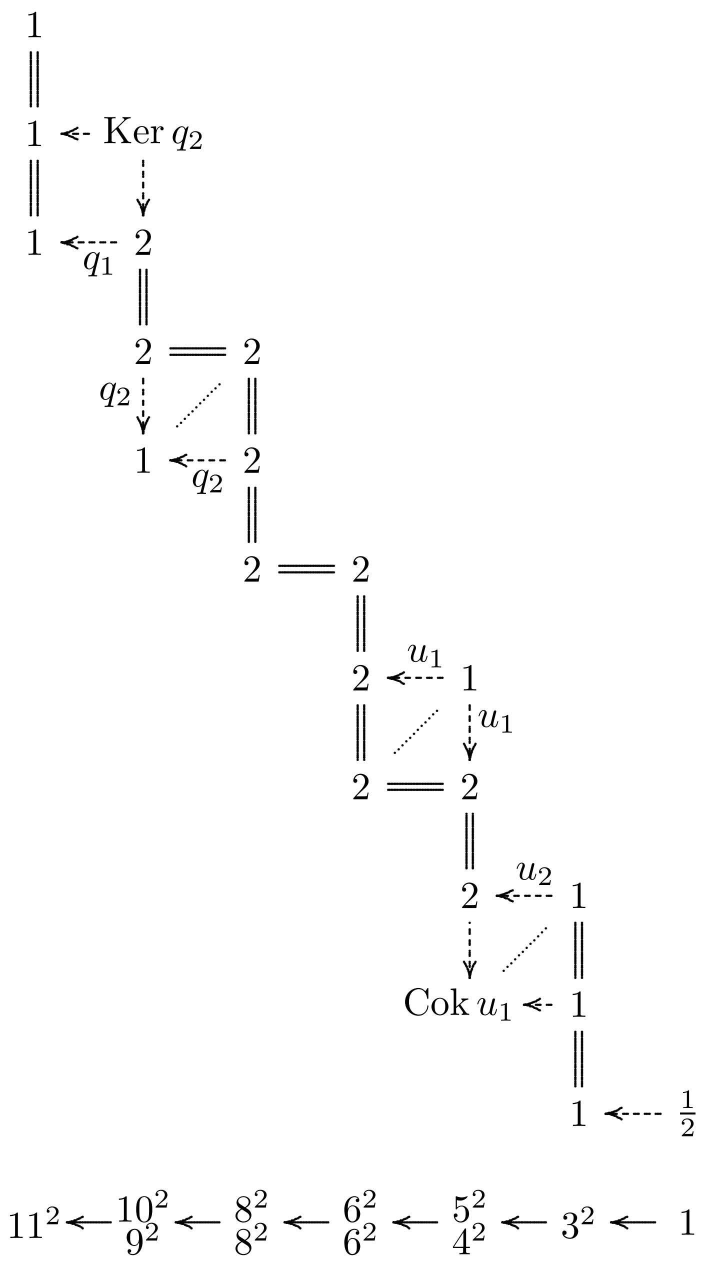
$$9 \leftarrow \begin{matrix}8\\7\end{matrix} \leftarrow \begin{matrix}6\\6\end{matrix} \leftarrow \begin{matrix}4\\4\end{matrix} \leftarrow \begin{matrix}3\\2\end{matrix} \leftarrow 1$$

Better: For any column index t, there is a full exact embedding functor η_t.

For $n = 10$: a second one-parameter family, shifting the stair one step up (and modifying the lower part in order to obtain a perfect complex):



For $n = 11$, we obtain a strictly wild category:



$A(1)$	A_1	
$A(2)$	A_2	
$A(3)$	A_3	
$A(4)$	D_4	
$A(5)$	D_5	
$A(6)$	E_6	
$A(7)$	E_7	
$A(8)$	E_8	
$A(9)$	$\widetilde{E}_8$	$C(5,3,2)$
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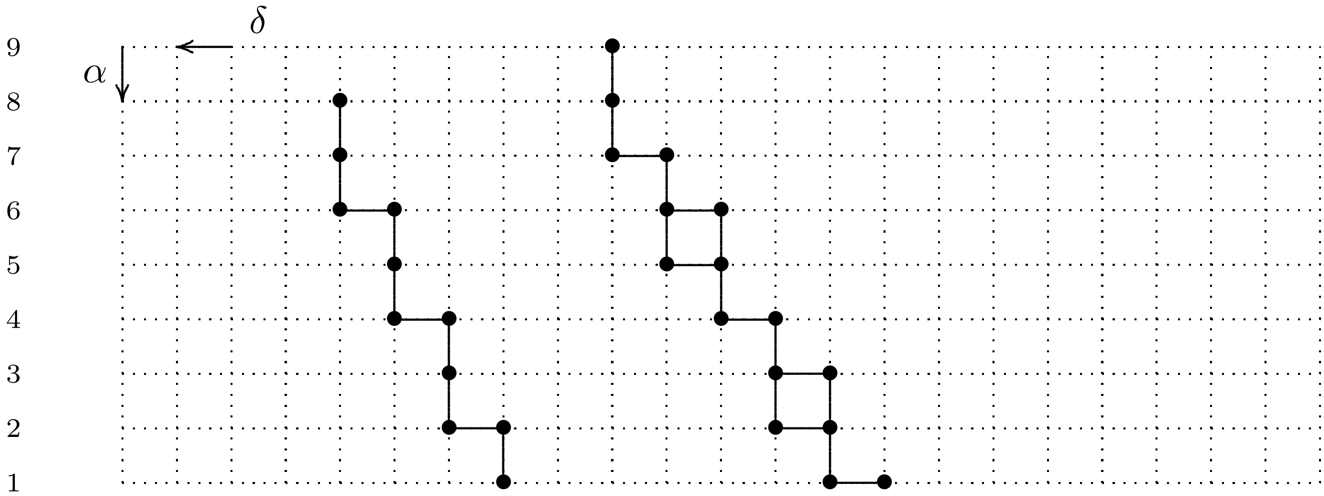
$A(13)$	$\cdots$	
$A(14)$	$\cdots$	

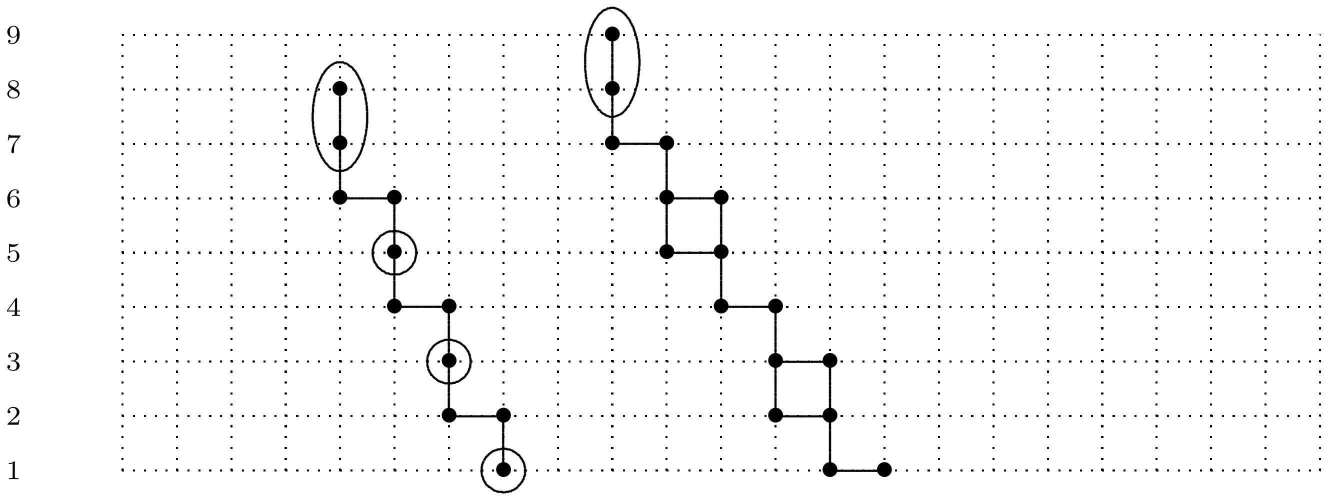
2.1. The basic wings.

Consider thin objects.

Two extreme cases:



jumps resolution



encircled is the homology

Remark: Recall that we consider $\underline{\mathcal{P}}(\Lambda) \simeq D^b(\operatorname{mod}\Lambda)$.

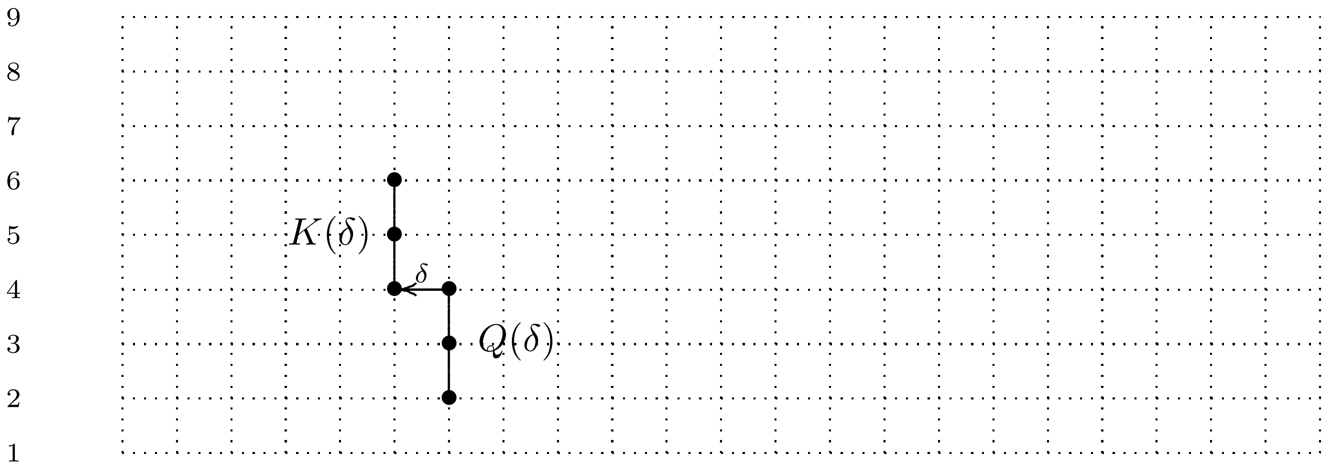
Any Λ-module (considered as an object of $D^b(\operatorname{mod}\Lambda)$)
is thus identified with its projective resolution (considered as an object in $\underline{\mathcal{P}}(\Lambda)$)
(the right complex is such a resolution).

Border sequences. We consider the jumps in more detail.

Recall (Butler-Ringel, 1987): Any arrow $\gamma\colon x \to y$ gives rise to an Auslander-Reiten sequence

$$0 \to K(\gamma) \to M(\gamma) \to Q(\gamma) \to 0$$

$Q(\gamma)$ is the cokernel of $P(\gamma)\colon P(y) \to P(x)$ and
$K(\gamma)$ is the kernel of $I(\gamma)\colon I(y) \to I(x)$.
and $M(\gamma)$ is **indecomposable**.

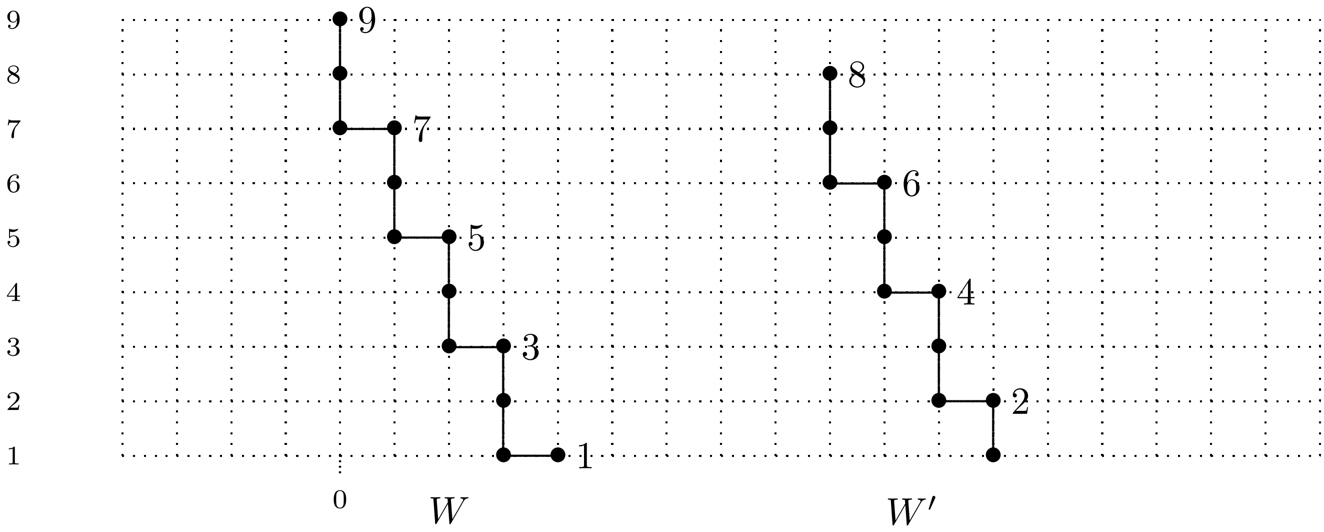


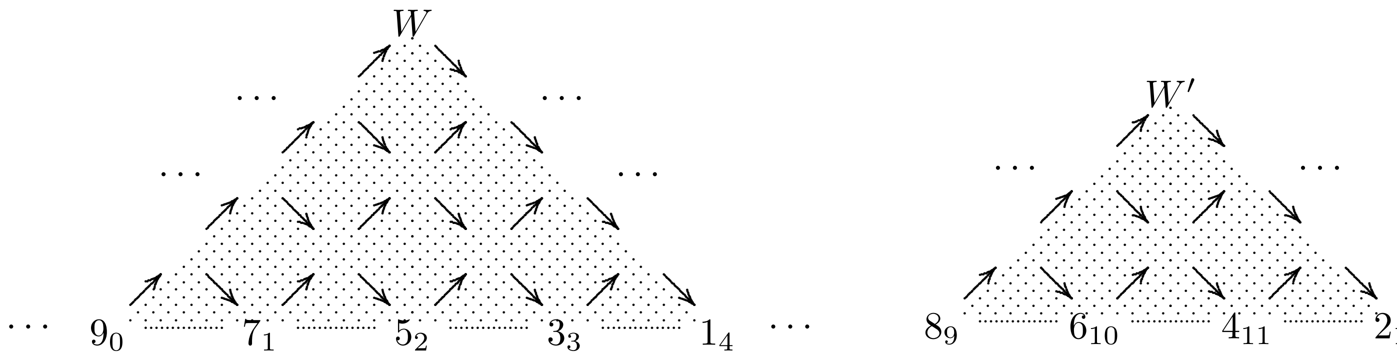
$M(\delta)$

$6_t \leftarrow 4_{t+1}$

This is an Auslander-Reiten sequence in $\mathcal{C}(\Lambda)$,
If δ does not belong to the upper two rows, then the sequence belongs to $\mathcal{P}(\Lambda)$, and then is an Auslander-Reiten sequence in $\mathcal{P}(\Lambda)$.

The basic wings for n odd.





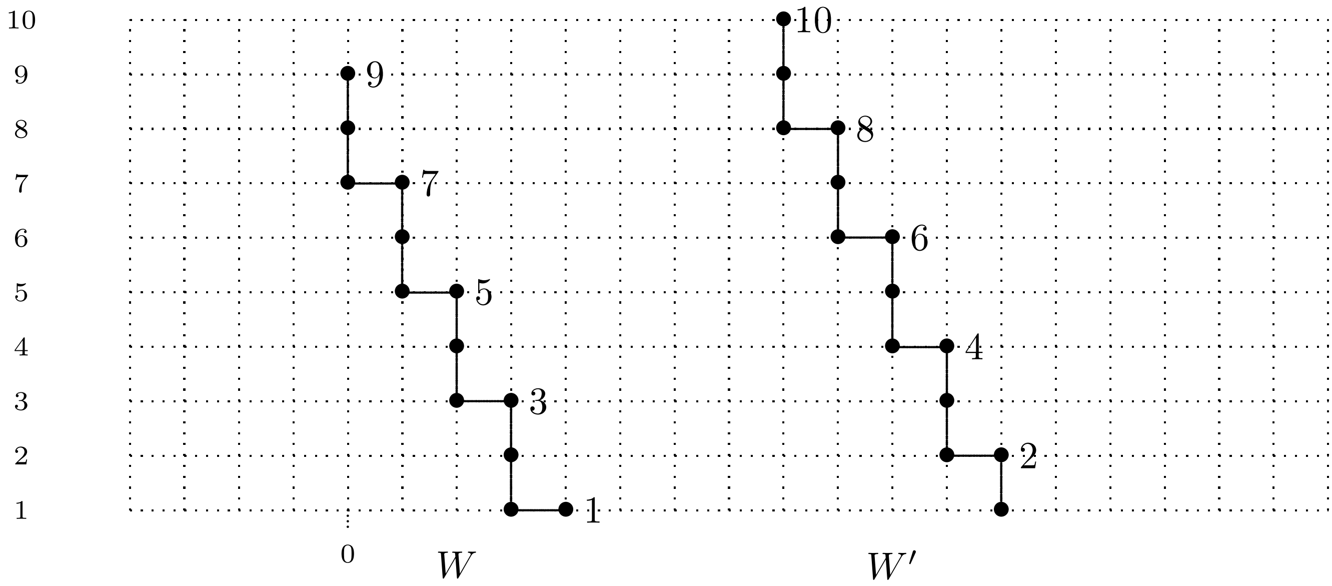
"even" "odd"

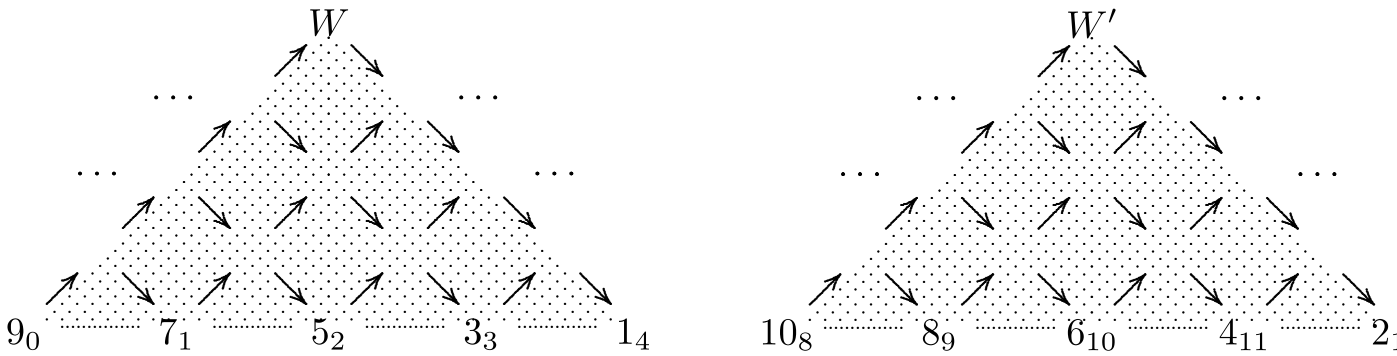
Note that we deal (on the lower boundary) with pairwise orthogonal bricks!

Why do we call the wing on the left "even", the wing on the right "odd"? This concerns the homology:

All but one composition factors of the homology of W have an even label. All but one composition factors of the homology of W' have an odd label.

The basic wings for n even.



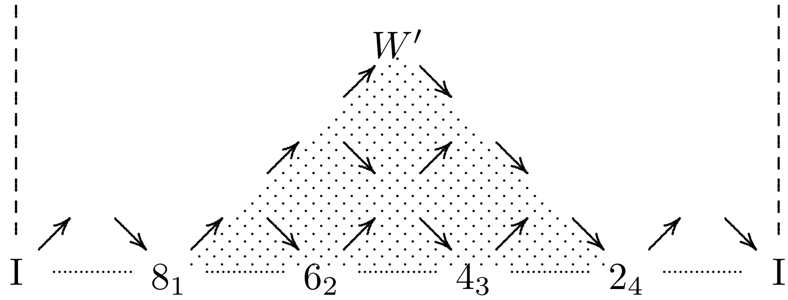


"even" "odd"

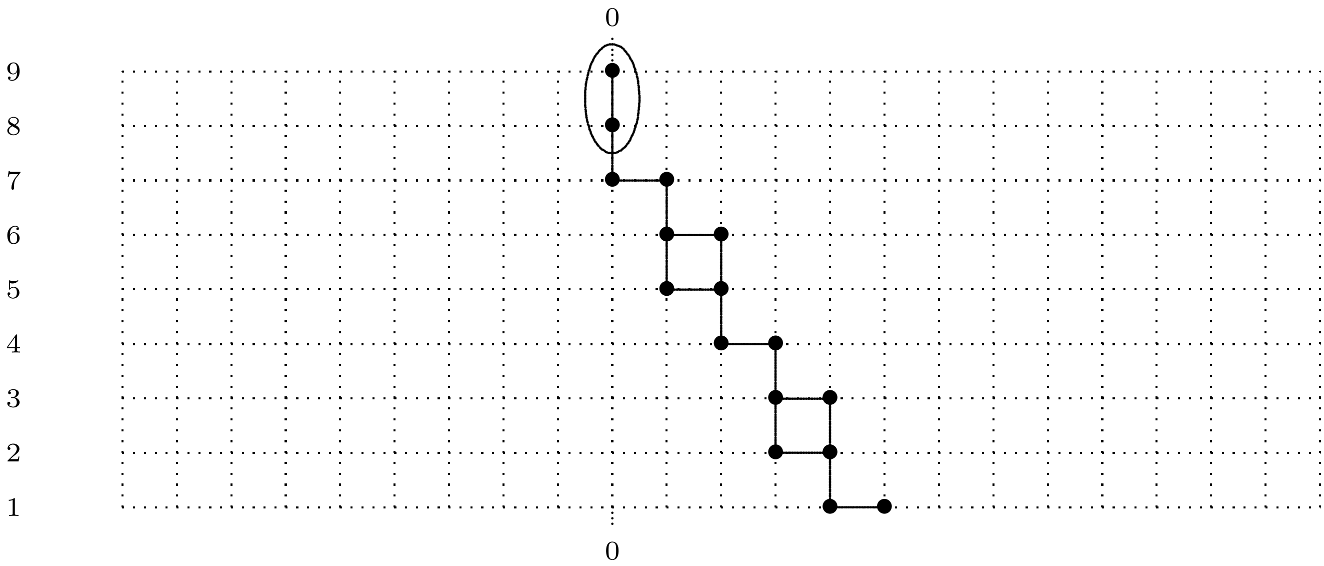
The even wing is known already, new is the odd wing on the right.

2.2. The neighbors of a basic wing

An example: The "odd" wing for $n = 9$.



Here, $\mathrm{I} = \mathrm{I}_0$ is the projective resolution of the module I in the column 0 (I indecomposable, with composition factors $S(n)$ and $S(n-1)$), thus:

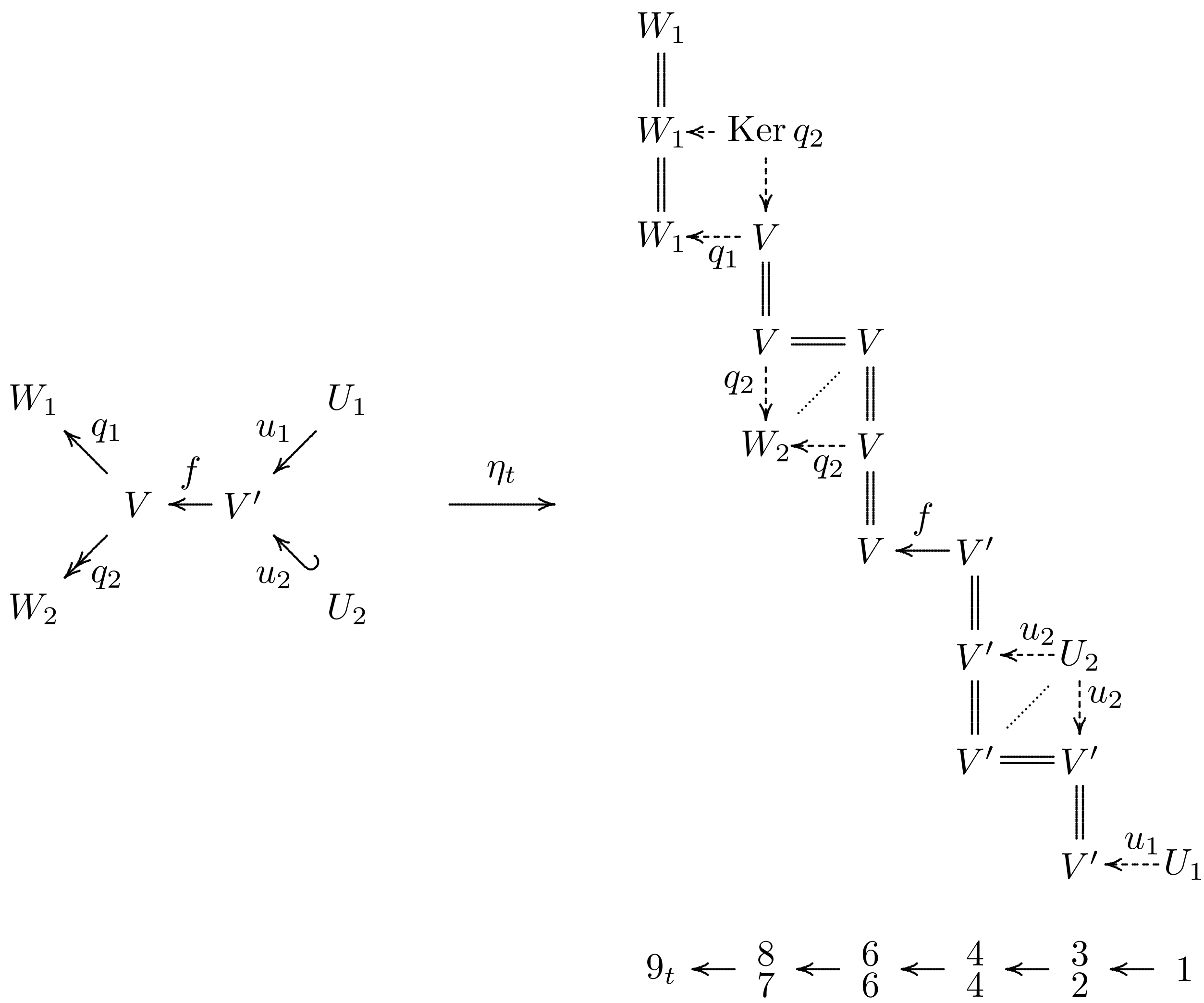


Note that we obtain a stable tube.

For $n \leq 8$ we deal with categories of locally finite type: there are no tubes.

For $n \geq 9$, the basic wings belong to components which are modulo the shift (not necessarily stable) tubes.

Recall: For any column index t, there is a full exact embedding functor η_t.



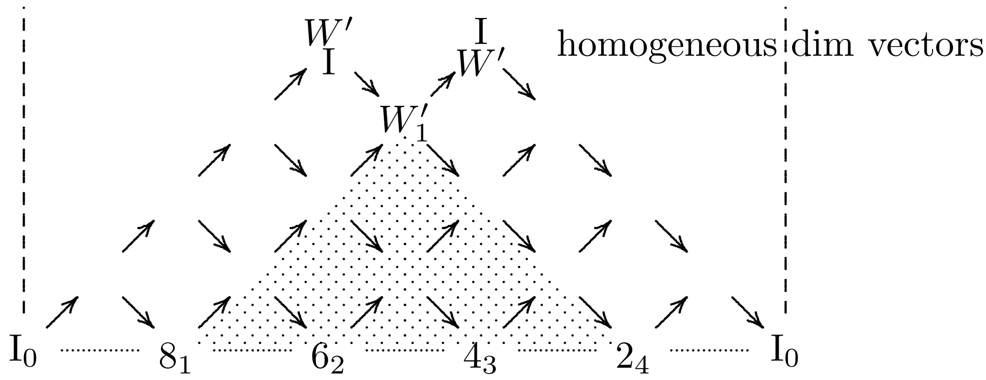
and we have

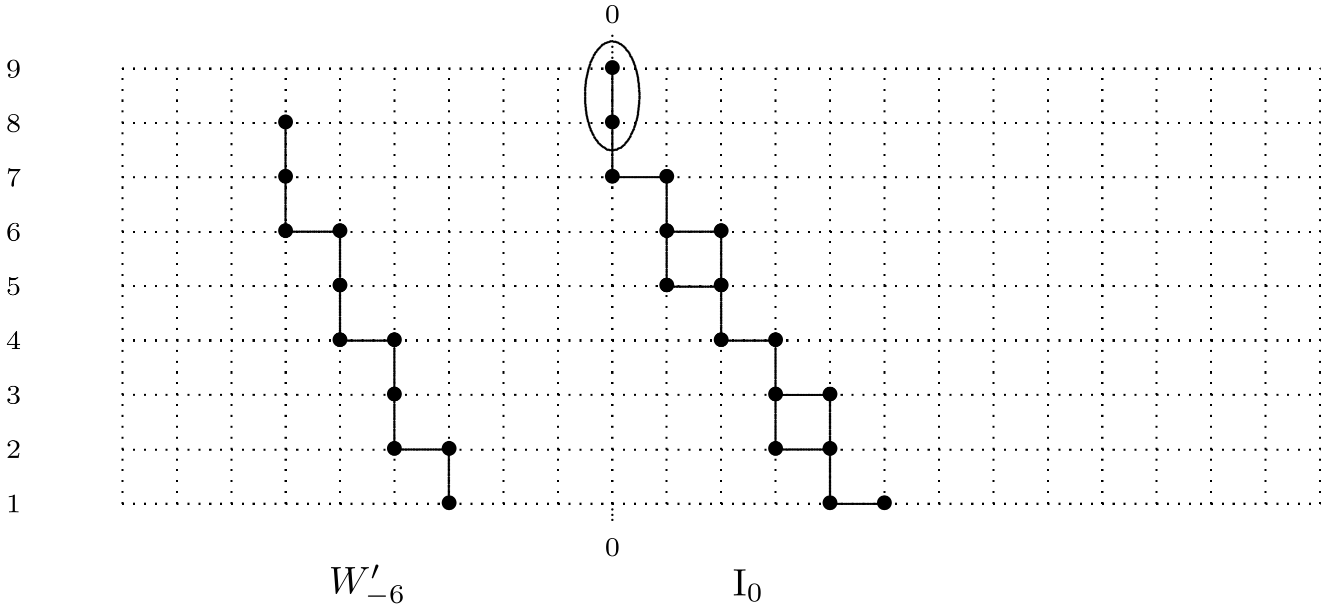
$$\eta \begin{pmatrix} 0 & & & 0 \\ & 1 & 1 & \\ 0 & & & 0 \end{pmatrix} = W'$$

$$\eta \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ 1 & & & 1 \end{pmatrix} = \mathrm{I}$$

The objects W' and I are orthogonal and have extensions in both ways, the extensions are exhibited in the next picture.

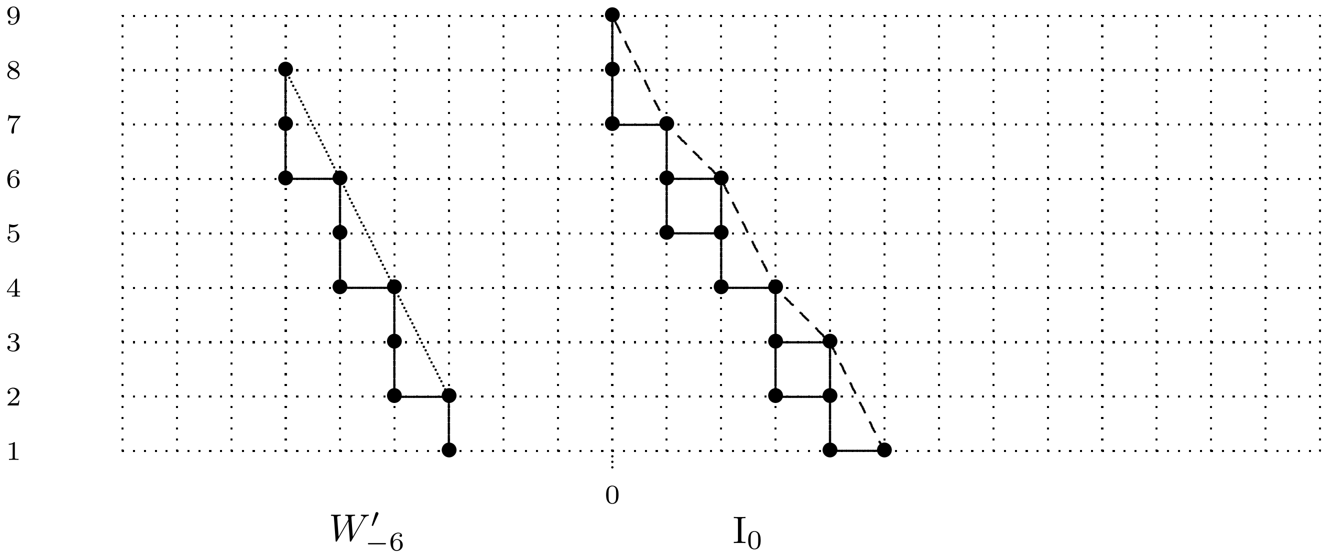
The odd-homology wing for $n = 9$.

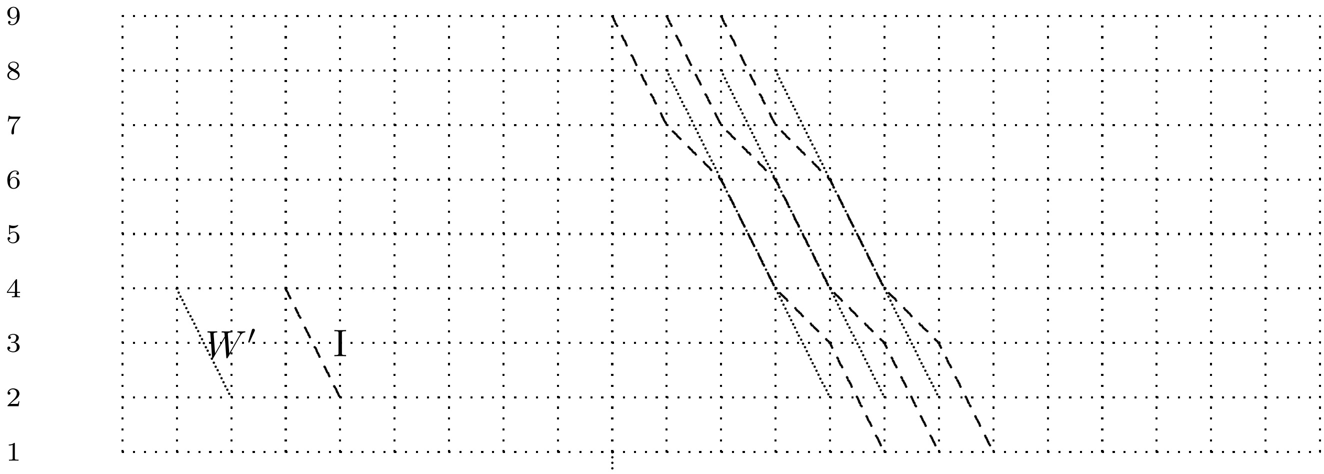




Of particular interest is object $\genfrac{}{}{0pt}{}{W'}{I}$, since as composition factors of its homology all odd labels (1,3,5,7,9) occur, and each with multiplicity one.

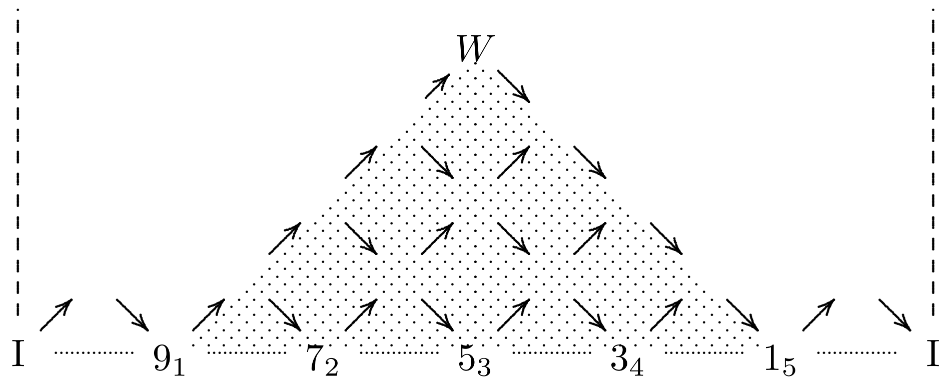
$n = 9$

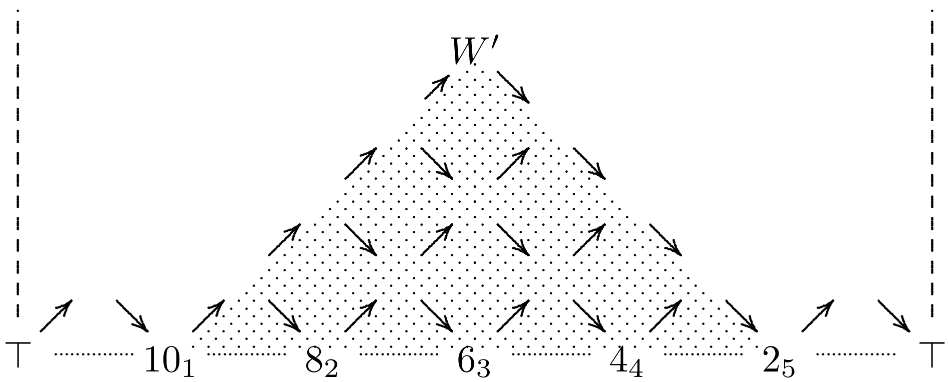




Next example: $n = 10$. This is a tubular catgeory.

There are (up to shift) two basic one-parameter families, the tubes of rank 6 are obtained from our wings:



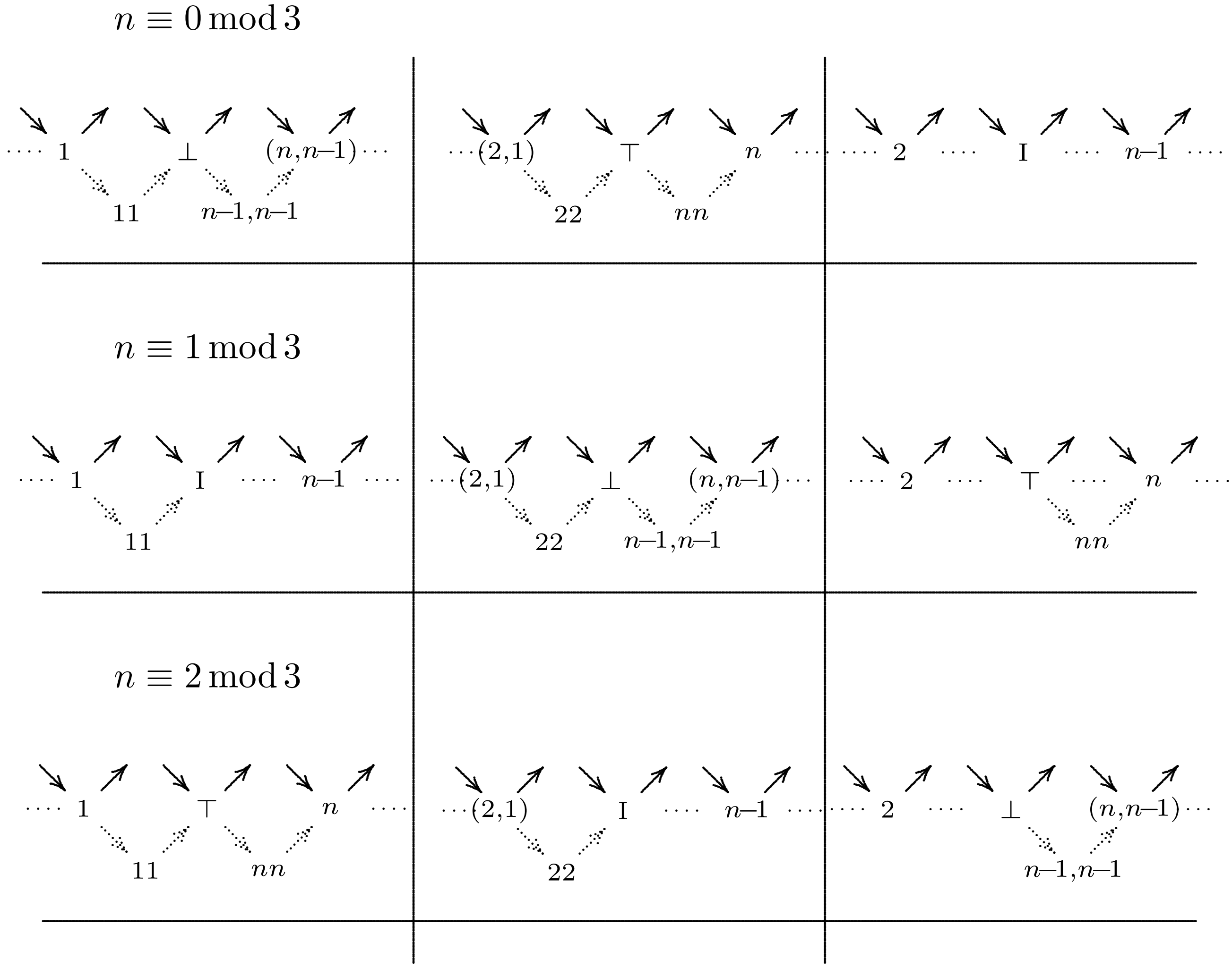


In general: Objects which are neighbors of the basic wings.

$\top = S(n)$
$\bot = S(n-1)$
I indecomposable of length 2, with composition factors $\top$ and $\bot$

Lemma. *The left neighbor of a basic wing is either* I *or* $\top$*,*
the right neighbor is I *or* $\top$ *or* $\bot$*.*

Alternatively: here are the AR-translates of $\top, \bot,$ and I:



Additional objects appear: $(t+1, t)$ with $t = 1, n-1$,
as well as projective-injectives $tt = (t,t)$ with $t = 1, 2, n-1, n$.

The main table

	Even Homology	Odd Homology
$n = 3$	$\top \overset{3}{\text{———}} \bot[1]$	$\mathrm{I} \overset{2}{\text{———}} \mathrm{I}[1]$
4	$\mathrm{I} \overset{3}{\text{———}} \mathrm{I}[1]$	$\top \overset{3}{\text{———}} \top[1]$
5	$\top \overset{4}{\text{———}} \top[1]$	$\mathrm{I} \overset{3}{\text{———}} \bot[1]$
6	$\mathrm{I} \overset{4}{\text{———}} \bot[1]$	$\top \overset{4}{\text{———}} \mathrm{I}[1]$
7	$\top \overset{5}{\text{———}} \mathrm{I}[1]$	$\mathrm{I} \overset{4}{\text{———}} \top[0]$
8	$\mathrm{I} \overset{5}{\text{———}} \top[0]$	$\top \overset{5}{\text{———}} \bot[0]$
9	$\top \overset{6}{\text{———}} \bot[0]$	$\boxed{\mathrm{I} \overset{5}{\text{———}} \mathrm{I}[0]}$
10	$\boxed{\mathrm{I} \overset{6}{\text{———}} \mathrm{I}[0]}$	$\boxed{\top \overset{6}{\text{———}} \top[0]}$
11	$\boxed{\top \overset{7}{\text{———}} \top[0]}$	$\mathrm{I} \overset{6}{\text{———}} \bot[0]$
12	$\mathrm{I} \overset{7}{\text{———}} \bot[0]$	$\top \overset{7}{\text{———}} \mathrm{I}[0]$
13	$\top \overset{8}{\text{———}} \mathrm{I}[0]$	$\mathrm{I} \overset{7}{\text{———}} \top[-1]$
14	$\mathrm{I} \overset{8}{\text{———}} \top[-1]$	$\top \overset{8}{\text{———}} \bot[-1]$
15	$\top \overset{9}{\text{———}} \bot[-1]$	$\mathrm{I} \overset{8}{\text{———}} \mathrm{I}[-1]$
	$\vdots$	$\vdots$

For $n = 9, 10, 11$ we obtain tubes!

The main table (1) $\qquad n = 3t + 1$

	Even Homology	Odd Homology
$n = 3$	$\top \overset{3}{-\!-\!-\!-} \bot[1]$	$\mathrm{I} \overset{2}{-\!-\!-\!-} \mathrm{I}[1]$
4	$\mathrm{I} \overset{3}{-\!-\!-\!-} \mathrm{I}[1]$	$\top \overset{3}{-\!-\!-\!-} \top[1]$
5	$\top \overset{4}{-\!-\!-\!-} \top[1]$	$\mathrm{I} \overset{3}{-\!-\!-\!-} \bot[1]$
6	$\mathrm{I} \overset{4}{-\!-\!-\!-} \bot[1]$	$\top \overset{4}{-\!-\!-\!-} \mathrm{I}[1]$
7	$\top \overset{5}{-\!-\!-\!-} \mathrm{I}[1]$	$\mathrm{I} \overset{4}{-\!-\!-\!-} \top[0]$
8	$\mathrm{I} \overset{5}{-\!-\!-\!-} \top[0]$	$\top \overset{5}{-\!-\!-\!-} \bot[0]$
9	$\top \overset{6}{-\!-\!-\!-} \bot[0]$	$\mathrm{I} \overset{5}{-\!-\!-\!-} \mathrm{I}[0]$
10	$\mathrm{I} \overset{6}{-\!-\!-\!-} \mathrm{I}[0]$	$\top \overset{6}{-\!-\!-\!-} \top[0]$
11	$\top \overset{7}{-\!-\!-\!-} \top[0]$	$\mathrm{I} \overset{6}{-\!-\!-\!-} \bot[0]$
12	$\mathrm{I} \overset{7}{-\!-\!-\!-} \bot[0]$	$\top \overset{7}{-\!-\!-\!-} \mathrm{I}[0]$
13	$\top \overset{8}{-\!-\!-\!-} \mathrm{I}[0]$	$\mathrm{I} \overset{7}{-\!-\!-\!-} \top[-1]$
14	$\mathrm{I} \overset{8}{-\!-\!-\!-} \top[-1]$	$\top \overset{8}{-\!-\!-\!-} \bot[-1]$
15	$\top \overset{9}{-\!-\!-\!-} \bot[-1]$	$\mathrm{I} \overset{8}{-\!-\!-\!-} \mathrm{I}[-1]$
	$\vdots$	$\vdots$

For $n = 3t + 1$ all $P(t)_j$ are shift-periodic!

2.3. Further wings.

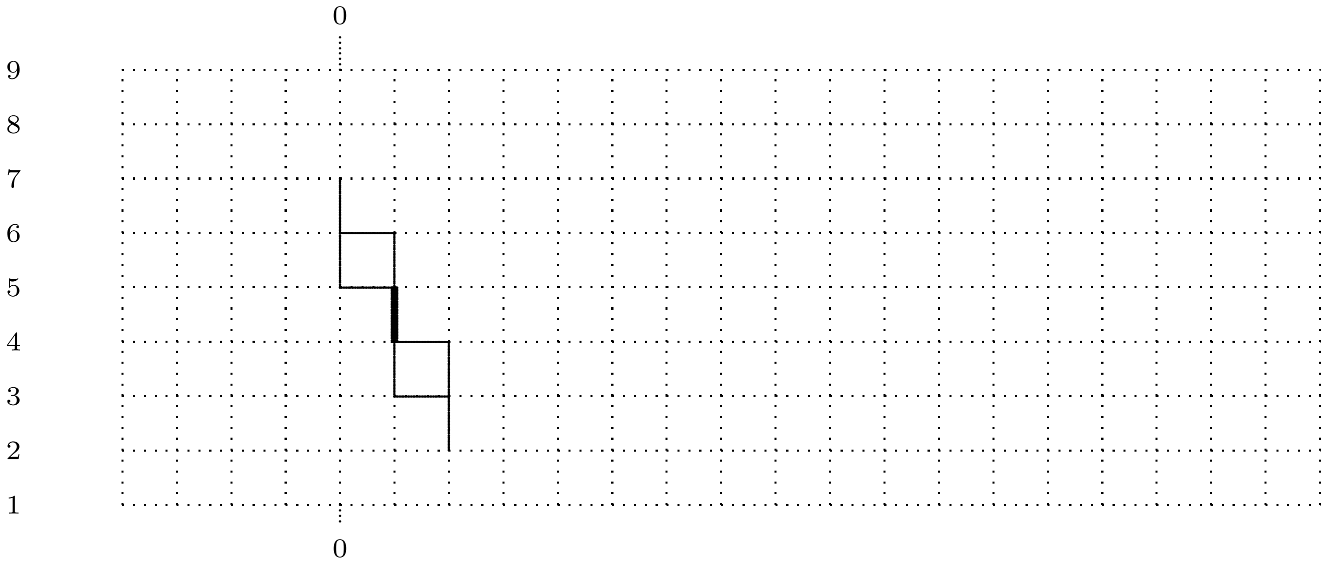
We consider the additional objects $(t+1, t)$ with $1 \leq t < n$, these are just the steps.

And they yield again Auslander-Reiten-sequences in $\mathcal{C}(A(n))$

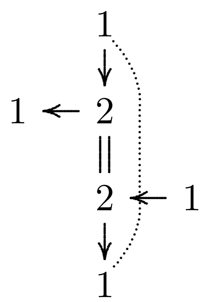
$$0 \to (t+3, t+2)_0 \to X \to (t+1, t)_1 \to 0$$

with indecomposable middle term.

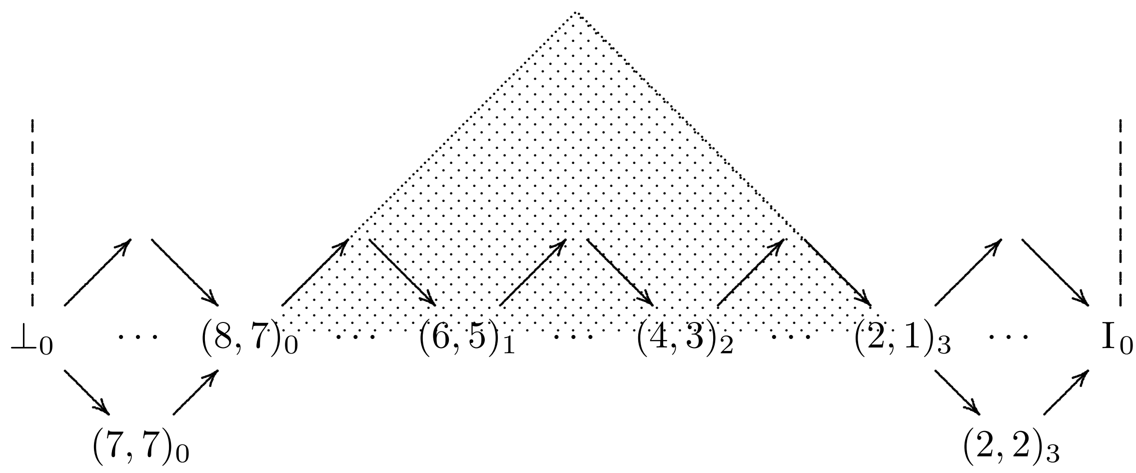
Here X is as follows (for $t = 4$):



Such an indecomposable complex X is unique, since it comes from



Again, we obtain a wing!
Always, the left neighbor is $\perp$.
For example for $n = 8$:



For n even, the right neighbor is one of $\top, \perp$ and I.
For n odd, the right neighbor is some new object.

The main table (2) $\quad n$ even

	Even Homology	Odd Homology	
$n = 3$	$\top \overset{3}{\text{———}} \bot[1]$	$\mathrm{I} \overset{2}{\text{———}} \mathrm{I}[1]$	
4	$\mathrm{I} \overset{3}{\text{———}} \mathrm{I}[1]$	$\top \overset{3}{\text{———}} \top[1]$	$\bot \overset{3}{\text{———}} \bot[1]$
5	$\top \overset{4}{\text{———}} \top[1]$	$\mathrm{I} \overset{3}{\text{———}} \bot[1]$	
6	$\mathrm{I} \overset{4}{\text{———}} \bot[1]$	$\top \overset{4}{\text{———}} \mathrm{I}[1]$	$\bot \overset{4}{\text{———}} \top[0]$
7	$\top \overset{5}{\text{———}} \mathrm{I}[1]$	$\mathrm{I} \overset{4}{\text{———}} \top[0]$	
8	$\mathrm{I} \overset{5}{\text{———}} \top[0]$	$\top \overset{5}{\text{———}} \bot[0]$	$\bot \overset{5}{\text{———}} \mathrm{I}[0]$
9	$\top \overset{6}{\text{———}} \bot[0]$	$\mathrm{I} \overset{5}{\text{———}} \mathrm{I}[0]$	
10	$\mathrm{I} \overset{6}{\text{———}} \mathrm{I}[0]$	$\top \overset{6}{\text{———}} \top[0]$	$\bot \overset{6}{\text{———}} \bot[0]$
11	$\top \overset{7}{\text{———}} \top[0]$	$\mathrm{I} \overset{6}{\text{———}} \bot[0]$	
12	$\mathrm{I} \overset{7}{\text{———}} \bot[0]$	$\top \overset{7}{\text{———}} \mathrm{I}[0]$	$\bot \overset{7}{\text{———}} \top[-1]$
13	$\top \overset{8}{\text{———}} \mathrm{I}[0]$	$\mathrm{I} \overset{7}{\text{———}} \top[-1]$	
14	$\mathrm{I} \overset{8}{\text{———}} \top[-1]$	$\top \overset{8}{\text{———}} \bot[-1]$	$\bot \overset{8}{\text{———}} \mathrm{I}[-1]$
15	$\top \overset{9}{\text{———}} \bot[-1]$	$\mathrm{I} \overset{8}{\text{———}} \mathrm{I}[-1]$	
	$\vdots$	$\vdots$	$\vdots$

For n even, all $P(t)_j$ are shift-periodic!

The main table (3)

$n = 6t+3,\ 6t+5$
(the remaining cases)

	Even Homology	Odd Homology	
$n=3$	$\top \overset{3}{\text{———}} \bot[1]$	$\boxed{\mathrm{I} \overset{2}{\text{———}} \mathrm{I}\,[1]}$	
4	$\mathrm{I} \overset{3}{\text{———}} \mathrm{I}\,[1]$	$\top \overset{3}{\text{———}} \top[1]$	$\bot \overset{3}{\text{———}} \bot[1]$
5	$\boxed{\top \overset{4}{\text{———}} \top[1]}$	$\mathrm{I} \overset{3}{\text{———}} \bot[1]$	
6	$\mathrm{I} \overset{4}{\text{———}} \bot[1]$	$\top \overset{4}{\text{———}} \mathrm{I}\,[1]$	$\bot \overset{4}{\text{———}} \top[0]$
7	$\top \overset{5}{\text{———}} \mathrm{I}\,[1]$	$\mathrm{I} \overset{4}{\text{———}} \top[0]$	
8	$\mathrm{I} \overset{5}{\text{———}} \top[0]$	$\top \overset{5}{\text{———}} \bot[0]$	$\bot \overset{5}{\text{———}} \mathrm{I}\,[0]$
9	$\top \overset{6}{\text{———}} \bot[0]$	$\boxed{\mathrm{I} \overset{5}{\text{———}} \mathrm{I}\,[0]}$	
10	$\mathrm{I} \overset{6}{\text{———}} \mathrm{I}\,[0]$	$\top \overset{6}{\text{———}} \top[0]$	$\bot \overset{6}{\text{———}} \bot[0]$
11	$\boxed{\top \overset{7}{\text{———}} \top[0]}$	$\mathrm{I} \overset{6}{\text{———}} \bot[0]$	
12	$\mathrm{I} \overset{7}{\text{———}} \bot[0]$	$\top \overset{7}{\text{———}} \mathrm{I}\,[0]$	$\bot \overset{7}{\text{———}} \top[-1]$
13	$\top \overset{8}{\text{———}} \mathrm{I}\,[0]$	$\mathrm{I} \overset{7}{\text{———}} \top[-1]$	
14	$\mathrm{I} \overset{8}{\text{———}} \top[-1]$	$\top \overset{8}{\text{———}} \bot[-1]$	$\bot \overset{8}{\text{———}} \mathrm{I}\,[-1]$
15	$\top \overset{9}{\text{———}} \bot[-1]$	$\boxed{\mathrm{I} \overset{8}{\text{———}} \mathrm{I}\,[-1]}$	
	$\vdots$	$\vdots$	$\vdots$

In these remaining cases, at least some $P(t)_j$ are shift-periodic.
For $n \geq 9$, we obtain one shift-tube of rank

$$\frac{n+3}{2} \qquad\qquad \frac{n+1}{2}$$

Consequences

Theorem 1.

For any n, some of the $P(t)_j$ are shift-periodic.

If $\equiv 0, 1, 2, 4 \mod 6$, then any $P(t)_j$ is¡ shift-periodic.

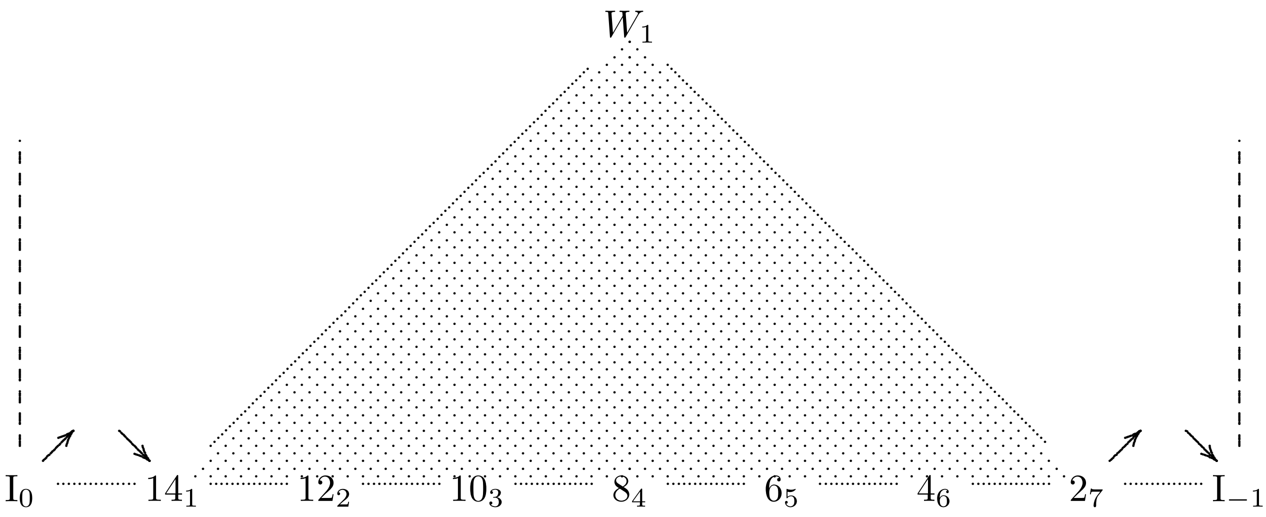
Theorem 2.

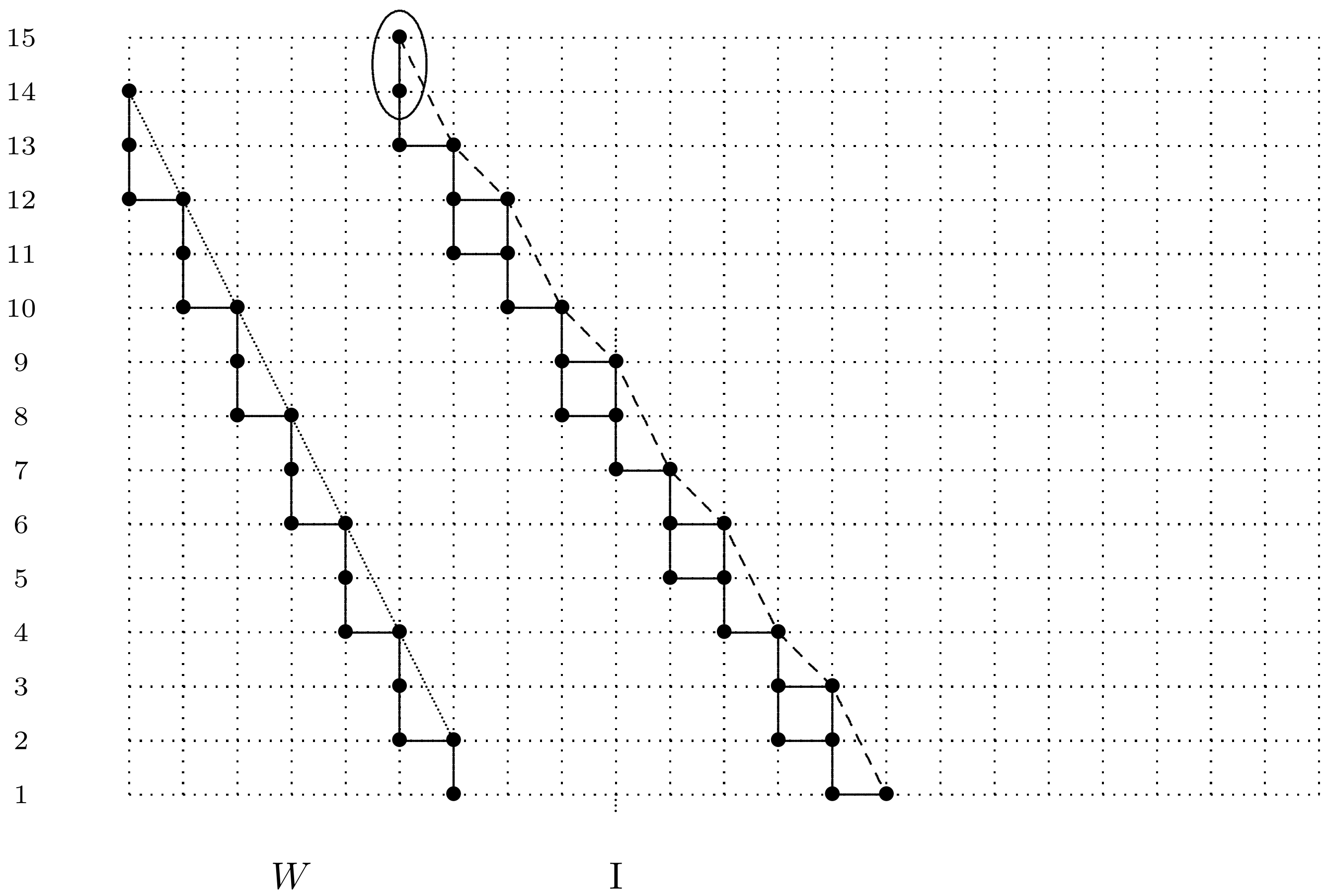
(a) *If $n = 6t + 4$, then $\tau_n^{(n+2)/2}$ is the shift functor $[\frac{n-10}{6}]$.*

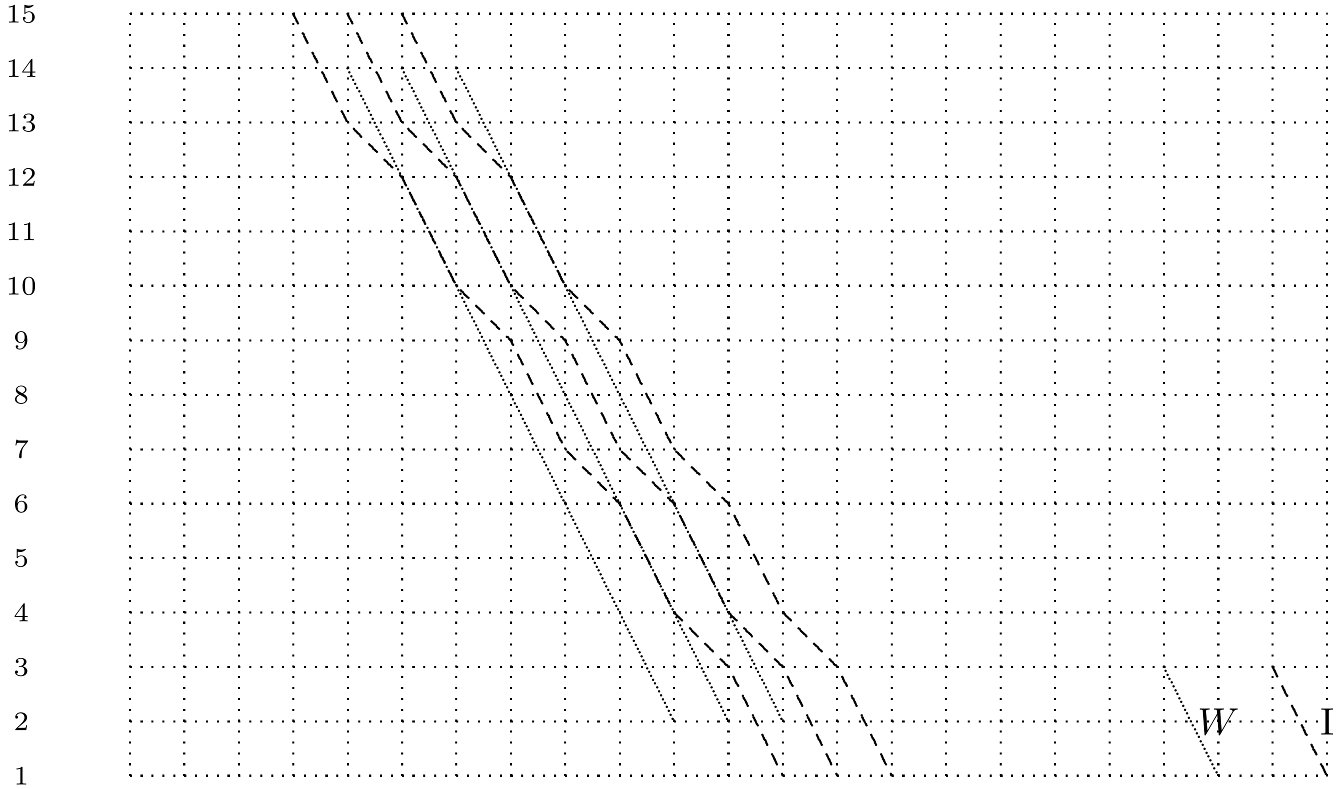
(b) *If $n = 6t + 1$, then τ_n^{n+2} is the shift functor $[\frac{n-10}{3}]$.*

(c) *If n is even, $\tau_n^{3(n+2)/2}$ is the shift functor $[\frac{n-10}{2}]$.*

3. Indecomposable perfect complexes of large width. The odd-homology wing for $n = 15$.







This yields indecomposable perfect complexes of arbitrary width.
(note: the wing is part of a component of type $\mathbb{A}_\infty$).
Thus: The strong global dimension of $A(15)$ is ∞.

Even: The strong global dimension of $A(12)$ is ∞.
Namely, for $A(12)$, there are 3 wings which together show that all $P(t)_j$ are shift-periodic with period 21.

Again these wings are part of a component of type $\mathbb{A}_\infty$
and thus they can be used to construct indecomposable perfect complexes of arbitrary width.

The main table (4) $\quad n = 12$

	Even Homology	Odd Homology	
$n = 3$	$\top \overset{3}{\text{———}} \bot[1]$	$\mathrm{I} \overset{2}{\text{———}} \mathrm{I}[1]$	
4	$\mathrm{I} \overset{3}{\text{———}} \mathrm{I}[1]$	$\top \overset{3}{\text{———}} \top[1]$	$\bot \overset{3}{\text{———}} \bot[1]$
5	$\top \overset{4}{\text{———}} \top[1]$	$\mathrm{I} \overset{3}{\text{———}} \bot[1]$	
6	$\mathrm{I} \overset{4}{\text{———}} \bot[1]$	$\top \overset{4}{\text{———}} \mathrm{I}[1]$	$\bot \overset{4}{\text{———}} \top[0]$
7	$\top \overset{5}{\text{———}} \mathrm{I}[1]$	$\mathrm{I} \overset{4}{\text{———}} \top[0]$	
8	$\mathrm{I} \overset{5}{\text{———}} \top[0]$	$\top \overset{5}{\text{———}} \bot[0]$	$\bot \overset{5}{\text{———}} \mathrm{I}[0]$
9	$\top \overset{6}{\text{———}} \bot[0]$	$\mathrm{I} \overset{5}{\text{———}} \mathrm{I}[0]$	
10	$\mathrm{I} \overset{6}{\text{———}} \mathrm{I}[0]$	$\top \overset{6}{\text{———}} \top[0]$	$\bot \overset{6}{\text{———}} \bot[0]$
11	$\top \overset{7}{\text{———}} \top[0]$	$\mathrm{I} \overset{6}{\text{———}} \bot[0]$	
12	$\mathrm{I} \overset{7}{\text{———}} \bot[0]$	$\top \overset{7}{\text{———}} \mathrm{I}[0]$	$\bot \overset{7}{\text{———}} \top[-1]$
13	$\top \overset{8}{\text{———}} \mathrm{I}[0]$	$\mathrm{I} \overset{7}{\text{———}} \top[-1]$	
14	$\mathrm{I} \overset{8}{\text{———}} \top[-1]$	$\top \overset{8}{\text{———}} \bot[-1]$	$\bot \overset{8}{\text{———}} \mathrm{I}[-1]$
15	$\top \overset{9}{\text{———}} \bot[-1]$	$\mathrm{I} \overset{8}{\text{———}} \mathrm{I}[-1]$	
	$\vdots$	$\vdots$	$\vdots$