The $(n-1)$-antichains in a root poset of width n.

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Abstract. Let Δ be a Dynkin diagram of rank $n \geq 2$ and Φ_+ the corresponding root poset (it consists of all positive roots with respect to some root basis). Antichains in Φ_+ of cardinality t will be called t-antichains. There always exists a unique maximal $(n-1)$-antichain A. We will show that for $\Delta \neq \mathbb{E}_6$ all roots in A have the same length with respect to the root basis, whereas for $\Delta = \mathbb{E}_6$, this antichain consists of roots of length 4 and 5. It follows from these considerations that in general one cannot recover the poset structure of Φ_+ by looking at the set of $(n-1)$-antichains in Φ_+.

Given a root system Φ of Dynkin type Δ, the root poset $\Phi_+ = \Phi_+(\Delta)$ is the set of positive roots in Φ with respect to some fixed choice of a root basis; here, one takes the following partial ordering: $x \leq y$ provided $y - x$ is a non-negative linear combination of elements of the root basis. The root posets and its antichains play an important role in many parts of mathematics.

If P is a poset, a subset A of cardinality t which consists of pairwise incomparable elements is called a *t-antichain.* The width of a finite poset P is the maximal number t such that there exists a t-antichain in P. It is well-known that the width of Φ_+ is the rank $n = n(\Delta)$ of the root system (thus the number of vertices of Δ). Since Φ_+ has an n-antichain, it has many $(n-1)$-antichains. We say that an $(n-1)$-antichain A in a poset P is *maximal*, provided it does not lie in the ideal generated by any other $(n-1)$-antichain. Of course, in a finite poset P, any $(n-1)$-antichain is contained in a maximal $(n-1)$-antichain.

For any positive root x, we denote by $l(x)$ the sum of the coefficients when written as a linear combination with respect to the root basis, thus $l(x) - 1$ is the height of x in the root poset Φ_+. For any natural number t, let Φ_t be the set of roots x with $l(x) = t$ (thus these are the elements of height $t-1$ in the root poset). Clearly, Φ_t is an antichain.

Theorem. *For any Dynkin diagram Δ of rank $n \geq 2$, there is a number $h = h(\Delta)$ with the following property: If $\Delta \neq \mathbb{E}_6$, then Φ_h is the unique maximal $(n-1)$-antichain of Φ_+. If $\Delta = \mathbb{E}_6$, then Φ_h is an $(n-1)$-antichain, but not maximal; the unique maximal $(n-1)$-antichain lies in $\Phi_h \cup \Phi_{h+1}$; and all $(n-1)$-antichains but the maximal one belong to the ideal generated by Φ_h.*

Δ	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$h(\Delta)$	2	3	3	3	4	5	7	5	5

Remark 1. Let us stress that in general, a poset of width n will have several maximal $(n-1)$-antichains, here are two examples for $n = 3$.

Remark 2. For a given Dynkin diagram Δ, we denote by $r(i) = r_\Delta(i)$ the number of positive roots x with $l(x) = i$. Then, the number $h(\Delta)$ listed above is the largest number i such that $r(i) = n-1$. It follows that $r(i)$ is also the largest number i such that $r(g-i+1) = 1$, where g is the Coxeter number for Δ. Namely, for all $1 \le i \le g$, one knows that

$$r(i) + r(g-i+1) = n$$

(see Humphreys [H, Theorem 3.20] and Armstrong [Ar, Theorem 5.4.1]; we are indebted to H. Thomas for pointed out these references).

The proof of the theorem will be given case by case, starting with the case $\mathbb{A}_n$ and ending with $\mathbb{F}_4$, the case $\mathbb{G}_2$ is of course trivial. If necessary, we will refer to the visualization of the various root posets as exhibited in the appendix.

The case $\mathbb{A}_n$. The root poset has the form of a triangle, it may be identified with the poset P of the real intervals $[i,j]$, where $1 \le i \le j \le n$ are integers, and the partial ordering is just given by inclusion. (The reader should be aware: For the interval $[i,j]$ considered as an element of $P = \Phi_+$, we have $l([i,j]) = j-i+1$, and the height of $[i,j]$ in $P = \Phi_+$ is equal to the (Euclidean) length $j-i$ of the interval $[i,j]$). The set P_2 consists of the elements $[1,2],[2,3],\dots[n-1,n]$, it is an $(n-1)$-antichain. If $n=2$, then P_2 is the only 2-antichain. Thus, we may assume that $n \ge 3$.

For $1 \le i \le j \le n$, let $X[i,j]$ be the set of elements in P which are comparable with $[i,j]$. Let $L[i,j]$ be the set of elements of the form $[i,t]$ with $i \le t \le j$ and $[s,j]$ with $1 \le s \le i$. Similarly, let $R[i,j]$ be the set of elements of the form $[i,t]$ with $j \le t \le n$ and $[s,i]$ with $i \le s \le j$. Note that both $L[i,j]$ and $R[i,j]$ are chains.

Assume that $j-i = a$. The complement of $X[i,j]$ in P can be covered by the sets $L[s,s+a]$ with $1 \le s < i$ and the sets $R[s,s+a]$ with $i < s \le n-a$. These are $(i-1)+(n-a-i) = n-a-1$ chains, thus if $[i,j]$ belongs to a b-antichain, then $b \le n-a$. This shows that the elements $[i,j]$ with $j-i \ge 2$ cannot belong to an $(n-1)$-antichain.

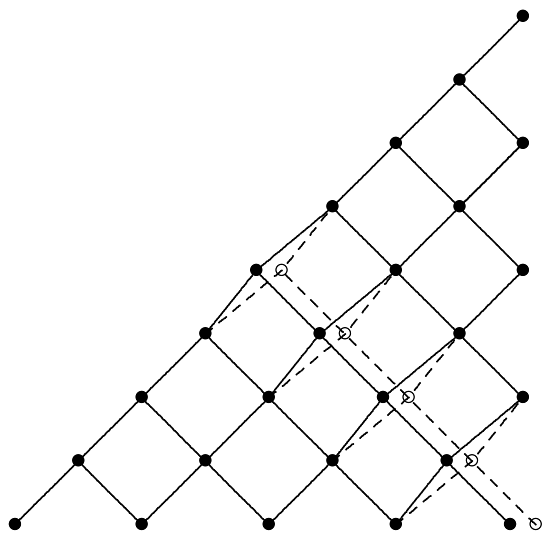
The cases $\mathbb{B}_n$ and $\mathbb{C}_n$. The root posets P of type $\mathbb{B}_n$ and $\mathbb{C}_n$ both can be identified with the subset of $\Phi_+(\mathbb{A}_{2n-1})$ consisting of all intervals $[i,j]$ which satisfy the additional condition $i+j \le 2n$. The set P_3 consists of the elements $[1,3],[2,4],\cdots,[n-1,n+1]$, it clearly is an $(n-1)$-antichain.

Let us show that no element $[i,j]$ with $l([i,j]) \ge 4$ belongs to an $(n-1)$-antichain. Again, we denote by $X[i,j]$ the set of elements which are comparable with $X[i,j]$.

Let $j-i = a$. The complement of $X[i,j]$ in P can be covered by the sets $L[s,s+a]$ with $1 \le s < i$ and the sets $R[s,s+a]$ with $i < s$ such that $[s,s+a]$ still belongs to P. But $[s,s+a]$ belongs to P means that $2s+a \le 2n$, thus $s \le n-\frac{a}{2}$. Thus the complement is covered by $(i-1)+(\lfloor n-\frac{a}{2}\rfloor - i) = \lfloor n-\frac{a}{2}\rfloor - 1 = n - \lceil\frac{a}{2}\rceil - 1$ chains. As a consequence, if $[i,j]$ belongs to a b-antichain, then $b \le \lfloor n-\frac{a}{2}\rfloor$.

This shows that for $a \ge 3$, the element $[i,j]$ cannot belong to an $(n-1)$-antichain. Therefore P_3 is the maximal $(n-1)$-antichain.

The case $\mathbb{D}_n$. For a visualization of the root poset $\Phi_+(\mathbb{D}_n)$ see the appendix. We see there nicely the two levels, the lower one is the root poset for one of the subdiagrams $\mathbb{A}_{n-1}$, the upper level has again the shape of a triangle, but rotated by 90^o. Unfortunately, the picture seems to conceal the action of an automorphism of order 2, thus let us "squeeze" the picture slightly. Here the case $n = 6$:



What we see immediately is the existence of a poset map π onto the root poset of type $\mathbb{B}_{n-1}$ or $\mathbb{C}_{n-1}$. For the further calculations it seems convenient to consider instead of $\Phi(\mathbb{D}_n)$ the root poset $P = \Phi(\mathbb{D}_{n+1})$. In terms of roots the map π is best expressed by looking at type $\mathbb{C}$ (not $\mathbb{B}$), thus let $P' = \Phi_+(\mathbb{C}_n)$ and define $\pi\colon P \to P'$ by

$$\pi\left(\begin{smallmatrix} & & & x_n & \\ x_1 & \cdots & x_{n-3} & x_{n-2} & x_{n-1}\end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1, & \cdots, & x_{n-3}, & x_{n-2}, & x_{n-1}+x_n \end{smallmatrix}\right).$$

Let us introduce now labels for the vertices of $P = \Phi_+(\mathbb{D}_{n+1})$ (having in mind the map π). We will use as vertices the pairs $[i, j]$ with $1 \le i \le j \le 2n-1$ such that $i+j \le 2n$ (these are represented in the picture by bullets) as well as additional pairs $[i, n-i]'$ with $1 \le i \le n$ (they are represented by small circles). The pairs $[i, j]$ are ordered in the same way as in P' (thus via the inclusion relation of intervals). Also, the pairs $[i, j]$ with $i + j \ne n$ together with the new vertices $[i, n-i]'$ form another copy of P' inside P (and there are no additional relations). Of course, the map π is now defined by $\pi([i, j]) = [i, j]$ and $\pi([i, n-i]') = [i, n-i]$.

Consider a vertex $[i, j]$ or $[i, j]'$ of P and assume that it belongs to some antichain A. Let $j - i = a$. We want to show that $|A| \ge n$ implies that $a \le 2$.

First, consider $[i, j]$ with $i + j \ne n$. We consider in P' the subsets $X[i, j]$, $L[s, s+a]$ with $1 \le s < i$ and $R[s, s+a]$ with $i < s \le n - \frac{a}{2}$, and form the inverse images under π. As we know from the last section, the number of these sets is $n - \lceil \frac{a}{2} \rceil$.

Clearly, $\pi^{-1}X[i, j]$ are the elements in P which are comparable with $[i, j]$ thus, A contains no other element from this set. Let us say that a subset of a poset is almost a chain provided it contains at most one pair of incomparable elements. The sets $\pi^{-1}L[s, s+a]$ and $\pi^{-1}R[s, s+a]$ are almost chains, thus the intersection of A with any such set consists of at most 2 elements and then these elements are of the form $[u, n-u]$ and $[u, n-u]'$ for some u. However, if A contains at most one element of the form $[u, n-u]$ and at most one element of the form $[u, n-u]'$. This shows: If A intersects one of the sets $\pi^{-1}L[s, s+a]$ and $\pi^{-1}R[s, s+a]$ in two elements, then all the others in at most one element. Thus the number of elements of A is bounded by $n - \lceil \frac{a}{2} \rceil + 1$. It follows that $|A| \geq n$ implies that $\lceil \frac{a}{2} \rceil + 1 \leq 1$, thus $a \leq 2$.

Next, consider a pair of the form $[i, j]$ with $j = n - i$ (the case of $[i, n-i]'$ is similar). Again, we consider in P' the subsets $X[i, j]$, $L[s, s+a]$ with $1 \leq s < i$ and $R[s, s+a]$ with $i < s \leq n - \frac{a}{2}$, and form the inverse images under π. Again, the number of these sets is $n - \lceil \frac{a}{2} \rceil$.

Now, $\pi^{-1}X[i, j]$ are the elements in P which are comparable with $[i, j]$ or $[i, j]'$, thus the intersection with A can consist of two elements. But now all the sets $\pi^{-1}L[s, s+a]$ and $\pi^{-1}R[s, s+a]$ are actually chains, thus A intersects any of these sets in at most one element, therefore we have again $|A| \leq n - \lceil \frac{a}{2} \rceil + 1$ (instead of verifying that the almost chains are chains, we also could use argue as in the case $i + j \neq n$; in particular, if A contains both $[i, j]$ and $[i, j]'$, then A intersects any set $\pi^{-1}L[s, s+a]$ and $\pi^{-1}R[s, s+a]$ in at most one element). As before, $n \leq |A| \leq n - \lceil \frac{a}{2} \rceil + 1$ implies that $a \leq 2$. This completes the proof.

The cases $\mathbb{E}_n$ and $\mathbb{F}_4$. We use the following criterion:

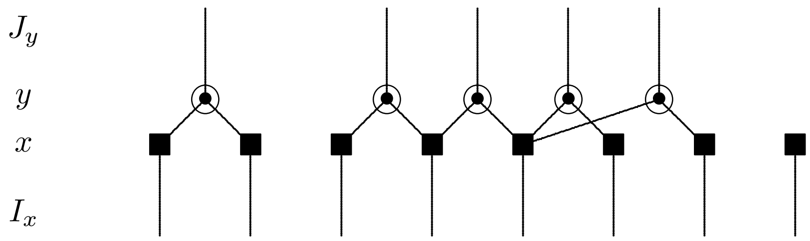
Lemma. *Let P be a finite poset. Let X be an n-antichain in P, let Y be an m-antichain with $m < n$, such that the ideal I generated by X and the coideal J generated by Y are complements of each other. Assume that I has width n and that J has width m. Finally, assume that for any element $y \in Y$ (with $1 \leq i \leq m$) we can choose two elements $f(y), g(y) \in X$ with $f(y) < y$ and $g(y) < y$ such that the graph with vertices the elements of $X \cup Y$ and edges $(y, f(y)), (y, g(y))$ for $y \in Y$ is a forest.*

Then P has width n and X is the unique maximal n-chain.

Proof: We show that for any element $z \in J$, the set $P(z)$ of elements not comparable with z has width at most $n - 2$.

Write I as a union of n chains and label the chain containing $x \in X$ by I_x. Similarly, J is the union of m chains, and the chain containing $y \in Y$ will be labeled J_y.

Thus we deal with the following situation: The elements of X are marked by black squares ■, the elements of Y are encircled. The lower lines indicate the chains I_i, the upper lines the chains J_i and the lines connecting the elements of X and Y form the forest which exists by assumption:



Thus, assume that $z \in J$, say $z \in J_y$ for some $y \in Y$. Then the elements of $P(z)$ do not belong to $J_y, I_{f(y)}, I_{g(y)}$. But deleting from P these chains, the remaining subset is the union of $n - 2$ chains, thus $P(z)$ has width at most $n - 2$.

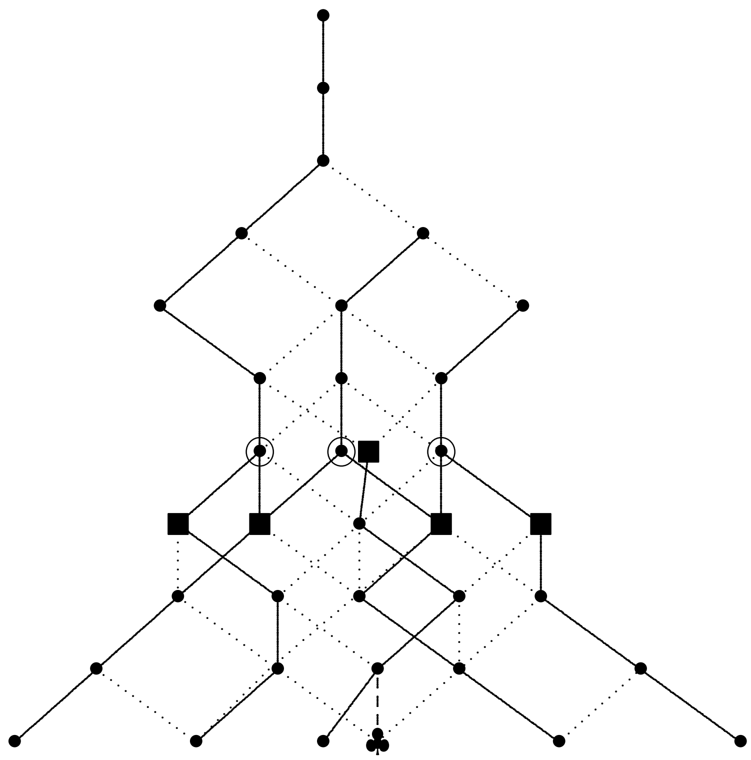
If B is any antichain in P, then either P contains an element $z \in J$, then we have shown that the cardinality of B is at most $n-1$. If B does not contain such an element, then B is contained in I and therefore the cardinality of B is at most n by assumption. This shows the width of P is at most n. Since A is an n-antichain in P, we see that the width of P is precisely n.

We consider now the case $\mathbb{E}_n$ with $n = 6, 7, 8$ and $\mathbb{F}_n$ with $n = 4$ case by case. Always, we mark $n-1$ vertices using black squares, they clearly form an $(n-1)$-antichain X and we claim that this is the unique maximal $(n-1)$-antichain. Let I be the ideal generated by X and $J = P \setminus I$. The encircled vertices generate the coideal J, they form an $(n-2)$-antichain which we call Y.

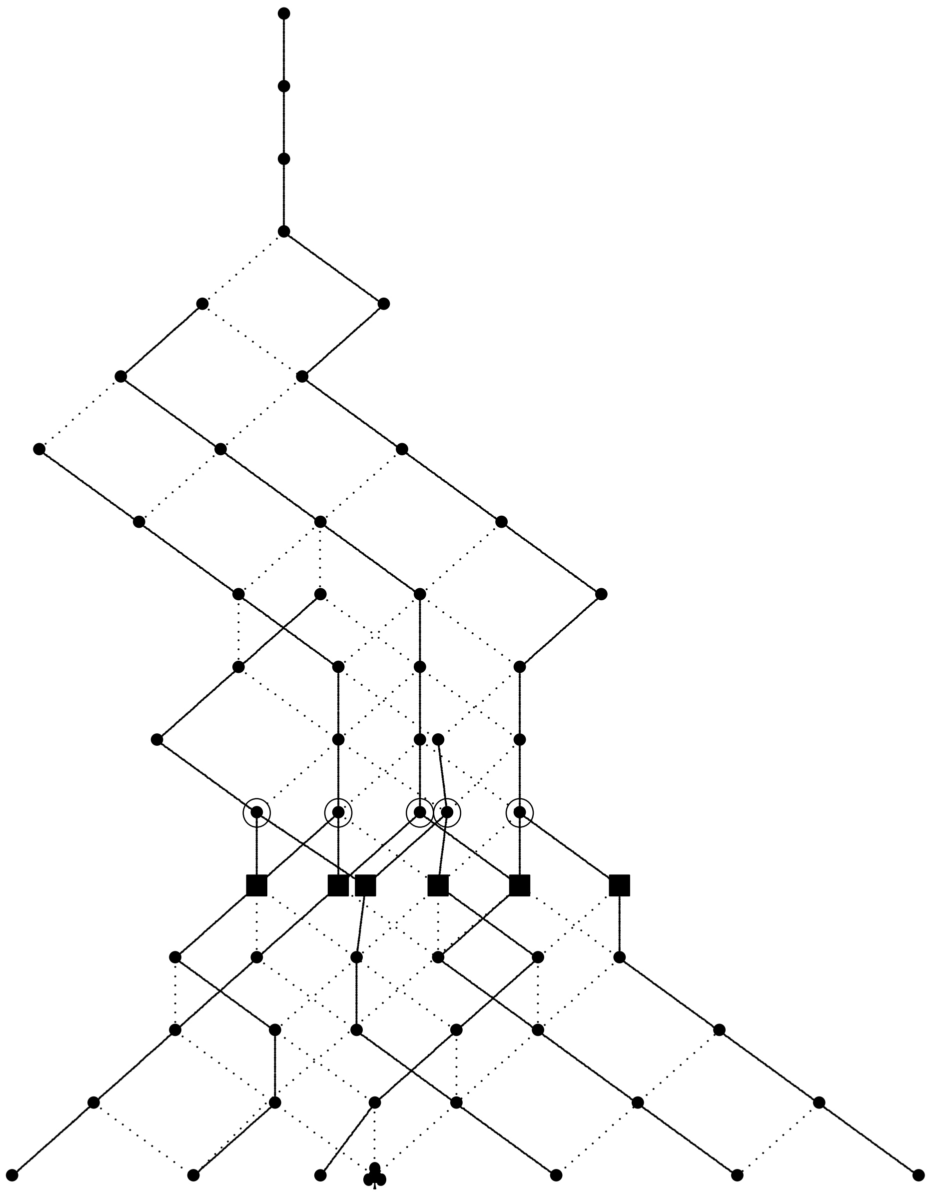
Since we want to show that no element of J belongs to an $(n-1)$-antichain, we may delete from the beginning any element which is smaller than all the elements of Y. Thus, we always delete the basis root with label c (marked by a ♣). Since it is well-known that the root basis is the only n-antichain of the root poset, we see that in this way we obtain as $P \setminus \{c\}$ a poset of width $n-1$.

In order to be able to apply the Lemma (with n replaced by $n-1$), we exhibit in all four cases in the lower part chains I_x, for any $x \in X$, in the upper part chains J_y, for any $y \in Y$, and in-between X and Y a forest given by the vertices $X \cup Y$ and edges $(y, f(y))$ and $(y, g(y))$, for $y \in Y$.

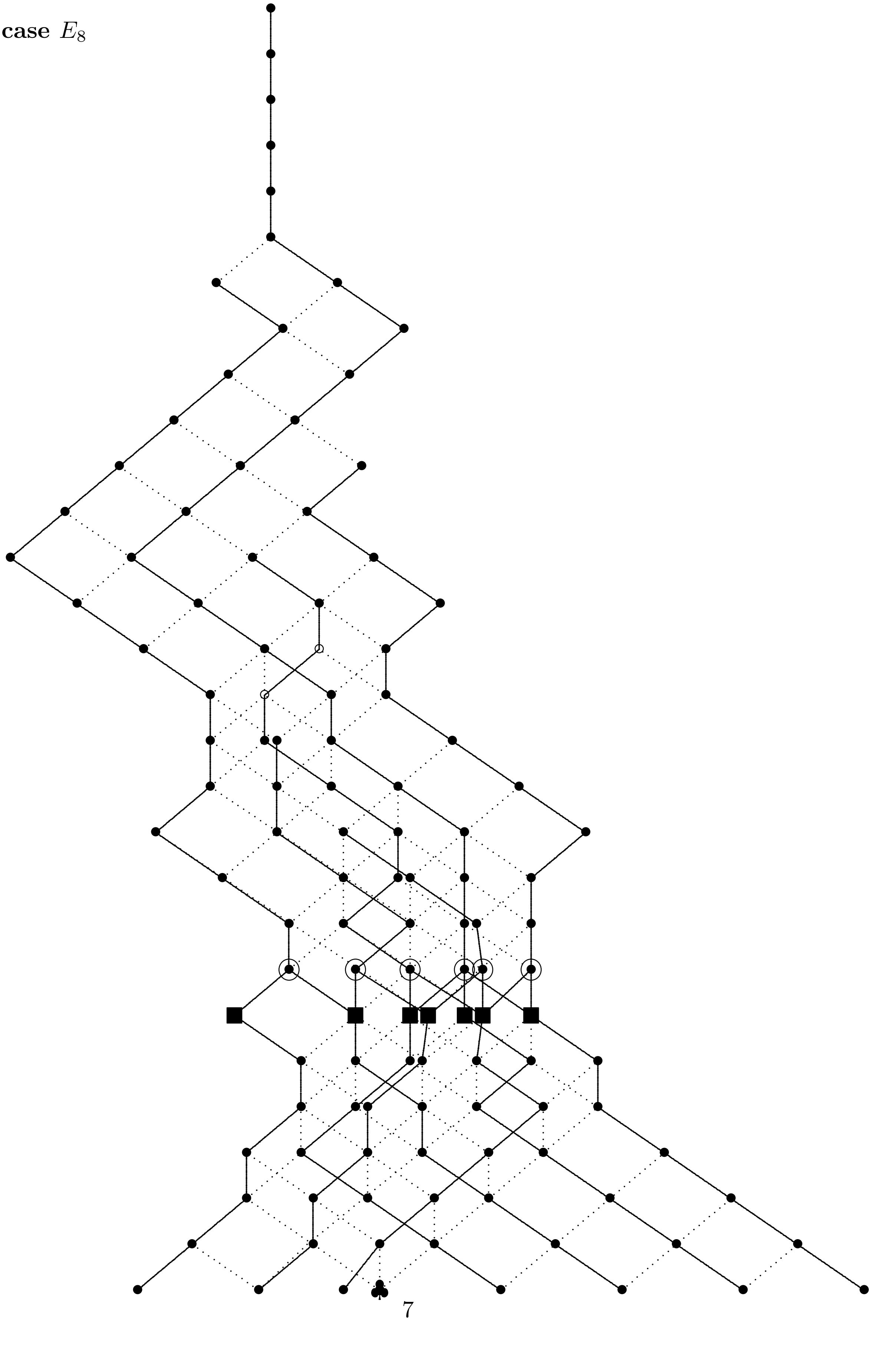
The case $\mathbb{E}_6$.



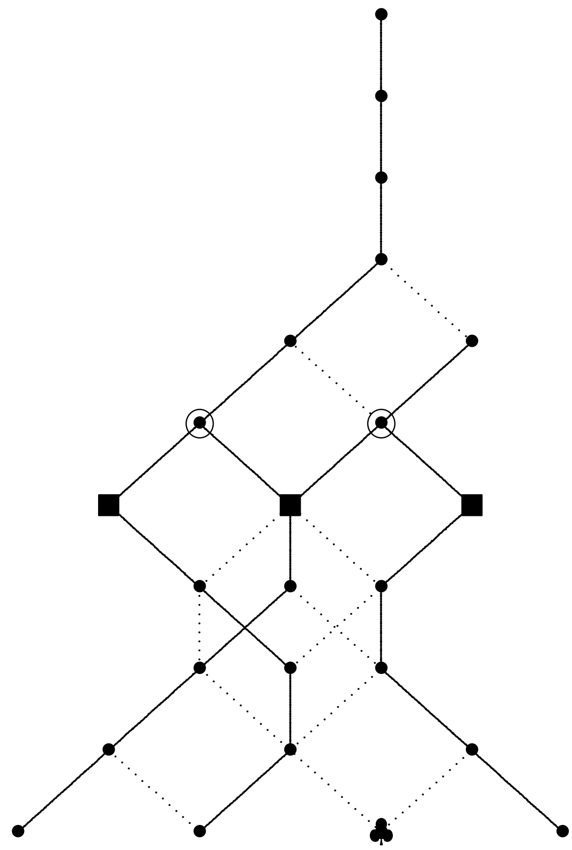
The case $\mathbb{E}_7$.



The case E_8



The case $\mathbb{F}_4$.



Final remark.

If P is a finite poset and t a non-negative integer, let $\mathcal{A}_t(P)$ be the set of t-antichains in P.

Given any Dynkin diagram Δ of rank n with root poset Φ_+, one knows that $|\mathcal{A}_t(\Phi_+)| = |\mathcal{A}_{n-t}(\Phi_+)|$, for $0 \le t \le n$, see [At]. In particular, since $A_1(\Phi_+) = \Phi_+$, we always have

$$|\mathcal{A}_{n-1}(\Phi_+)| = |\Phi_+|,$$

thus one may ask whether it is possible to recover also the partial ordering of Φ_+ by looking at the set of $(n-1)$-antichains. Note that given a poset P, there are several ways to consider $\mathcal{A}_t(P)$ as a poset (for example, given two antichains x, y, one may define $x \le y$ provided x lies in the ideal generated by y; alternatively, one also could use the property that y is contained in the coideal generated by x).

As we want to point out, *it is not possible in general to recover the partial ordering of* Φ_+ *by looking at the set of* $(n-1)$*-antichains.* Namely, consider the Dynkin type $\mathbb{F}_4$ and let I be the ideal generated by Φ_5. As we have shown above, $\mathcal{A}_3(\Phi_+) = \mathcal{A}_3(I)$. Now the poset I has an automorphism ϕ of order 2, and ϕ induces an automorphism on $\mathcal{A}_3(I)$ which is non-trivial (since obviously there are 3-antichains which are not invariant under ϕ). But Φ_+ itself has no non-trivial automorphisms.

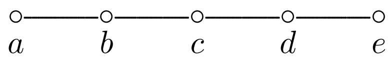
Appendix: Some pictures of the root posets.

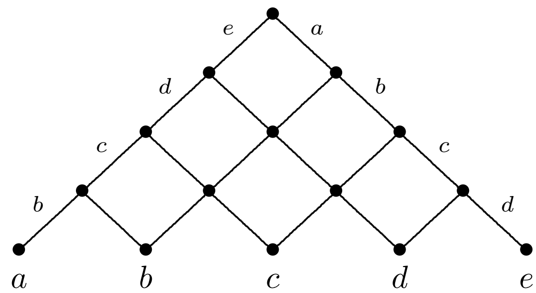
We exhibit a visualization of the root posets $\mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ and $\mathbb{F}_4$ which we found useful when preparing this note. Indeed, the pictures which we found in the literature and in the web did not seem to be quite convincing (for us). The pictures shown below draw the attention to the fact that for any pair of elements $x, z \in \Phi_+$, the interval $\{y \in \Phi_+ \mid x \leq y \leq z\}$ is a distributive lattice which can be constructed in a convenient way using 3-dimensional cubes. A drawback of our visualization is that it does not take into account the diversity of the various edges (which otherwise could be indicated by using different slopes). As a remedy, we label the edges by the corresponding basis vectors (it is sufficient to do this at the boundary, since the modularity transfers this information to the remaining edges).

For the sake of completeness, we also include the cases $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n$ and $\mathbb{G}_2$. It is well-known that the root posets $\mathbb{B}_n$ and $\mathbb{C}_n$ do not differ as long as we do not refer to the labels of the edges. But since we present the cases $\mathbb{D}_n, \mathbb{E}_n$ and $\mathbb{F}_4$ with labels, we do it also for $\mathbb{B}_n$ and $\mathbb{C}_n$, thus we have to present these types separately.

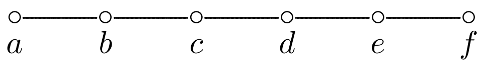
The cubical pictures for $\mathbb{D}_n, \mathbb{E}_n$ and $\mathbb{F}_4$ stress a division of the positive roots into "levels" which seems to be a kind of measure of the complexity of a positive root. The roots belonging to a fixed level form a planar graph, often a rectangle. All levels have a unique maximal element. Level 1 has $n-1$ minimal elements, all other levels have a unique minimal element. We list the minimal and the maximal elements, as well as the number of roots belonging to the level.

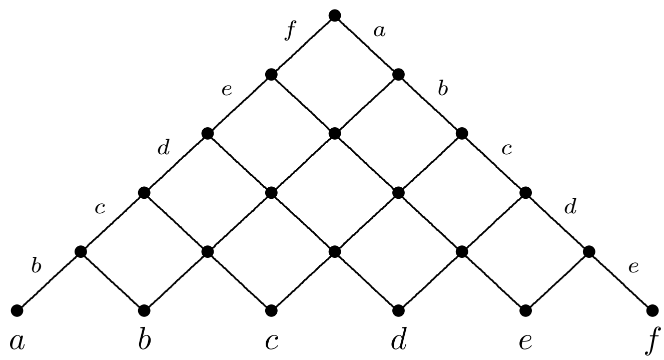
The root poset $\mathbb{A}_5$.



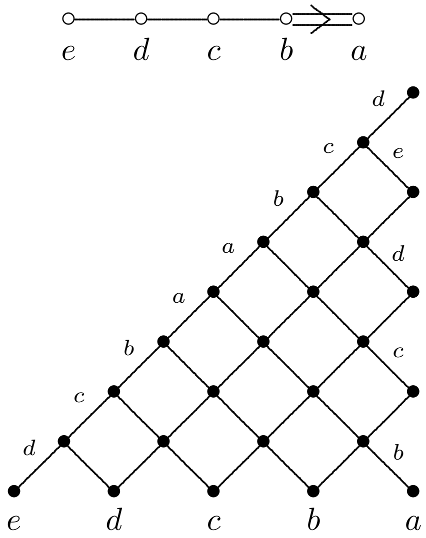


The root poset $\mathbb{A}_6$.

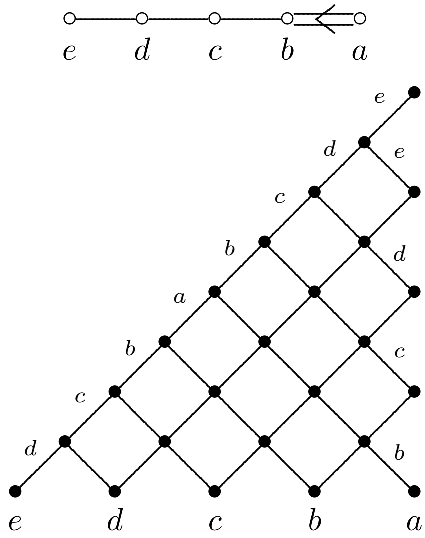




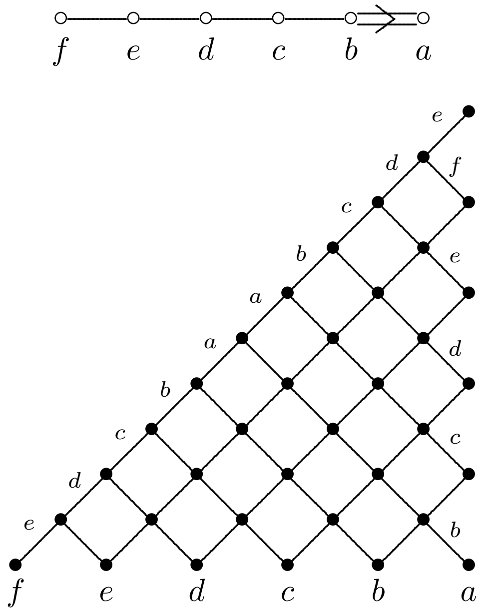
The root poset $\mathbb{B}_5$.



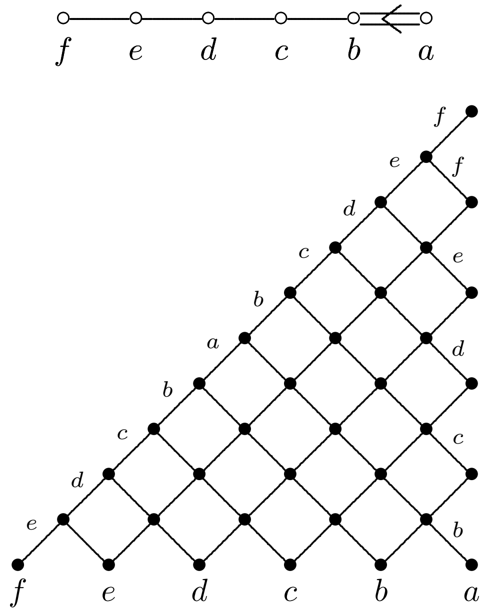
The root poset $\mathbb{C}_5$.



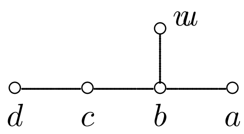
The root poset $\mathbb{B}_6$.

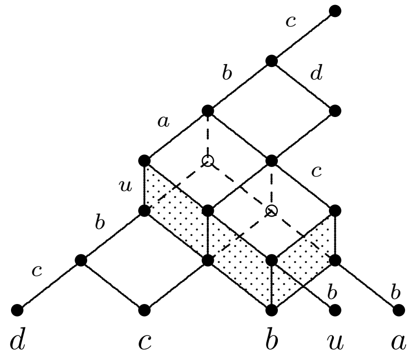


The root poset $\mathbb{C}_6$.

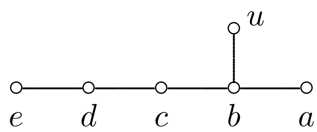


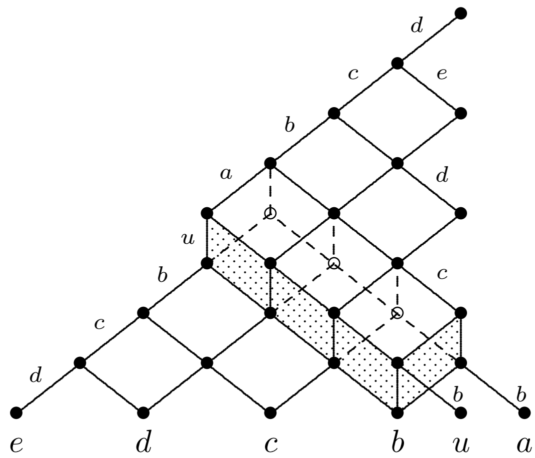
The root poset $\mathbb{D}_5$.





The root poset $\mathbb{D}_6$.

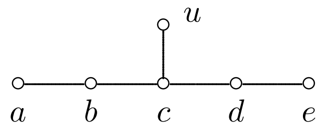


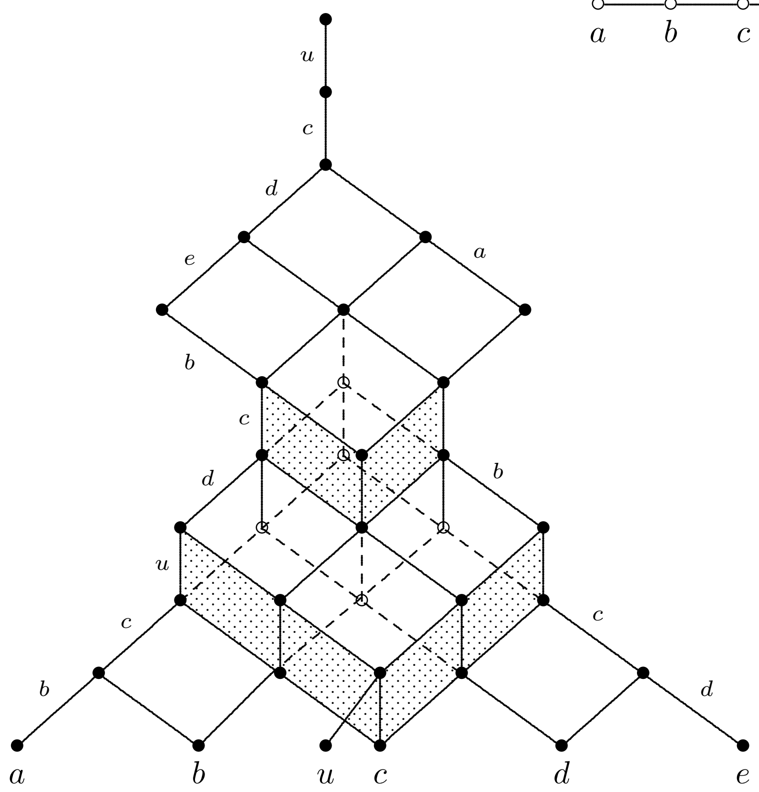


The levels for $\mathbb{D}_n$.

Level	Conditions	Minimal elements	Maximal element	number
1	$u = 0$	$n-1$ simple roots	$\begin{smallmatrix} & & & & 0 & \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{smallmatrix}$	$\binom{n}{2}$
2	$u = 1$	$\begin{smallmatrix} & & & & 1 & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & & & 1 & \\ 1 & 2 & \cdots & 2 & 2 & 1 \end{smallmatrix}$	$\binom{n}{2}$

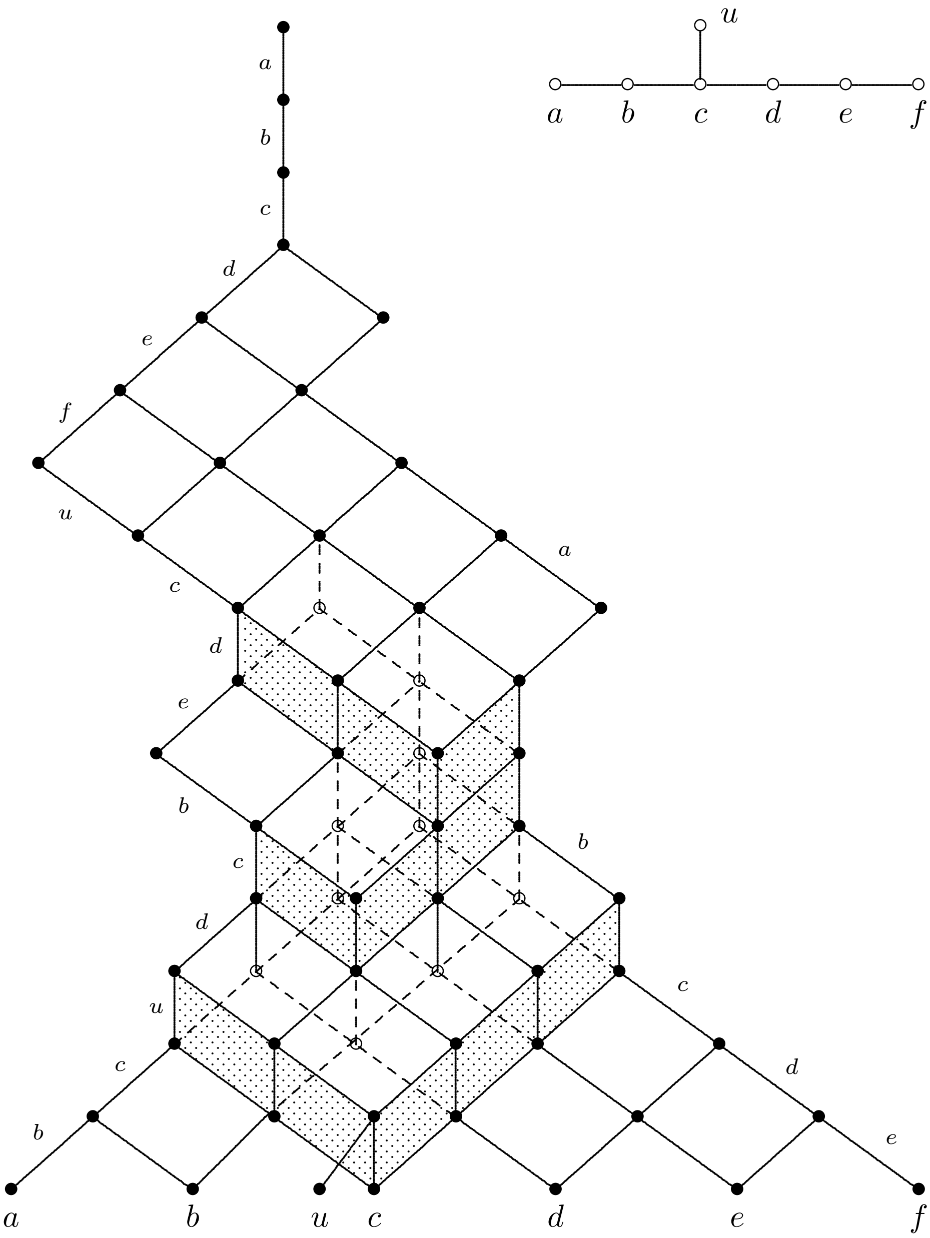
The root poset $\mathbb{E}_6$.





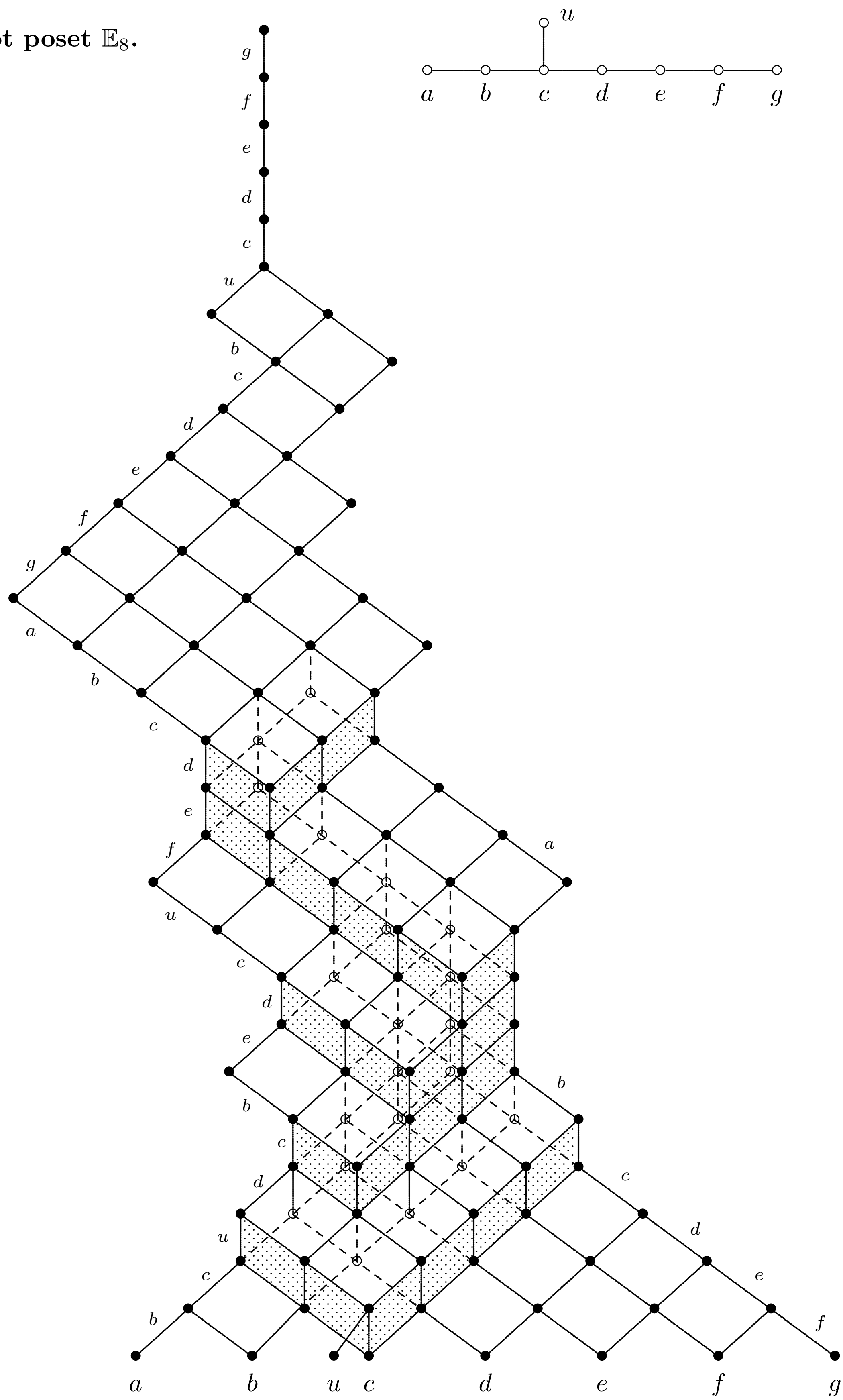
Level	Conditions	Minimal elements	Maximal element	number
1	$u = 0$	5 simple roots	$\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	15
2	$u = 1$	$\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	$3 \times 3 + 1$
3	$c \geq 2$	$\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$	$3 \times 3 + 2$

The root poset $\mathbb{E}_7$.



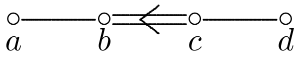
Level	Conditions	Minimal elements	Maximal element	number
1	$u = 0$	6 simple roots	0 1 1 1 1 1 1	21
2	$u = 1,\ c \leq 1$	1 0 0 0 0 0 0	1 1 1 1 1 1 1	$3 \times 4 + 1$
3	$c = 2,\ d = 1$	1 0 1 2 1 0 0	1 1 2 2 1 1 1	3×3
4	$d \geq 2$	1 0 1 2 2 1 0	2 2 3 4 3 2 1	$5 \times 3 + 5$

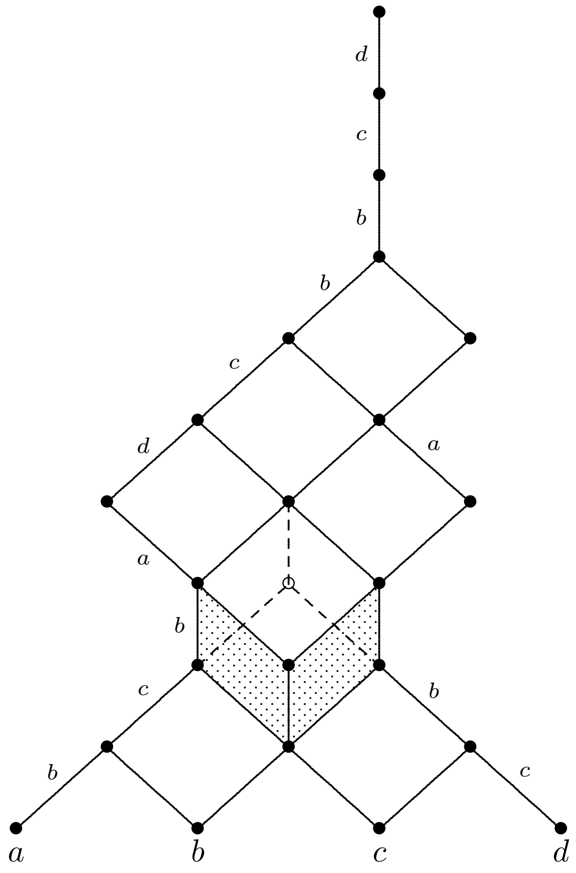
The root poset $\mathbb{E}_8$.



Level	Conditions	Minimal elements	Maximal element	number
1	$u=0$	7 simple roots	0 1 1 1 1 1 1 1	28
2	$u=1,\ c\le 1$	1 0 0 0 0 0 0 0	1 1 1 1 1 1 1 1	$3\times 5+1$
3	$c=2,\ d=1$	1 0 1 2 1 0 0 0	1 1 2 2 1 1 1 1	3×4
4	$d=2,\ e=1$	1 0 1 2 2 1 0 0	2 1 2 3 2 1 1 1	5×3
5	$d=2,\ e=3$	1 0 1 2 2 2 1 0	2 1 2 3 2 2 2 1	5×3
6	$d\ge 3$	1 1 2 3 3 2 1 0	3 2 4 6 5 4 3 12	$5\times 4+14$

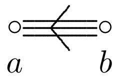
The root poset $\mathbb{F}_4$.

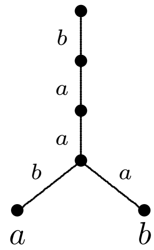




Level	Conditions	Minimal elements	Maximal element	number
1	$b\le 1$	4 simple roots	1 1 1 1	10
2	$b\ge 2$	0 2 1 0	2 4 3 2	$3\times 3+5$

The root poset $\mathbb{G}_2$.





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