# Algebraically Compact Modules Arising from Tubular Families. A Survey.

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ABSTRACT. Algebraically compact modules play an important role in representation theory: they have been introduced both by logicians and by algebraists as the basic class of representations of any ring. We will concentrate on algebraically compact modules which are related to tubes in the Auslander-Reiten quiver of an artin algebra. We are going to survey part of the present knowledge and present typical examples. The decomposition results are similar to those known for algebraically compact abelian groups, whereas some of the embedding properties of the relevant categories provide completely different features.

The main theme of this survey is the transfer of knowledge from finite length modules to arbitrary ones, a theme which has attracted a lot of interest in recent years, see for example the proceedings [KR] of a corresponding conference held at Bielefeld in 1998, in particular the introduction [R4]. The special case of dealing with a finite-dimensional hereditary algebra has been treated in [R1] and [R2], and it was observed recently [RR] by Reiten and the author that the structure theory developed in these papers can be obtained in the more general setting of concealed canonical algebras: any sincere stable separating tubular family $\mathbf{t}$ yields a subcategory $\mathcal{M} = \mathcal{M}(\mathbf{t})$ which strongly resembles the category of abelian groups. In between, Crawley-Boevey in his Trondheim lectures 1996 [CB] provided all the details for dealing with the indecomposable algebraically compact modules which arise from such a family $\mathbf{t}$; the present survey aims to complement this work by describing the structure of all the algebraically compact modules in $\mathcal{M}$. While the decomposition results presented here are direct analogues of those known for abelian groups [F,K], the proofs have to be screened quite carefully due to the fact that the decisive subcategory $\mathcal{M}$ is not abelian. On the other hand, we focus the attention to the embedding properties of the relevant categories and some of these results have no analogy at all in the abelian group case.

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[1] The lecture was devoted to constructions of algebraically compact modules for tame algebras. In the second part, the structure of algebraically compact modules for special biserial algebras was outlined, the third one dealt again with tubular algebras and provided a construction of infinite dimensional indecomposable representations with non-rational slope.

fruitful discussions. He is indebted to Idun Reiten and the referee for a careful reading of the manuscript.

### 1.1. Algebraic Compactness.

Let $\Lambda$ be a ring (associative, with 1). All the modules mentioned will be left modules. The basic notion is that of a direct sum: given submodules $M'$, $M''$ of a module $M$, one says that $M$ is the *direct sum* of $M'$ and $M''$ provided $M'+M'' = M$ and $M' \cap M'' = 0$ and one writes $M = M' \oplus M''$ in this case and calls $M', M''$ *direct summands* of $M$. The module $M$ is said to be *indecomposable* provided $M$ is non-zero and the only direct decompositions $M = M' \oplus M''$ are those with $M' = 0$ or $M'' = 0$.

In order to formulate the concept of algebraic compactness, one considers the solvability of linear equations. *Linear equations* are of the form $\sum_{j\in J} \lambda_j X_j = m$; here, $J$ is some index set, for every $j \in J$ there is given an element $\lambda_j \in \Lambda$, but only finitely many elements $\lambda_j$ are allowed to be non-zero, and $m$ is an element of the $\Lambda$-module $M$; as usually when dealing with equations, the symbols $X_j$ are called variables, and a solution of this equation is a sequence of elements $x_j$ with $j \in J$ such that $\sum_{j\in J} \lambda_j x_j = m$. Thus, the basic data for writing down a linear equation is the element $m \in M$ and the indexed family $(\lambda_j)_j$, this is just an element in the free $\Lambda$-module $\Lambda^{(J)}$ with basis indexed by $J$. Note that whereas we assume that almost all coefficients $\lambda_j$ are zero, there is no corresponding condition formulated for the elements $x_j \in M$, thus the solution is an element of the product module $M^J$. The reason for looking for elements in the product $M^J$ and not only in the direct sum module $M^{(J)}$ is the following: usually we will consider not just a single linear equation $\sum_{j\in J} \lambda_j X_j = m$, but a system

$$(*) \qquad \sum\nolimits_{j\in J} \lambda_{ij} X_j = m_i$$

of linear equations indexed by $i \in I$; here one uses two index sets $I, J$ (the set $I$ indexes the equations, the set $J$ the variables). By definition, a *solution* of this system is just a common solution of the various equations. Such a system is said to be *solvable* if at least one solution exists, and it is called *finitely solvable* provided for every finite subset $I' \subseteq I$, the system consisting of the equations with index in $I'$ has a solution. A $\Lambda$-module M is said to be *algebraically compact* provided every finitely solvable system of equations $(*)$ with values $m_i \in M$ has a solution (in $M$).

As typical examples, consider $\Lambda = \mathbb{Z}$, let $p$ be a prime number and consider the system $X_0 = m$, $X_{i-1} - pX_i = 0$ for $i \in \mathbb{N}_1 = \{1, 2, 3, \dots\}$, where $m$ is some element in an abelian group $M$. We construct two different factor groups of $N = \bigoplus_{i\geq 1} \mathbb{Z}a_i$, where $\mathbb{Z}a_i$ is a cyclic group of order $p^i$ (here, $a_i$ is a generator for the group $\mathbb{Z}a_i = \mathbb{Z}/p^i\mathbb{Z}$, for example $a_i = \overline{1} = 1 + p^i\mathbb{Z}$). First, let $U$ be the subgroup of $N$ generated by the elements $pa_{i+1} - a_i$, with $i \in \mathbb{N}_1$. The group $N/U$ is called a Prüfer group and is a well known example of an algebraically compact abelian group. Whatever element $m \in N/U$ we take, there is $m' \in N/U$ with $pm' = m$, thus the system mentioned at the beginning of the paragraph is solvable, for any $m \in N/U$. On the other hand, let $U'$ be the subgroup of $N$ generated by the elements $p^i a_{i+1} - p^{i-1} a_i$, for all $i \in \mathbb{N}_1$, and let $m = a_1 + U' \in N/U'$. Then it is easy to see that the system is finitely solvable, but not solvable. As a consequence, $N/U'$ is not algebraically compact.

In order to formulate the concept of pure injectivity, let us recall that a module homomorphism $M' \to M''$ is said to be a *pure monomorphism* provided the induced map $C \otimes_\Lambda M' \to C \otimes_\Lambda M''$ is injective for any $\Lambda^{\mathrm{op}}$-module $C$ (and it is sufficient to require this for all the finitely presented $\Lambda^{\mathrm{op}}$-modules $C$). The $\Lambda$-module $M$ is said to be *pure injective* provided any pure monomorphism $M \to M''$ is a split monomorphism.

*Let $\Lambda$ be any ring. A $\Lambda$-module $M$ is algebraically compact if and only if it is pure injective.*

The algebraically compact modules can be characterized in many different ways. We refer to the book of Jensen and Lenzing [JL] for a very readable account.

A module $M$ is said to be *discrete* provided any non-zero direct summand $M'$ of $M$ has an indecomposable direct summand $M''$. On the other hand, $M$ is said to be *superdecomposable* provided $M$ does not have any indecomposable direct summand.

*Any algebraically compact module $M$ is a direct sum of a discrete module $M_d$ and a superdecomposable module $M_s$.* Such a decomposition is unique in a quite strong sense: If $M = M_d \oplus M_s = M'_d \oplus M'_s$ with discrete modules $M_d, M'_d$ and superdecomposable modules $M_s, M'_s$, then also $M = M_d \oplus M'_s$ (and, consequently $M_d$ and $M'_d$ are isomorphic, and $M_s$ and $M'_s$ are isomorphic).

This decomposition property holds in general for algebraically compact modules (not only for artin algebras). For the proof, one uses the fact that one may embed the category $\operatorname{Mod}\Lambda$ into a Grothendieck category $\mathcal{G}$ such that the full subcategory $\mathcal{C}$ of $\operatorname{Mod}\Lambda$ given by all the algebraically compact modules is sent to the full subcategory $\mathcal{I}$ of all injective objects of $\mathcal{G}$ and yields an equivalence between $\mathcal{C}$ and $\mathcal{G}$. Such an embedding has been constructed by Gruson and Jensen. The properties of discreteness and superdecomposability are properties reflected by the endomorphism rings, thus they are preserved under categorical equivalences. Now the theorem of Gabriel and Oberst asserts that injective objects in $\mathcal{C}$ satisfy the corresponding statement: any such object is the direct sum of a discrete and a superdecomposable object and these decompositions have the required unicity property. (The reader should be aware that in general none of the direct summands $M_d$ and $M_s$ is unique as a subset of $M$, in contrast to frequent assertions in the literature; see the examples in [R4].) The Gruson-Jensen equivalence can be used also to obtain additional information on algebraically compact modules. For example, the endomorphism ring of an indecomposable injective object in $\mathcal{C}$ is local, thus:

*Any indecomposable algebraically compact module has a local endomorphism ring.*

We usually will deal with an artin algebra $\Lambda$ (recall that this means that $\Lambda$ is a $k$-algebra where $k$ is a commutative artinian ring $k$ such that $\Lambda$ as a $k$-module is finitely generated).

*Let $\Lambda$ be an artin algebra, and $M$ a $\Lambda$-module. Then $M$ is algebraically compact if and only if $M$ is a direct summand of a product of $\Lambda$-modules of finite length.*

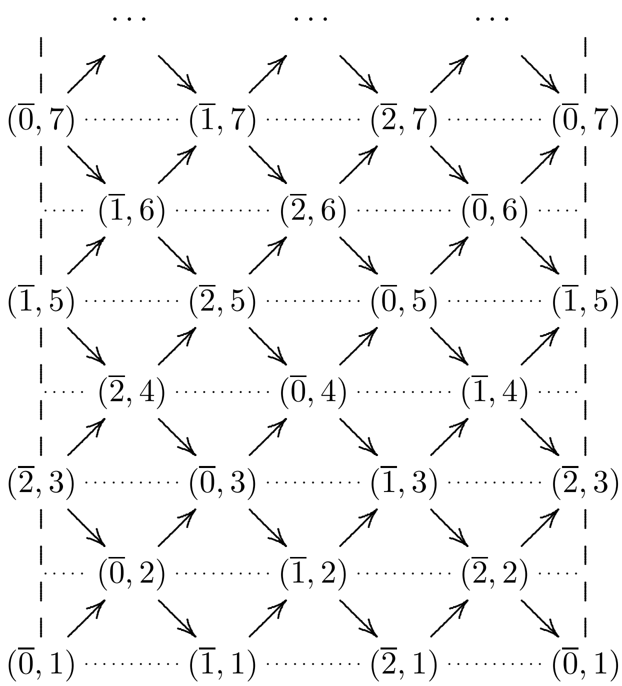
This characterization is due to Couchot [C], for a proof, we may refer for example to [R4].

Given a class $\mathbf{x}$ of modules we denote by $\Pi(\mathbf{x})$ the class of direct summands of products of modules in $\mathbf{x}$. Using this notation, we may reformulate the last assertion as follows: For an artin algebra $\Lambda$, the class $\Pi(\operatorname{mod}\Lambda)$ is the class of all algebraically compact modules.

**1.2. Tubes.**

A *translation quiver* (without multiple arrows) is of the form $\Gamma = (\Gamma_0, \Gamma_1, \tau)$, where $(\Gamma_0, \Gamma_1)$ is a quiver (thus, $\Gamma_0$ is a set, its elements are called vertices, and $\Gamma_1$ a subset of $\Gamma_0^2$, such an element $\gamma = (a, b)$ with $a, b \in \Gamma_0$ will be called an arrow and written in the form $\gamma\colon a \to b$), and there is given an injective map $\tau\colon \Gamma_0' \to \Gamma_0$, where $\Gamma_0'$ is a subset of $\Gamma_0$, such that for any vertex $z \in \Gamma_0'$, there exists an arrow $y \to z$ if and only if there exists an arrow $\tau(z) \to y$. The translation quivers we are dealing with have the property that for any vertex $z \in \Gamma_0'$ there exists an arrow $y \to z$, thus $\tau$ is uniquely determined by the set of triples $(x, y, z)$ where $z \in \Gamma_0', x = \tau(z)$ and $(y, z) \in \Gamma_1$, these triples will be considered as 2-simplices (and the elements of $\Gamma_1$ as 1-simplices). In this way, a translation quiver becomes a simplicial complex (with a fixed orientation of the 1-simplices).

A *stable tube of rank* $n$ is the following translation quiver: $\Gamma_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{N}_1$, thus the vertices are pairs of the form $(a, i)$, where $a$ is a residue class modulo $n$ and $i \geq 1$ is a natural number. There are arrows $(a, i) \to (a, i+1)$ and $(a, i+1) \to (a+1, i)$ and $\tau(a, i) = (a-1, i)$. The stable tubes of rank 1 are said to be *homogeneous*. Here is the stable tube of rank 3 (its simplicial complex):

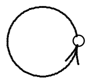


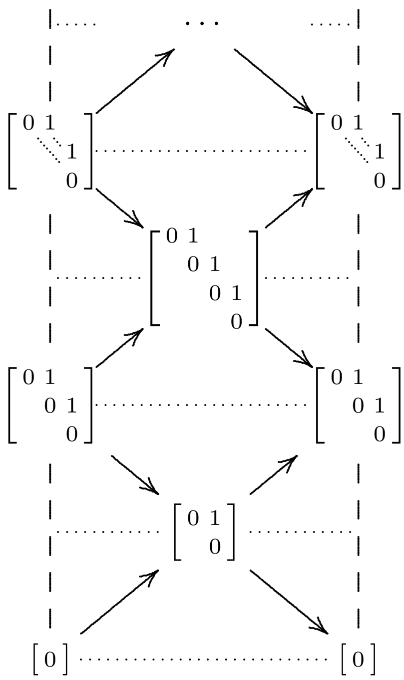

(the vertical boundary lines have to be identified, so that one really obtains a "tube" or a cylinder). The vertices with second index equal to 1 (those on the lower boundary line) will be said to lie on the *mouth* of the tube.

### 1.3. The Auslander-Reiten Quiver.

Tubes arise quite frequently in representation theory, namely as components of some Auslander-Reiten quivers.

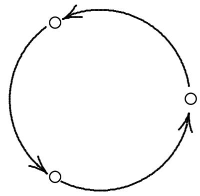
We recall the notion of an Auslander-Reiten quiver of an exact category $\mathcal{C}$ with Auslander-Reiten sequences. We need the notion of an irreducible map $f\colon X \to Z$, this is a non-invertible map such that for any factorization $f = f'f''$, the map $f''$ is split mono or the map $f'$ is split epi. An exact sequence $0 \to X \to Y \to Z \to 0$ is said to be an Auslander-Reiten sequence (starting in $X$ and ending in $Z$), provided both maps $X \to Y$ and $Y \to Z$ are irreducible, and $\mathcal{C}$ is said to have Auslander-Reiten sequences provided for any indecomposable non-injective object $X$ in $\mathcal{C}$, there exists an Auslander-Reiten sequence starting in $X$, and for any indecomposable non-projective object $Z$ in $\mathcal{C}$, there exists an Auslander-Reiten sequence ending in $Z$. For example, if $\mathcal{C} = \operatorname{mod}\Lambda$ is the category of all finite length modules, where $\Lambda$ is an artin algebra, then $\mathcal{C}$ has Auslander-Reiten sequences. The same is true for the category $\mathcal{C} = \operatorname{mod}_0 k[[T]]$ just considered. Given an exact category with Auslander-Reiten sequences, its *Auslander-Reiten quiver* $\Gamma(\mathcal{C})$ has as vertices the isomorphism classes $[X]$ of indecomposable objects $X$ in $\mathcal{C}$; given indecomposable objects $X, X'$, there is an arrow $[X] \to [X']$ in $\Gamma(\mathcal{C})$, provided there exists an irreducible map $X \to X'$ and finally, we draw a dotted edge connecting $[X]$ and $[Z]$ provided there is an Auslander-Reiten sequence starting in $X$ and ending in $Z$. (Sometimes it seems to be convenient to endow the arrows with multiplicities or "weights" in order to encode further information on the so-called "bimodule of irreducible maps", but in the present paper, there is no need to do so.)

As an example, consider the category $\operatorname{mod}_0 k[[T]]$ of finite length $k[[T]]$-modules, where $k[[T]]$ is the ring of formal power series in one variable $T$ with coefficients in the field $k$. Note that the $k[[T]]$-modules of finite length are the pairs $(V, f)$, where $V$ is a finite dimensional $k$-space (this is the underlying $k$-space of the module) and $f$ a nilpotent endomorphism of $V$ (given by the multiplication by $T$); such pairs are just the nilpotent representations of the one-loop quiver . The Auslander-Reiten quiver of $\operatorname{mod}_0 k[[T]]$ has the following form:



Here, we have exhibited the $k[[T]]$-module $(V, f)$ by just writing down $f$, or better a matrix which describes $f$ with respect to some basis; for example, in the lowest line, we see the $1 \times 1$-matrix $[\,0\,]$, thus the $k[[T]]$-module $(k, [\,0\,])$; note that this is the only simple $k[[T]]$-module.

As an obvious generalization of the category $\mathbf{n}(1) = \operatorname{mod}_0 k[[T]]$, one may consider the category $\mathbf{n}(t)$ of nilpotent representations of the cyclic quiver with $t$ vertices. The *cyclic quiver with $t$ vertices* has vertices indexed by the elements of $\mathbb{Z}/t\mathbb{Z}$, and there is one arrow $\overline{a} \to \overline{a+1}$, for any residue class $\overline{a} \in \mathbb{Z}/t\mathbb{Z}$. Here is the cyclic quiver with $t = 3$ vertices:



A representation of such a quiver is said to be *nilpotent* provided the endomorphisms which arise when composing maps going around the circle are nilpotent. Note that the categories $\mathbf{n}(t)$ all have Auslander-Reiten sequences and the Auslander-Reiten quiver of $\mathbf{n}(t)$ is just a stable tube of rank $t$. In particular, the stable tube of rank 3 exhibited in section 1.2 is the Auslander-Reiten quiver of the cyclic quiver with 3 vertices shown above.

### 1.4. Dedekind Rings.

Let $\Lambda$ be a **Dedekind ring** (for example $\Lambda = \mathbb{Z}$ or the polynomial ring $\Lambda = k[T]$ in one variable, where $k$ is a field). Note that the category of all $\Lambda$-modules of finite length is an abelian category which is serial: any indecomposable object $M$ has a unique composition series. Let $S$ be a simple $\Lambda$-module. There is an indecomposable $\Lambda$-module of length $i$ with all composition factors isomorphic to $S$, we denote it by $S[i]$ (stressing in this way that its socle is of the form $S$) or by $[i]S$ (in order to stress that its top is of the form $S$), it is unique up to isomorphism and all indecomposable $\Lambda$-modules are obtained in this way. There are chains of inclusions

$$S = S[1] \to S[2] \to \cdots,$$

the direct limit $S[\infty] = \varinjlim S[i]$ does not depend on the choice of the inclusion maps, it is called a *Prüfer* module. Dually, there are chains of epimorphisms

$$S = [1]S \leftarrow [2]S \leftarrow \cdots,$$

the inverse limit $\widehat{S} = \varprojlim S[i]$ does not depend on the choice of the epimorphisms, it is called an *adic* module.

(a) *All algebraically compact modules are discrete.*

(b) *Here is the list of all indecomposable algebraically compact modules*

- *The indecomposable modules of finite length, they are of the form* $\Lambda/\mathbf{m}^i$, *where* $\mathbf{m}$ *is a maximal ideal and* $i \in \mathbb{N}_1$.
- *The Prüfer modules* $\varinjlim \Lambda/\mathbf{m}^i$, *where* $\mathbf{m}$ *is a maximal ideal.*
- *The* $\mathbf{m}$*-adic modules* $\varprojlim \Lambda/\mathbf{m}^i$, *where* $\mathbf{m}$ *is a maximal ideal.*
- *The quotient field* $\operatorname{Quot}(\Lambda)$ *of* $\Lambda$.

**1.5. Tame hereditary algebras.**

Let $k$ be a field and $\Lambda$ be a finite dimensional $k$-algebra which is **connected, hereditary and tame.** Let us recall from [DR1] and [DR2] the structure of the category $\operatorname{mod}\Lambda$. There are two Auslander-Reiten components which are not stable, namely a preprojective component and a preinjective component. The remaining indecomposable modules belong to a $\mathbb{P}^1$-family of stable tubes (with $\mathbb{P}^1$ being the projective line over $k$), where all but at most three of these tubes are homogeneous. The structure theory for the algebraically compact $\Lambda$-modules is very similar to the case of a Dedekind ring. Actually, the full subcategory of all direct sums of indecomposables of finite length which are neither in the preprojective nor in the preinjective component is an abelian category which we denote by $\mathbf{t}(\Lambda)$. If $M$ belongs to $\mathbf{t}(\Lambda)$, the length of $M$ as an object of $\mathbf{t}(\Lambda)$ will be called its *regular length.* Note that the category $\mathbf{t}(\Lambda)$ is serial: any indecomposable object $M$ in $\mathbf{t}(\Lambda)$ has a unique composition series (inside $\mathbf{t}(\Lambda)$). Let $S$ be a simple object in $\mathbf{t}(\Lambda)$. There is an indecomposable object in $\mathbf{t}(\Lambda)$ of regular length $i$ which has $S$ as a submodule, we denote it by $S[i]$ (it is unique up to isomorphism). Dually, there is an indecomposable object in $\mathbf{t}(\Lambda)$ of regular length $i$ which has $S$ as a factor module, we denote it by $[i]S$ (again, it is unique up to isomorphism). And there are chains of inclusions

$$S = S[1] \to S[2] \to \cdots ,$$

the direct limit $\varinjlim S[i]$ does not depend on the choice of the inclusion maps, it is called a *Prüfer* module. Dually, there are chains of epimorphisms

$$S = [1]S \leftarrow [2]S \leftarrow \cdots ,$$

the inverse limit $\varprojlim S[i]$ does not depend on the choice of the epimorphisms, it is called an *adic* module.

(a) *All algebraically compact modules are discrete.*

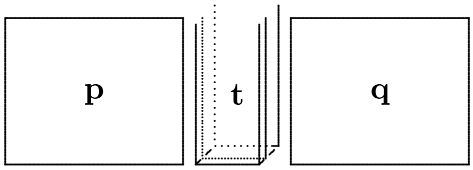
(b) *Here is the list of all indecomposable algebraically compact modules*

- *The indecomposable modules of finite length.*
- *The Prüfer modules* $\varinjlim S[i]$, *where* $S$ *is a simple object in* $\mathbf{t}(\Lambda)$
- *The adic modules* $\varprojlim [i]S$ *where* $S$ *is a simple object in* $\mathbf{t}(\Lambda)$
- *The unique indecomposable torsionfree divisible* $\Lambda$*-module* $G$ (as constructed in [R1]).

**2.1. Sincere stable separating tubular families**

In this part 2, we assume that $\Lambda$ is an artin algebra, and that $\mathbf{t}$ is a sincere stable separating tubular family. Recall that this means the following: a *tubular family* consists of all the indecomposables belonging to a set of tubes in the Auslander-Reiten quiver of $\Lambda$ (in particular, all the modules in $\mathbf{t}$ are of finite length). We use $\Omega$ as index set for the tubes: thus, for $\lambda \in \Omega$, $\mathbf{t}(\lambda)$ will be the corresponding tube, and we denote by $\mathcal{S}(\lambda)$ the set of isomorphism classes of objects on the mouth of the tube $\mathbf{t}(\lambda)$. Such a tubular family is said to be *stable* provided all the tubes are stable, thus provided it does not contain any indecomposable module which is projective or injective. A family of modules is said to be *sincere* provided every simple $\Lambda$-module occurs as the composition factor of at least one of the given modules. Finally, let us say that the tubular family $\mathbf{t}$ is *separating* provided the following two properties are satisfied: first, if $M$ is indecomposable with $\mathrm{Hom}(\mathbf{t}, M) \neq 0$ and $\mathrm{Hom}(M, \mathbf{t}) \neq 0$, then $M$ belongs to $\mathbf{t}$; and second, if $M, M'$ are indecomposables which do not belong to $\mathbf{t}$, and such that $\mathrm{Hom}(M, \mathbf{t}) \neq 0$ and $\mathrm{Hom}(\mathbf{t}, M') \neq 0$, then for any $\lambda \in \Omega$, any map $M \to M'$ has a factorization $M \to N \to M'$, where $N$ is indecomposable and belongs to $\mathbf{t}(\lambda)$.

Now let $\mathbf{t}$ be a separating tubular family. We denote by $\mathbf{p}$ the class of indecomposables $M$ of finite length with $\mathrm{Hom}(M, \mathbf{t}) \neq 0$, but which do not belong to $\mathbf{t}$, and by $\mathbf{q}$ the class of indecomposables $M$ of finite length with $\mathrm{Hom}(\mathbf{t}, M) \neq 0$, which again do not belong to $\mathbf{t}$. Then any indecomposable module of finite length belongs either to $\mathbf{p}$, $\mathbf{t}$ or $\mathbf{q}$




and one says that $\mathbf{t}$ *separates* $\mathbf{p}$ *from* $\mathbf{q}$. Note that there are no maps "backwards":

$$\mathrm{Hom}(\mathbf{t}, \mathbf{p}) = \mathrm{Hom}(\mathbf{q}, \mathbf{p}) = \mathrm{Hom}(\mathbf{q}, \mathbf{t}) = 0$$

and that any map from a module in $\mathbf{p}$ to a module in $\mathbf{q}$ can be factored through a module in $\mathbf{t}$ (even through an indecomposable lying in a prescribed tube inside $\mathbf{t}$). In case $\mathbf{t}$ is in addition sincere and stable, then all the indecomposable projective modules belong to $\mathbf{p}$, whereas the indecomposable injective modules belong to $\mathbf{q}$. As a consequence, in this case the modules which belong to $\mathbf{p}$ or $\mathbf{t}$ have projective dimension at most 1, those which belong to $\mathbf{t}$ or $\mathbf{q}$ have injective dimension at most 1. Also note that a stable separating tubular family $\mathbf{t}$ always is an exact abelian subcategory of $\mathrm{mod}\,\Lambda$ (and all the indecomposables in $\mathbf{t}$ are serial when considered as objects in this subcategory; those on the mouth of the tube are the simple objects).

The algebras $\Lambda$ with a sincere stable separating tubular family are the *concealed canonical algebras*. (They have been studied in [LM], [LP] and [RS], a review of the main steps of the construction can be found in [RR].)

Let $\mathcal{C}$ be the full subcategory of all modules $M$ with $\mathrm{Hom}(\mathbf{q}, M) = 0$ and $\mathcal{B}$ the full subcategory of all modules $M$ with $\mathrm{Hom}(M, \mathbf{p}) = 0$. The subcategory we are interested in is

$$\mathcal{M} = \mathcal{B} \cap \mathcal{C} = \{M \mid \mathrm{Hom}(M, \mathbf{p}) = 0 = \mathrm{Hom}(\mathbf{q}, M)\}$$

(let us stress that here we deal with arbitrary, not necessarily finitely generated modules).

Our aim is to describe the algebraically compact $\Lambda$-modules in $\mathcal{M}$. This category $\mathcal{M}$ has properties which are very similar to the complete category $\mathrm{Mod}\, R$ where $R$ is a Dedekind ring. However, one should be aware that $\mathcal{M}$ is not abelian in contrast to $\mathrm{Mod}\, R$ (but note that the full subcategory of $\mathcal{M}$ of all objects of finite length is just $\mathrm{add}\,\mathbf{t}$, thus this subcategory is abelian).

**2.2. The reduced and the divisible modules.**

We denote by $\mathcal{D}$ the full subcategory of all modules $M$ with $\mathrm{Hom}(M, \mathbf{t}) = 0$ and we set

$$\omega = \mathcal{C} \cap \mathcal{D} = \{M \mid \mathrm{Hom}(\mathbf{q}, M) = 0 = \mathrm{Hom}(M, \mathbf{t})\}.$$

Since $\mathbf{p}$ is cogenerated by $\mathbf{t}$, it follows that $\omega \subseteq \mathcal{M}$. The structure of the subcategory $\omega$ is completely known: all the modules in $\omega$ can be written as direct sums of indecomposable objects, and there are two kinds of indecomposables in $\omega$, namely there is (unique up to isomorphism) an indecomposable module $G$ in $\omega$ with a division ring as endomorphism ring (this module $G$ is called the generic module in $\omega$, note that $G$ has finite length when considered as a module over its endomorphism ring), the remaining indecomposables are the so called Prüfer modules.

Also, we recall that $\mathcal{D}$ is the torsion class of a split torsion pair, the corresponding torsionfree class will be denoted by $\mathcal{R}$, note that $\mathcal{R} = \{M \mid \mathrm{Hom}(G, M) = 0\}$, according to [RR]. The modules in $\mathcal{D}$ are said to be *divisible,* those in $\mathcal{R}$ *reduced.* Since the torsion pair $(\mathcal{R}, \mathcal{D})$ is split we see that *any object in $\mathcal{M}$ is the direct sum of a module in $\mathcal{R} \cap \mathcal{M}$ and a module in $\mathcal{D} \cap \mathcal{M}$.* Now the modules in $\mathcal{D} \cap \mathcal{M}$ are known:

$$\mathcal{D} \cap \mathcal{M} = \omega$$

(namely, $\mathcal{D}\cap\mathcal{M} = \mathcal{D}\cap\mathcal{B}\cap\mathcal{C} = \mathcal{D}\cap\mathcal{C}$, since $\mathcal{D} \subset \mathcal{B}$). The challenge which remains is to find a convenient description of the modules in $\mathcal{R}\cap\mathcal{M}$. Part (b) of the Theorem presented below will provide such a description for those modules in $\mathcal{R}\cap\mathcal{M}$ which are algebraically compact.

Couchot's theorem asserts that the algebraically compact modules are the modules in $\Pi(\mathrm{mod}\,\Lambda)$. Since any indecomposable module of finite length belongs either to $\mathbf{p}$, $\mathbf{t}$ or $\mathbf{q}$, it is of interest to determine the intersection of the classes $\Pi(\mathbf{p}), \Pi(\mathbf{t})$ and $\Pi(\mathbf{q})$ with $\mathcal{M}$. This is established in part (a) of the following Theorem.

**Theorem.**
(a) $\Pi(\mathbf{p}) \cap \mathcal{M} = 0, \quad \Pi(\mathbf{t}) \subseteq \mathcal{M}$ and $\Pi(\mathbf{q}) \cap \mathcal{M} = \omega$.
(b) *The class $\Pi(\mathbf{t})$ is the class of all algebraically compact modules in $\mathcal{R} \cap \mathcal{M}$.*
(c) *The following conditions are equivalent for a $\Lambda$-module $M$.*
  (i) $M = M' \oplus M''$, where $M'$ belongs to $\Pi(\mathbf{t})$ and $M''$ to $\omega$.
  (ii) *$M$ is algebraically compact and belongs to $\mathcal{M}$.*

We should stress that the last assertion of part (a)

$$\Pi(\mathbf{q}) \cap \mathcal{M} = \omega$$

is of special interest: it provides a description of $\omega$ in terms of finite length modules. Note that there is no analogue in the case of a Dedekind ring $R$, since there does not exist any class of $R$-modules which could play the role of $\mathbf{q}$. The possibility for such a formula relies on Couchot's theorem, and this result uses strong finiteness conditions on the ring $\Lambda$. For a Dedekind ring $R$, only the reduced algebraically compact modules are direct sums of products of modules of finite length, whereas the divisible ones have not even non-zero maps to modules of finite length.

Proof of (a): First, assume that $M$ belongs to $\Pi(\mathbf{p})\cap\mathcal{M}$, thus $M$ is a submodule of a product of modules in $\mathbf{p}$. However, $\mathrm{Hom}(M, \mathbf{p}) = 0$, thus $M = 0$.

Second, assume that $M$ belongs to $\Pi(\mathbf{t})$. Then $M$ is algebraically compact. It is easy to see that $\mathcal{M}$ is closed under products (see Proposition 9 of [RR]) and of course also under direct summands, thus $M$ belongs to $\mathcal{M}$.

Third, we are going to show the third assertion $\Pi(\mathbf{q}) \cap \mathcal{M} = \omega$. One of the inclusions is obtained as follows:

$$\Pi(\mathbf{q}) \cap \mathcal{M} \subseteq \mathcal{D} \cap \mathcal{M} = \mathcal{D} \cap \mathcal{C} \cap \mathcal{B} = \mathcal{D} \cap \mathcal{C} = \omega,$$

where the first inclusion follows from the fact that $\mathcal{D}$ is closed under products, see Lemma 9 of [RR]. For the reverse inclusion $\omega \subseteq \Pi(\mathbf{q}) \cap \mathcal{M}$, it is clear that $\omega \subset \mathcal{M}$.

Thus, it remains to see that any module in $\omega$ belongs to $\Pi(\mathbf{q})$. First let us show that such a module $M$ is algebraically compact. This is easy to see in case $M$ is indecomposable, since then $M$ is artinian when considered as a module over its endomorphism ring. In general, write $M$ as a direct sum of indecomposables, say $M = \bigoplus_i M_i$. Since $\omega$ is closed under products, also the product $N = \prod_i M_i$ belongs to $\omega$. But any inclusion map in $\omega$ splits [RR], thus $M$ is a direct summand of $N$. Since all the modules $M_i$ are algebraically compact, also $N$ and therefore $M$ is algebraically compact.

As an algebraically compact module, $M$ can be written as $M = M_{\mathbf{p}} \oplus M_{\mathbf{t}} \oplus M_{\mathbf{q}}$ with $M_{\mathbf{p}} \in \Pi(\mathbf{p})$, $M_{\mathbf{t}} \in \Pi(\mathbf{t})$ and $M_{\mathbf{q}} \in \Pi(\mathbf{q})$. However, $\mathrm{Hom}(M, \mathbf{p}) = 0$ and $\mathrm{Hom}(M, \mathbf{t}) = 0$, thus $M_{\mathbf{p}} = 0 = M_{\mathbf{t}}$ and therefore $M = M_{\mathbf{q}}$ belongs to $\Pi(\mathbf{q})$.

Proof of (b). Of course, the modules in $\Pi(\mathbf{t})$ are algebraically compact, thus let us show that they belong both to $\mathcal{R}$ and $\mathcal{M}$. Now, $\mathrm{Hom}(G, \mathbf{t}) = 0$ and $\mathrm{Hom}(\mathbf{q}, \mathbf{t}) = 0$ imply $\mathrm{Hom}(G, \Pi(\mathbf{t})) = 0$ and $\mathrm{Hom}(\mathbf{q}, \Pi(\mathbf{t})) = 0$. Finally, it is well-known [RR] that also $\mathcal{B}$ is closed under products, thus $\mathrm{Hom}(\Pi(\mathbf{t}), \mathbf{p}) = 0$.

For the converse, let $N$ be an algebraically compact module in $\mathcal{R} \cap \mathcal{M}$. Since $N$ is algebraically compact, it is a direct summand of a product of finite dimensional modules, thus there are modules $X_{\mathbf{p}} \in \Pi(\mathbf{p})$, $X_{\mathbf{t}} \in \Pi(\mathbf{t})$, $X_{\mathbf{q}} \in \Pi(\mathbf{q})$ and maps

$$N \xrightarrow{\begin{bmatrix} g_{\mathbf{p}} \\ g_{\mathbf{t}} \\ g_{\mathbf{q}} \end{bmatrix}} X_{\mathbf{p}} \oplus X_{\mathbf{t}} \oplus X_{\mathbf{q}} \xrightarrow{[\, f_{\mathbf{p}}\ f_{\mathbf{t}}\ f_{\mathbf{q}} \,]} N \qquad \text{with } [\, f_{\mathbf{p}}\ f_{\mathbf{t}}\ f_{\mathbf{q}} \,] \begin{bmatrix} g_{\mathbf{p}} \\ g_{\mathbf{t}} \\ g_{\mathbf{q}} \end{bmatrix} = 1.$$

Since $\mathrm{Hom}(N, \mathbf{p}) = 0$ and $X_{\mathbf{p}}$ is a submodule of a product of modules in $\mathbf{p}$, we have $g_{\mathbf{p}} = 0$. On the other hand, we claim that also $f_{\mathbf{q}} = 0$. First of all, we have $\mathrm{Hom}(G, N) = 0$. Second, $\mathcal{D}$ is closed under products, thus $X_{\mathbf{q}}$ belongs to $\mathcal{D}$. According to [RR], all the modules in $\mathcal{D}$ are generated by $G$. It follows from $\mathrm{Hom}(G, N) = 0$ that $\mathrm{Hom}(X_{\mathbf{q}}, N) = 0$. Since $g_{\mathbf{p}} = 0$ and $f_{\mathbf{q}} = 0$, we see that $f_{\mathbf{t}} g_{\mathbf{t}} = 1$. This shows that $N$ is a direct summand of $X_{\mathbf{t}}$, thus $N$ belongs to $\Pi(\mathbf{t})$.

Proof of (c). The implication (i) $\Longrightarrow$ (ii) is a direct consequence of (a). For the implication (ii) $\Longrightarrow$ (i), let $M$ be an algebraically compact module in $\mathcal{M}$. Since $(\mathcal{R}, \mathcal{D})$ is a split torsion pair, we may decompose $M = M' \oplus M''$ with $M'$ in $\mathcal{R}$ and $M''$ in $\mathcal{D}$. According to (b), the module $M'$ belongs to $\mathcal{R} \cap \mathcal{M} = \Pi(\mathbf{t})$; and we have noted above that $\mathcal{D} \cap \mathcal{M} = \omega$. This completes the proof of Theorem.

It is one of the essential observations that any algebraically compact module $M$ in $\mathcal{M}$ decomposes as $M = M' \oplus M''$, where $M'$ belongs to $\Pi(\mathbf{t})$ and $M''$ to $\omega$. The proof given above uses the fact that the torsion pair $(\mathcal{R}, \mathcal{D})$ splits. It seems to be worthwhile to focus the attention on the argument, since it provides additional information. Let $M''$ be the sum of the images of maps $G \to M$, thus $M''$ is generated by $G$ and therefore belongs to $\mathcal{D}$. Since $M''$ is a submodule of $M \in \mathcal{C}$, and $\mathcal{C}$ is closed under submodules, it follows that $M'' \in \mathcal{C}$. Altogether we see that $M''$ belongs to $\omega$. The inclusion map $M'' \to M$ splits, since $M''$ belongs to $\omega$ and $M$ to $\mathcal{C}$, thus there is a submodule $M'$ with $M = M' \oplus M''$. Note that the image of any map $G \to M$ maps into $M''$, thus $\operatorname{Hom}(G, M') = 0$, thus $M'$ belongs to $\mathcal{R}$. Of course, $M'$ as a direct summand of $M$ belongs to $\mathcal{M}$, thus to $\mathcal{R} \cap \mathcal{M} = \Pi(\mathbf{t})$.

### 2.3. The $\lambda$-adic components.

Recall that add $\mathbf{t}$ is a direct sum of serial categories add $\mathbf{t}(\lambda)$, with $\lambda$ in some index set $\Omega$, and each subcategory $\mathbf{t}(\lambda)$ contains only finitely many isomorphism classes of simple objects. Let us denote by $\mathcal{A}(\lambda) = \Pi(\mathbf{t}(\lambda))$ the full subcategory of all direct summands of products in $\mathbf{t}(\lambda)$.

**Lemma 1.** *If $\lambda, \mu$ are different elements of $\Omega$, then* $\operatorname{Hom}(\mathcal{A}(\lambda), \mathcal{A}(\mu)) = 0$.

Proof: It is sufficient to show the following: If $X_i$ is a family of objects in $\mathbf{t}(\lambda)$ and $Y$ is indecomposable in $\mathbf{t}(\mu)$, then $\operatorname{Hom}(\prod_i X_i, Y) = 0$ Thus, assume there exists a non-zero map $f\colon \prod_i X_i \to Y$, let $I$ be its image and $Z$ its cokernel. We apply $\operatorname{Ext}^1(\tau^{-1}Y, -)$ to the exact sequence $0 \to I \to Y \to Z \to 0$, where $\tau$ is the Auslander-Reiten translation and obtain an exact sequence

$$\operatorname{Ext}^1(\tau^{-1}Y, I) \to \operatorname{Ext}^1(\tau^{-1}Y, Y) \to \operatorname{Ext}^1(\tau^{-1}Y, Z)$$

Since $Z$ is a proper factor module of $Y$, the Auslander-Reiten sequence goes under the right map to zero, thus $\operatorname{Ext}^1(\tau^{-1}Y, I) \neq 0$. Since the projective dimension of $\tau^{-1}Y$ is 1, the epimorphism $\prod_i X_i \to I$ yields an epimorphism

$$\operatorname{Ext}^1(\tau^{-1}Y, \prod\nolimits_i X_i) \to \operatorname{Ext}^1(\tau^{-1}Y, I).$$

In particular, we see that $\operatorname{Ext}^1(\tau^{-1}Y, \prod_i X_i)$ is non-zero. However

$$\operatorname{Ext}^1(\tau^{-1}Y, \prod\nolimits_i X_i) = \prod\nolimits_i \operatorname{Ext}^1(\tau^{-1}Y, X_i)$$

and all the factors $\operatorname{Ext}^1(\tau^{-1}Y, X_i)$ are zero, since $X_i \in \mathbf{t}(\lambda), Y \in \mathbf{t}(\mu)$ and $\lambda \neq \mu$.

**Lemma 2.** *For any $\lambda \in \Omega$, let $M_\lambda$ be a module in $\mathcal{A}(\lambda)$. Then $\prod_\lambda M_\lambda / \bigoplus_\lambda M_\lambda$ is a direct sum of copies of the generic module $G$.*

Proof: In order to show that $X = \prod_\lambda M_\lambda / \bigoplus_\lambda M_\lambda$ belongs to $\mathcal{C}$, consider any module $Q \in \mathbf{q}$ and a homomorphism $f\colon Q \to X$. We want to show that $f = 0$. Since $\mathbf{q}$ is closed under factor modules, we may assume that $f$ is an inclusion map. Since the embedding of $\bigoplus_\lambda M_\lambda$ into $\prod_\lambda M_\lambda$ is pure, it follows that $Q$ can be embedded into $\prod_\lambda M_\lambda$ (see Theorem 1.F in [R1]). But any $M_\lambda$ is a submodule of a product of modules in $\mathbf{t}$, so that $\operatorname{Hom}(Q, M_\lambda) = 0$. This shows that $Q = 0$. In order to show that $X$ belongs to $\mathcal{D}$, we show that $\operatorname{Ext}^1(S, X) = 0$ for any indecomposable module $S$ in $\mathbf{t}$. Assume that $S$ belongs to $\mathbf{t}(\mu)$, where $\mu \in \Omega$. Note that $X = \prod_{\lambda\neq\mu} M_\lambda / \bigoplus_{\lambda\neq\mu} M_\lambda$. For the $\lambda \in \Omega$ with $\lambda \neq \mu$, we have $\operatorname{Ext}^1(S, M_\lambda) = 0$, since $\operatorname{Ext}^1(S, -)$ commutes with products and $M_\lambda$ is in $\mathcal{A}(\lambda)$, whereas $S$ is in $\mathbf{t}(\mu)$. Again using that $\operatorname{Ext}^1(S, -)$ commutes with products, we see that $\operatorname{Ext}^1(S, \prod_{\lambda\neq\mu}, M_\lambda) = 0$.

Since $S$ has projective dimension at most 1, the group $\operatorname{Ext}^1(S, \prod_{\lambda\neq\mu} M_\lambda)$ maps onto the group $\operatorname{Ext}^1(S, X)$. This completes the proof that $X$ belongs to $\omega$.

The modules in $\omega$ are direct sums of Prüfer modules and of copies of the generic module. Thus, it remains to show that $X$ has no nonzero submodule which belongs to $\mathbf{t}$. Assume $S$ belongs to $\mathbf{t}(\mu)$ for some $\mu \in \Omega$ and embeds into $X$. As before, write $X = \prod_{\lambda\neq\mu} M_\lambda / \bigoplus_{\lambda\neq\mu} M_\lambda$, and lift the embedding of $S$ into $X$ to an embedding of $S$ into $\prod_{\lambda\neq\mu} M_\lambda$. Obviously, we obtain a contradiction.

**Theorem.** *The algebraically compact modules in $\mathcal{R} \cap \mathcal{M}$ are precisely the modules $M$ of the form $\prod_{\lambda\in\Omega} M_\lambda$ with $M_\lambda \in \mathcal{A}(\lambda)$ and this product decomposition is unique.*

Given such a module $M = \prod_{\lambda\in\Omega} M_\lambda$ with $M_\lambda \in \mathcal{A}(\lambda)$, one may call $M_\lambda$ the *$\lambda$-adic component* of $M$. Note that $M$ it is an extension of the submodule $\bigoplus_{\lambda\in\Omega} M_\lambda$ by a direct sum of copies of $G$, according to Lemma 2.

Proof of Theorem. Since $\mathcal{R} \cap \mathcal{M}$ is closed under products, all the modules of the form $\prod_{\lambda\in\Omega} M_\lambda$ with $M_\lambda \in \mathcal{A}(\lambda)$ belong to $\mathcal{R} \cap \mathcal{M}$.

Conversely, assume that $M$ is algebraically compact and belongs to $\mathcal{R} \cap \mathcal{M}$. Now $M$ is a direct summand of a module $A$ which is a product of modules $A_\lambda$ in $\mathcal{A}(\lambda)$, say $M \oplus M' = A = \prod A_\lambda$. According to Lemma 1, $A_\lambda$ is fully invariant in $A$, thus $A_\lambda = (M \cap A_\lambda) \oplus (M' \cap A_\lambda)$. As a consequence, $A = N \oplus N'$, where $N = \prod (M \cap A_\lambda)$ and $N' = \prod (M' \cap A_\lambda)$. We claim that $N \subseteq M$. We have $\bigoplus (M \cap A_\lambda) \subseteq M$. Let $Q = N / \bigoplus (M \cap A_\lambda)$. We may consider the following commutative diagram with exact rows, where all the maps labelled $\iota$ are inclusion maps, those labelled $\pi$ the canonical projections, and $f$ exists, since the left square is commutative and the maps $\pi$ are cokernel maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus(M\cap A_\lambda) & \xrightarrow{\iota} & N & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \xrightarrow{\iota} & A & \xrightarrow{\pi} & M' & \longrightarrow & 0 \end{array}$$

According to Lemma 2 we know that $Q$ is a direct sum of copies of $G$. But $M'$ is a direct summand of $M$, thus cogenerated by $\mathbf{t}$. Since $\operatorname{Hom}(G, \mathbf{t}) = 0$, we see that $f = 0$ and consequently $N \subseteq M$. Similarly, $N' \subseteq M'$. But this then implies that $N = M$ and $N' = M'$. Let $M_\lambda = M \cap A_\lambda$. Since $M_\lambda$ is a direct summand of $A_\lambda$, we see that $M_\lambda$ belongs to $\mathcal{A}(\lambda)$.

In order to show the unicity of the decomposition, let $\prod_\lambda M_\lambda = \prod_\lambda M'_\lambda$ with modules $M_\lambda, M'_\lambda$ in $\mathcal{A}(\lambda)$. Now, for any index $\mu$, the embedding $M'_\mu \subseteq \prod_\lambda M_\lambda$ induces an embedding $M'_\mu \to M_\mu$, according to Lemma 1. But then $M'_\mu = M_\mu$.

### 2.4. Basic submodules and completions.

It remains to study the modules in $\mathcal{A}(\lambda) = \Pi(\mathbf{t}(\lambda))$. The module classes of the form $\Pi(\mathbf{x})$ are often quite difficult to understand since the process of forming direct summands is mysterious in general. In order to overcome this problem in our case, we are going to provide a different characterization of the modules in $\mathcal{A}(\lambda) = \Pi(\mathbf{t}(\lambda))$ which avoids the need to deal with direct summands.

Actually, in order to deal with $\mathcal{A}(\lambda)$, we may be quite brief, since we are in a safer realm than before: instead of working with $\Lambda$, we may consider a suitable kind of localization $\Lambda_\lambda$ which behaves really like a Dedekind ring. Here, $\Lambda_\lambda$ is the endomorphism ring of the direct sum $F(\lambda)$ of all the modules $\widehat{S}$ with $S \in \mathcal{S}(\lambda)$, one from each isomorphism class. Denote by $\mathcal{F}(\lambda)$ the full subcategory of arbitrary direct sums of copies of $F(\lambda)$ and by $\mathcal{M}(\lambda)$ that of all cokernels of maps in $\mathcal{F}(\lambda)$. It is easy to see that $\mathcal{M}(\lambda)$ *is an exact abelian subcategory of* $\operatorname{Mod}\Lambda$ and that $F(\lambda)$ *is a progenerator in* $\mathcal{M}(\lambda)$*, thus* $\mathcal{M}(\lambda)$ *is equivalent to the category* $\operatorname{Mod}\Lambda_\lambda$. One may (and should) use this equivalence in oder to verify those assertions which are mentioned here without proof.

Denote by $\mathcal{B}(\lambda)$ the full subcategory of $\mathcal{M}$ of all direct sums of modules in $\mathbf{t}(\lambda)$ as well as of copies of $\widehat{S}$, where $S$ belongs to $\mathcal{S}(\lambda)$.

Let $A$ belong to $\mathcal{A}(\lambda)$. A $\lambda$-*basic* submodule of $A$ is a pure submodule $B$ of $A$ which belongs to $\mathcal{B}(\lambda)$ and such that $B/A$ is divisible.

If the $\Lambda$-module $B$ is cogenerated by $\mathbf{t}(\lambda)$, its $\lambda$-completion is defined as follows: Let ${}_nB$ be the intersection of the kernels of maps $B \to S[n]$, where $S$ belongs to $\mathcal{S}(\lambda)$. Then

$$\widehat{B} = \lim_{\leftarrow} \, B/{}_nB$$

is called the $\lambda$-*adic completion* of $B$.

**Theorem.** *There is a bijection between the isomorphism classes of the modules in* $\mathcal{A}(\lambda)$ *and those in* $\mathcal{B}(\lambda)$*, as follows: to every module in* $\mathcal{A}(\lambda)$*, attach a basic submodule; conversely, for every module* $B$ *in* $\mathcal{B}(\lambda)$*, form its completion.*

Proof: Starting with $B \in \mathcal{B}(\lambda)$, consider its $\lambda$-adic completion $\widehat{B}$. Of course, $\widehat{B}$ is algebraically compact, and the embedding $B \to \widehat{B}$ is a pure monomorphism. Since $\operatorname{Hom}(M, \widehat{B}) = 0$ for $M$ in $\mathbf{t}(\mu)$ with $\mu \neq \lambda$, as well as for $M$ in $\mathbf{q}$, whereas $\operatorname{Hom}(\widehat{B}, M) = 0$, for $M$ in $\mathbf{p}$, it follows that $\widehat{B}$ belongs to $\mathcal{A}(\lambda)$. We claim that $\widehat{B}/B$ belongs to $\omega$. First of all, since $B \to \widehat{B}$ is pure, and $\operatorname{Hom}(Q, \widehat{B}) = 0$, we see that $\widehat{B}/B$ belongs to $\mathcal{C}$. Second, in order to show that $\widehat{B}/B$ belongs to $\mathcal{D}$, we have to show that $\operatorname{Ext}^1(S, \widehat{B}/B) = 0$ for any simple object $S$ in $\mathbf{t}$. But this follows from the fact that the embedding $B \to \widehat{B}$ induces a bijection $\operatorname{Ext}^1(S, B) \to \operatorname{Ext}^1(S, \widehat{B})$ since $S$ has projective dimension 1. Thus we see that $B$ is a $\lambda$-basic submodule of $\widehat{B}$.

**Remark.** *For any module* $M$ *in* $\mathcal{M}$*, let* ${}_{n,\lambda}M$ *be the intersection of all maps* $M \to [n]S$*, with* $S \in \mathcal{S}(\lambda)$*. Then* $M/{}_{n,\lambda}M$ *is a direct sum of objects in* $\mathbf{t}(\lambda)$ *of length at most* $n$*.*

For any module $M$ in $\mathcal{M}$, its $\lambda$-completion is defined as follows: we have the sequence of canonical epimorphisms

$$\cdots \to M/{}_{3,\lambda}M \to M/{}_{2,\lambda}M \to M/{}_{1,\lambda}M$$

and we denote by $\widehat{M^\lambda}$ the corresponding inverse limit, we call it the $\lambda$-*completion* of $M$.

**Theorem.** *For any module* $M$ *in* $\mathcal{M}$*, the* $\lambda$*-completion belongs to* $\mathcal{A}(\lambda)$*. If* $M$ *belongs to* $\mathcal{A}(\lambda)$*, then* $M$ *coincides with its* $\lambda$*-completion.*

Proof. Note that all the modules ${}_{n,\lambda}M$ belong to $\mathcal{A}(\lambda)$ and that $\mathcal{A}(\lambda)$ is closed under inverse limits. This shows the first part.

For the second part, if $M$ belongs to $\mathbf{t}(\lambda)$, then clearly $M$ coincides with its $\lambda$-completion. But the class of modules which coincides with its $\lambda$-completion, is closed under products and direct summands. Thus the modules in $\mathcal{A}(\lambda)$ coincide with their $\lambda$-completions. This completes the proof.

## 3. Application: Tubular algebras

Typical situations for using the results presented above arise for tubular algebras, since they have plenty of separating tubular families.

Thus, let $\Lambda$ be a tubular algebra. The structure of $\operatorname{mod}\Lambda$ is known in detail (see [R3] and [LP]). There is a preprojective component $\mathbf{p}_0$ and a preinjective component $\mathbf{q}_\infty$. We denote by $I_0$ the ideal which is maximal with the property that it annihilates all the modules in $\mathbf{p}_0$ and by $I_\infty$ the ideal which is maximal with the property that it annihilates all the modules in $\mathbf{q}_\infty$. Then we obtain factor algebras $\Lambda_0 = \Lambda/I_0$ and $\Lambda_\infty = \Lambda/I_\infty$ which both are tame concealed algebras. Let $\mathbf{t}_0$ be the Auslander-Reiten components of $\operatorname{mod}\Lambda$ which contain regular $\Lambda_0$-modules, and $\mathbf{t}_\infty$ those which contain regular $\Lambda_\infty$-modules. Then both $\mathbf{t}_0$ and $\mathbf{t}_\infty$ are sincere separating tubular families, but both are not stable ($\mathbf{t}_0$ will contain indecomposable projective modules, $\mathbf{t}_\infty$ indecomposable injective ones). If we denote by $\mathbf{q}_0$ the indecomposable modules in $\operatorname{mod}\Lambda$ which do not belong to $\mathbf{p}_0$ or $\mathbf{t}_0$, then $\mathbf{t}_0$ separates $\mathbf{p}_0$ from $\mathbf{q}_0$. If we denote by $\mathbf{p}_\infty$ the indecomposable modules in $\operatorname{mod}\Lambda$ which do not belong to $\mathbf{t}_\infty$ or $\mathbf{q}_\infty$, then $\mathbf{t}_\infty$ separates $\mathbf{p}_\infty$ from $\mathbf{q}_\infty$. The modules in $\mathbf{q}_0 \cap \mathbf{p}_\infty$ fall into a countable number of sincere stable separating tubular families $\mathbf{t}_\alpha$, indexed by $\alpha \in \mathbb{Q}^+$, such that for $\alpha < \beta$ in $\mathbb{Q}^+$ the class $\mathbf{t}_\alpha$ generates $\mathbf{t}_\beta$, and also $\mathbf{t}_\alpha$ is cogenerated by $\mathbf{t}_\beta$. More generally, this generation and cogeneration property holds for all $\alpha < \beta$ in $\mathbb{Q}_0^\infty = \mathbb{Q}^+ \cup \{0, \infty\}$.

Let $\mathbb{R}_0^\infty = \mathbb{R}^+ \cup \{0, \infty\}$. For any $w \in \mathbb{R}_0^\infty$, we denote by $\mathbf{p}_w$ the modules which belong to $\mathbf{p}_0$ or to some $\mathbf{t}_\alpha$ with $\alpha < w$, and we denote by $\mathbf{q}_w$ the modules which belong to $\mathbf{t}_\gamma$ with $w < \gamma$ or to $\mathbf{q}_\infty$ (here, $\alpha, \gamma$ belong to $\mathbb{Q}_0^\infty$). For $\beta \in \mathbb{Q}_0^\infty$ we obtain in this way a trisection $(\mathbf{p}_\beta, \mathbf{t}_\beta, \mathbf{q}_\beta)$ of $\operatorname{mod}\Lambda$, with $\mathbf{t}_\beta$ a tubular family which separates $\mathbf{p}_\beta$ from $\mathbf{q}_\beta$, and $\mathbf{t}_\beta$ is stable provided $0 < \beta < \infty$. On the other hand, for $w \in \mathbb{R}_0^\infty \setminus \mathbb{Q}_0^\infty$, the two module classes $\mathbf{p}_w$ and $\mathbf{q}_w$ comprise all the indecomposables from $\operatorname{mod}\Lambda$.

Let us turn our attention again to arbitrary (not necessarily finite dimensional) modules. For any $w \in \mathbb{R}_0^\infty$, let $\mathcal{C}_w$ be the full subcategory of all modules $M$ with $\operatorname{Hom}(\mathbf{q}_w, M) = 0$ and $\mathcal{B}_w$ the full subcategory of all modules $M$ with $\operatorname{Hom}(M, \mathbf{p}_w) = 0$. The subcategory we are interested in is

$$\mathcal{M}(w) = \mathcal{B}_w \cap \mathcal{C}_w = \{M \mid \operatorname{Hom}(M, \mathbf{p}_w) = 0 = \operatorname{Hom}(\mathbf{q}_w, M)\},$$

and for $w \in \mathbb{Q}^+$, these are subcategories as discussed above.

The modules in $\mathcal{M}(w)$ are said to have *slope* $w$, thus *the results presented in this survey describe all the algebraically compact modules with slope* $w$ *in* $\mathbb{Q}^+$. We should note that similar considerations yield corresponding results for $w = 0$ and $w = \infty$, however at the moment not much is known about non-rational slopes.

In order to see the relevance of the subcategories $\mathcal{M}(w)$, we refer to the following joint result with Reiten [RR]:

**Theorem.** *Any indecomposable $\Lambda$-module which does not belong to $\mathbf{p}_0$ or $\mathbf{q}_\infty$ has a slope. For $0 \le w < w' \le \infty$, we have $\mathrm{Hom}(\mathcal{M}(w'), \mathcal{M}(w)) = 0$.*

Note that the second assertion immediately implies that $\mathcal{M}(w) \cap \mathcal{M}(w') = 0$, thus if a module has a slope, its slope is a well-defined element of $\mathbb{R}_0^\infty$.

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