Minimal infinite submodule-closed subcategories.

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Abstract. Let Λ be an artin algebra. We are going to consider full subcategories of $\operatorname{mod}\Lambda$ closed under finite direct sums and under submodules with infinitely many isomorphism classes of indecomposable modules. The main result asserts that such a subcategory contains a minimal one and we exhibit some striking properties of these minimal subcategories. These results have to be considered as essential finiteness conditions for such module categories.

Let Λ be an artin algebra, and $\operatorname{mod}\Lambda$ the category of Λ-modules of finite length. All the subcategories to be considered will be full subcategories of $\operatorname{mod}\Lambda$ closed under isomorphisms, finite direct sums and direct summands, but note that we also consider individual Λ-modules which may not be of finite length. If the Λ module X has finite length, we denote its length by $|X|$.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. We say that $\mathcal{C}$ is *finite* provided it contains only finitely many isomorphism classes of indecomposable modules, otherwise $\mathcal{C}$ is said to be *infinite.* Of course, $\mathcal{C}$ is said to be *submodule-closed* provided for any module C in $\mathcal{C}$ also any submodule of C belongs to $\mathcal{C}$.

The aim of this paper is to study infinite submodule-closed subcategories of $\operatorname{mod}\Lambda$. A subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ will be called *minimal infinite submodule-closed,* or (in this paper) just *minimal,* provided it is infinite and submodule-closed, and no proper subcategory of $\mathcal{C}$ is both infinite and submodule-closed. On a first thought, it is not at all clear whether minimal subcategories do exist: the existence is in sharp contrast to the usual properties of infinite structures (recall that in set theory, a set is infinite iff it contains proper subsets of the same cardinality).

Theorem 1. *Any infinite submodule-closed subcategory of* $\operatorname{mod}\Lambda$ *contains a minimal subcategory.*

Of course, the assertion is of interest only in case Λ is representation-infinite. But already the special case of looking at the category $\operatorname{mod}\Lambda$ itself, with Λ representation-infinite, should be stressed: *The module category of any representation-infinite artin algebra has minimal subcategories.*

Let M be a Λ-module, not necessarily of finite length. We write $\mathcal{S}_M$ for the class of finite length modules cogenerated by M. This is clearly a submodule-closed subcategory of $\operatorname{mod}\Lambda$. (Conversely, any submodule-closed subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is of this form: take for M the direct sum of all modules in $\mathcal{C}$, one from each isomorphism class; or else, it is sufficient to take just indecomposable modules in $\mathcal{C}$.).

Theorem 2. *Let* $\mathcal{C}$ *be a minimal subcategory of* $\operatorname{mod}\Lambda$*. Then*

(a) *For any natural number d, there are only finitely many isomorphism classes of indecomposabe modules in $\mathcal{C}$ of length at most d and even only finitely many isomorphism*

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classes of indecomposabe modules which are submodules of a direct sum of modules C_i in $\mathcal{C}$ with $|C_i| \le d$.

(b) *Any module in $\mathcal{C}$ is isomorphic to a submodule of an indecomposable module in $\mathcal{C}$.*

(c) *There is an infinite sequence of indecomposable modules C_i in $\mathcal{C}$ with proper inclusions*

$$C_1 \subset C_2 \subset \cdots \subset C_i \subset C_{i+1} \subset \cdots$$

such that also the union $M = \bigcup_i C_i$ is indecomposable and then $\mathcal{C} = \mathcal{S}_M$.

As we have mentioned, Theorem 1 asserts, in particular, that the module category of any representation-infinite artin algebra has a minimal subcategory $\mathcal{C}$, and the assertion (c) of Theorem 2 yields arbitrarily large indecomposable modules in $\mathcal{C}$. This shows that we are in the realm of the first Brauer-Thrall conjecture (formulated by Brauer and Thrall around 1940 and proved by Roiter in 1968): any representation-infinite artin algebra has indecomposable modules of arbitrarily large length. The proof of Roiter and its combinatorial interpretation by Gabriel are the basis of the Gabriel-Roiter measure on $\operatorname{mod}\Lambda$, see [R1, R2]. Using it, we have shown in [R1] that the module category of a representation-infinite artin algebras always has a so-called take-off part: this is an infinite submodule-closed subcategory with property (a) of Theorem 2, and there is an infinite inclusion chain of indecomposables such that also the union M is indecomposable, as in property (c) of Theorem 2. However, $\mathcal{S}_M$ usually will be a proper subcategory of the take-off part, and then the take-off part cannot be minimal. Of course, we can apply Theorem 1 to the take-off part in order to obtain a minimal subcategory inside the take-off part. The important feature of the minimal categories is the following: we deal with a countable set of indecomposable modules which are strongly interlaced as the assertions (b) and (c) of Theorem 2 assert. Typical examples to have in mind are the infinite preprojective components of hereditary algebras (see section 4).

The proof of Theorem 1 will be given in section 2, the proof of Theorem 2 in section 3. These proofs depend on the Gabriel-Roiter measure for Λ-modules, as discussed in [R1,R2]. The remaining section 4 provides examples. First, we will mention some procedures for obtaining submodule-closed subcategories. Then, following Kerner-Takane, we will show that the preprojective component of a representation-infinite connected hereditary algebra Λ is always a minimal subcategory. In case Λ is tame, this is the only one, but for wild hereditary algebras, there will be further ones.

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2. Proof of Theorem 1.

Given a class $\mathcal{X}$ of modules of finite length (or of isomorphism classes of modules), we denote by $\operatorname{add}\mathcal{X}$ the smallest subcategory containing $\mathcal{X}$. We denote by $\mathbb{N} = \mathbb{N}_1$ the natural numbers starting with 1.

The proof will be based on results concerning the Gabriel-Roiter measure for Λ-modules, see [R1, R2]. For the benefit of the reader, let us recall the inductive definition of the *Gabriel-Roiter measure* $\mu(M)$ of a Λ-module M: For the zero module $M = 0$, one sets $\mu(0) = 0$. If $M \neq 0$ is decomposable, then $\mu(M)$ is the maximum of $\mu(M')$ where M' is a proper submodule of M, whereas for an indecomposable module M, one sets

$$\mu(M) = 2^{-|M|} + \max_{M' \subset M} \mu(M').$$

If M is indecomposable and not simple, then there always exists an indecomposable submodule $M' \subset M$ such that $\mu(M) - \mu(M') = 2^{-|M|}$, such submodules are called *Gabriel-Roiter submodules* of M. Inductively, we obtain for any indecomposable module M a chain of indecomposable submodules

$$M_1 \subset M_2 \subset \cdots \subset M_{t-1} \subset M_t = M$$

such that M_1 is simple and M_{i-1} is a Gabriel-Roiter submodule of M_i, for $2 \leq i \leq t$. Note that

$$\mu(M) = \sum\nolimits_{j=1}^{t} 2^{-|M_j|},$$

and it will sometimes be convenient to call also the set $I = \{|M_1|, \ldots, |M_t|\}$ the Gabriel-Roiter measure of M. Thus the Gabriel-Roiter measure $\mu(M)$ of a module M will be considered either as a finite set I of natural numbers, or else as the rational number $\sum_{i \in I} 2^{-i}$, whatever is more suitable.

Given a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ and a finite set I of natural numbers, let $\mathcal{C}(I)$ be the set of isomorphism classes of indecomposable objects in $\mathcal{C}$ with Gabriel-Roiter measure I. An obvious adaption of one of the main results of [R1] asserts:

There is an infinite sequence of Gabriel-Roiter measures $I_1 < I_2 < \cdots$ such that $\mathcal{C}(I_t)$ is non-empty for any $t \in \mathbb{N}$ and such that for any J with $\mathcal{C}(J) \neq \emptyset$, either $J = I_t$ for some t or else $J > I_t$ for all t. Moreover, all the sets $\mathcal{C}(I_t)$ are finite. (Note that the sequence of measures I_t depends on $\mathcal{C}$, thus one should write $I_t^{\mathcal{C}} = I_t$; the papers [R1,R2] were dealing only with the case $\mathcal{C} = \operatorname{mod}\Lambda$, but the proofs carry over to the more general case of dealing with a submodule-closed subcategory $\mathcal{C}$).

Since $\operatorname{add}\bigcup_{t\in\mathbb{N}} \mathcal{C}(I_t)$ is an infinite submodule-closed subcategory of $\mathcal{C}$, we may assume that $\mathcal{C} = \operatorname{add}\bigcup_{t\in\mathbb{N}} \mathcal{C}(I_t)$. In order to construct a minimal subcategory $\mathcal{C}'$, we will construct a sequence of subcategories

$$\mathcal{C} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots$$

with the following properties:
(a) Any subcategory $\mathcal{C}_i$ is infinite and submodule-closed,

(b) $$\mathcal{C}_i(I_t) = \mathcal{C}_t(I_t) \qquad \text{for} \qquad t \leq i.$$

(c) If $\mathcal{D} \subseteq \mathcal{C}_i$ is infinite and submodule-closed, then

$$\mathcal{D}(I_t) = \mathcal{C}_t(I_t) \qquad \text{for} \qquad t \le i.$$

We start with $\mathcal{C}_0 = \mathcal{C}$ (the t in conditions (b) and (c) satisfies $t \ge 1$, thus nothing has to be verified). Assume, we have constructed $\mathcal{C}_i$ for some $i \ge 0$, satisfying the conditions (a), and the conditions (b), (c) for all pairs (i, t) with $t \le i$. We are going to construct $\mathcal{C}_{i+1}$.

Call a subset $\mathcal{X}$ of $\mathcal{C}_i(I_{i+1})$ *good*, provided there is a subcategory $\mathcal{D}_{\mathcal{X}}$ of $\mathcal{C}_i$ which is infinite and submodule-closed and such that $\mathcal{D}_{\mathcal{X}}(I_{i+1}) = \mathcal{X}$. For example $\mathcal{C}_i(I_{i+1})$ itself is good (with $\mathcal{D}_{\mathcal{X}} = \mathcal{C}_i$). Since $\mathcal{C}_i(I_{i+1})$ is a finite set, we can choose a minimal good subset $\mathcal{X}' \subseteq \mathcal{C}_i(I_{i+1})$. For $\mathcal{X}'$, there is an infinite and submodule-closed subcategory $\mathcal{D}_{\mathcal{X}'}$ of $\mathcal{C}_i$ such that $\mathcal{D}_{\mathcal{X}'}(I_{i+1}) = \mathcal{X}'$. (Note that in general neither $\mathcal{X}'$ nor $\mathcal{D}_{\mathcal{X}'}$ will be uniquely determined: usually, there may be several possible choices.) Let $\mathcal{C}_{i+1} = \mathcal{D}_{\mathcal{X}'}$. By assumption, $\mathcal{C}_{i+1}$ is infinite and submodule-closed, thus (a) is satisfied. In order to show (b) for all pairs $(i+1, t)$ with $t \le i+1$, we first consider some $t \le i$. We can apply (c) for $\mathcal{D} = \mathcal{C}_{i+1} \subseteq \mathcal{C}_i$ and see that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$, as required. But for $t = i+1$, nothing has to be shown. Finally, let us show (c). Thus let $\mathcal{D} \subseteq \mathcal{C}_{i+1}$ be an infinite submodule-closed subcategory. Since $\mathcal{D} \subseteq \mathcal{C}_i$, we know by induction that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$ for $t \le i$. It remains to show that $\mathcal{D}(I_{i+1}) = \mathcal{C}_{i+1}(I_{i+1})$. Since $\mathcal{D} \subseteq \mathcal{C}_{i+1}$, we have $\mathcal{D}(I_{i+1}) \subseteq \mathcal{C}_{i+1}(I_{i+1})$. But if this would be a proper inclusion, then $\mathcal{X} = \mathcal{D}(I_{i+1})$ would be a good subset of $\mathcal{C}_i(I_{i+1})$ which is properly contained in $\mathcal{C}_{i+1}(I_{i+1}) = \mathcal{D}_{\mathcal{X}'}(I_{i+1})$, a contradiction to the minimality of $\mathcal{X}'$. This completes the inductive construction of the various $\mathcal{C}_i$.

Now let

$$\mathcal{C}' = \bigcap\nolimits_{i \in \mathbb{N}} \mathcal{C}_i.$$

Of course, $\mathcal{C}'$ is submodule-closed. Also, we see immediately

(b′) $$\mathcal{C}'(I_t) = \mathcal{C}_t(I_t) \qquad \text{for all} \qquad t,$$

since $\mathcal{C}'(I_t) = \bigcap_{i \ge t} \mathcal{C}_i(I_t) = \mathcal{C}_i(I_t)$, according to (b).

First, we show that $\mathcal{C}'$ is infinite. Of course, $\mathcal{C}'(I_1) \neq \emptyset$, since $I_1 = \{1\}$ and a good subset of $\mathcal{C}_0(I_1)$ has to contain at least one simple module. Assume that $\mathcal{C}'(I_s) \neq \emptyset$ for some s, we want to see that there is $t > s$ with $\mathcal{C}'(I_t) \neq \emptyset$. For every Gabriel-Roiter measure I, let $n(I)$ be the minimal number n with $I \subseteq [1, n]$, thus $n(I)$ is the length of the modules in $\mathcal{C}(I)$. Let $n(s)$ be the maximum of $n(I_j)$ with $j \le s$, thus $n(s)$ is the maximal length of the modules in $\bigcup_{j \le s} \mathcal{C}(I_j)$. Let s' be a natural number such that $n(I_j) > n(s)pq$ for all $j > s'$ (such a number exists, since the modules in $\mathcal{C}(I_j)$ with j large, have large length); here p is the maximal length of an indecomposable projective module, q that of an indecomposable injective module.

We claim that $\mathcal{C}'(I_j) \neq \emptyset$ for some j with $s < j \le s'$. Assume for the contrary that $\mathcal{C}'(I_j) = \emptyset$ for all $s < j \le s'$. We consider $\mathcal{C}_{s'}$. Since $\mathcal{C}_{s'}$ is infinite, there is some $t > s$ with $\mathcal{C}_{s'}(I_t) \neq \emptyset$, and we choose t minimal. Now for $s < j \le s'$, we know that $\mathcal{C}_{s'}(I_j) = \mathcal{C}_j(I_j) = \mathcal{C}'(I_j) = \emptyset$, according to (b) and (b′). This shows that $t > s'$. Let Y be an indecomposable module with isomorphism class in $\mathcal{C}_{s'}(I_t)$. Let X be a Gabriel-Roiter submodule of Y. Then X belongs to $\mathcal{C}_{s'}(I_j)$ with $j < t$. If $j \leq s$, then the length of X is bounded by $n(s)$, and therefore Y is bounded by $n(s)pq$ (see [R2], 3.1 Corollary), in contrast to the fact that $n(I_t) > n(s)pq$. This is the required contradiction. Thus $\mathcal{C}'$ is infinite.

Now, let $\mathcal{D}$ be an infinite submodule-closed subcategory of $\mathcal{C}'$. We show that $\mathcal{D}(I_t) = \mathcal{C}'(I_t)$ for all t. Consider some fixed t and choose an i with $i \geq t$. Since $\mathcal{C}' \subseteq \mathcal{C}_i$, we see that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$ for the given t, according to (b) for $\mathcal{C}_i$. But according to (b′), we also know that $\mathcal{C}'(I_t) = \mathcal{C}_t(I_t)$. This completes the proof.

3. Proof of Theorem 2.

We refer to [R1] for the proof of (a) and for the construction of an inclusion chain

$$C_1 \subset C_2 \subset \cdots \subset C_i \subset C_{i+1} \subset \cdots$$

with indecomposable union, as asserted in (c). In [R1] these assertions have been shown for the take-off part of $\operatorname{mod}\Lambda$, but the same proof with only minor modifications, carries over to minimal categories.

To complete the proof of (c), we only have to note the following: By construction, $\mathcal{S}_M$ contains all the modules C_i, thus $\mathcal{S}_M$ is not finite. But of course, $\mathcal{S}_M \subseteq \mathcal{C}$. Namely, if X is a finite length module which is cogenerated by M, then there are finitely many maps $f_i \colon X \to M$ such that the intersection of the kernels is zero. But there is some j such that the images of all the maps f_i are contained in C_j, therefore X is cogenerated by C_j and thus belongs to $\mathcal{C}$. The minimality of $\mathcal{C}$ implies that $\mathcal{S}_M = \mathcal{C}$.

It remains to proof part (b) of Theorem 2. We will need some general observations which may be of independent interest. Recall that a module is said to be of *finite type,* provided it is the direct sum of (may-be infinitely many) copies of a finite number of modules of finite length).

(1) *If $\mathcal{S}_M$ is minimal, then M is not of finite type.*

Proof: Assume that M is of finite type, let $M_1, \dots, M_t$ be the indecomposable direct summands of M, one from each isomorphism class. We may assume that they are indexed with increasing Gabriel-Roiter measures, thus $\mu(M_i) \leq \mu(M_j)$ for $i \leq j$. Let $\mathcal{S}'$ be the subcategory of $\operatorname{mod}\Lambda$ such that N belongs to $\mathcal{S}'$ if and only if any indecomposable direct summand of N belongs to $\mathcal{S}_M$ and is not isomorphic to M_t. Thus $\mathcal{S}'$ is a proper subcategory of $\mathcal{S}_M$ and infinite. We claim that $\mathcal{S}'$ is submodule closed (this then contradicts the minimality of $\mathcal{S}_M$).

Let N be in $\mathcal{S}'$. We want to show that any indecomposable submodule U of N belongs to $\mathcal{S}'$. Since $\mathcal{S}' \subset \mathcal{S}_M$, we know that U belongs to $\mathcal{S}_M$, thus we have to exclude that U is isomorphic to M_t. Thus, let us assume that $U = M_t$ and let $u \colon M_t \to N$, be the embedding. Since N belongs to $\mathcal{S}_M$, there is an embedding $u' \colon N \to M^r$ for some r. Altogether, there is the embedding $u'u \colon M_t \to M^r$. Now $\mu(M_t) = \max_{1 \leq i \leq t} \mu(M_i)$, and therefore $u'u$ is a split monomorphism. Consequently, also $u \colon M_t \to N$ is a split monomorphism. But since $N \in \mathcal{S}'$, no direct summand of N is isomorphic to M_t. This shows that $\mathcal{S}'$ is submodule closed.

(2) *If $\mathcal{S}_M$ is minimal and $M' \subseteq M$ is a cofinite submodule, then $\mathcal{S}_{M'} = \mathcal{S}_M$.*

Proof: Of course, $\mathcal{S}_{M'} \subseteq \mathcal{S}_M$. Since we assume that $\mathcal{S}_M$ is minimal, we only have to show that $\mathcal{S}_{M'}$ is infinite. Assume, for the contrary, that $\mathcal{S}_{M'}$ is finite. This implies that M' is of finite type (see [R5]), say $M' = \bigoplus_{i\in I} M'_i$, so that the modules M'_i belong to only finitely many isomorphism classes. Let U be a submodule of M of finite length such that $M' + U = M$. Now $M' \cap U$ is a submodule of M' of finite length, thus it is contained in some $M'' = \bigoplus_{i\in J} M'_i$, where J is a finite subset of I. Of course, $M'' + U$ is a submodule of finite length. We claim that $M = (M''+U)\oplus M'''$, where $M''' = \bigoplus_{i\in I\setminus J} M'_i$. Namely, on the one hand, $M'' + U + M''' = M' + U = M$, whereas, on the other hand, $(M'' + U) \cap M''' \subseteq (M'' + U) \cap M' = M'' + (U \cap M') \subseteq M''$, thus $(M'' + U) \cap M'''$ is contained both in M'' and M''', therefore in $M'' \cap M''' = 0$. Since both modules $M'' + U$ and M''' are of finite type, also $M = (M'' + U) \oplus M'''$ is a module of finite type. But this contradicts (1).

(3) *Assume that $\mathcal{C} = \mathcal{S}_M$ is minimal and let M_0 be a submodule of M of finite length. If X belongs to $\mathcal{C}$, then there is an embedding $u\colon X \to M$ such that $M_0 \cap u(X) = 0$.*

Proof. Let X be of finite length and cogenerated by M. We want to construct inductively maps $f\colon X \to M$ such that $M_0 \cap f(X) = 0$ and such that the length of $\mathrm{Ker}(f)$ decreases. As start, we take as f the zero map. The process will end when $\mathrm{Ker}(f) = 0$.

Thus, assume that we have given some map $f\colon X \to M$ with $M_0 \cap f(X) = 0$ and $\mathrm{Ker}(f) \neq 0$. We are going to construct a map $g\colon X \to M$ such that first $M_0 \cap g(X) = 0$ and second, $\mathrm{Ker}(g)$ is a proper submodule of $\mathrm{Ker}(f)$. Let $M_1 = M_0 + f(X)$, this is a submodule of finite length of M. Choose a submodule M' of M with $M_1 \cap M' = 0$, and maximal with this property. Note that M' is a cofinite submodule of M (namely, M/M' embeds into the injective hull of M_1, and with M_1 also its injective hull has finite length). According to (2), we know that $\mathcal{S}_{M'} = \mathcal{S}_M = \mathcal{C}$, thus X belongs to $\mathcal{S}_{M'}$. This means that X is cogenerated by M'. In particular, since $\mathrm{Ker}(f) \neq 0$, there is a map $f'\colon X \to M'$ such that $\mathrm{Ker}(f)$ is not contained in $\mathrm{Ker}(f')$. Let $g = (f, f')\colon X \to M_1 \oplus M' \subseteq M$. Then $\mathrm{Ker}(g) = \mathrm{Ker}(f) \cap \mathrm{Ker}(f')$ is a proper submodule of $\mathrm{Ker}(f)$. Also, the image $g(X)$ is contained in $f(X) + f'(X) \subseteq f(X) + M'$. Since $M_1 + M' = M_0 \oplus f(X) \oplus M'$, we see that $M_0 \cap g(X) = 0$.

This completes the induction step. After finitely many steps, we obtain in this way an embedding u of X into M such that $u(X) \cap M_0 = 0$.

(3′) *Assume that $\mathcal{C} = \mathcal{S}_M$ is minimal. If X, Y are submodules of M of finite length, then also $X \oplus Y$ is isomorphic to a submodule of M.*

Proof: If X, Y are submodules of M, then $X \oplus Y$ is cogenerated by M.

(3″) *Assume that $\mathcal{C} = \mathcal{S}_M$ is minimal. If C belongs to $\mathcal{C}$, then the direct sum of countably many copies of C can be embedded into M.*

Proof: Assume, there is given an embedding $u_t\colon C^t \to M$, where $t \geq 0$ is a natural number. Let $M_0 = u_t(C^t)$. According to (3), we find an embedding $u\colon C \to M$ such that $M_0 \cap u(C) = 0$. Thus, let $u_{t+1} = u_t \oplus u\colon C^{t+1} = C^t \oplus C \to M$.

Proof of part (b) of Theorem 2. Let C be a module in $\mathcal{C}$. Let $M = \bigcup_i C_i$ be as constructed in (c), thus all the C_i are indecomposable and $\mathcal{S}_M = \mathcal{C}$. According to (3), there is an embedding $u\colon C \to M$. Now the image of u lies in some C_i, thus u embeds C into the indecomposable module C_i.

Some consequences of Theorem 2 (b) should be mentioned. If S is a simple Λ-module, write $[X\colon S]$ for the Jordan-Hölder multiplicity of S in the Λ-module X.

Corollary 1. *Let $\mathcal{C}$ be a minimal subcategory. For any natural number d, there is an indecomposable module C in $\mathcal{C}$ with the following property: if S is a simple Λ-module with $[Y\colon S] \neq 0$ for some Y in $\mathcal{C}$, then $[C\colon S] \geq d$.*

Proof: We consider the simple Λ-modules S such that there exists a module $Y(S)$ in $\mathcal{C}$ with $[Y(S)\colon S] \neq 0$, and let $Y = \bigoplus Y(S)$ where the summation extends over all isomorphism classes of such simple modules S. Given a natural numer d, let us consider Y^d. According to assertion (b) of Theorem 2, there is an indecomposable Λ-module C such that Y^d embeds into C. But this implies that $[C\colon S] \geq [Y^d\colon S] = d[Y\colon S] \geq d[Y(S) : S] \geq d$.

Note that the corollary provides a strengthening of the assertion of the first Brauer-Thrall conjecture:

Corollary 2. *Let Λ be representation-infinite. Let $P = \Lambda e$ be indecomposable projective (e an idempotent in Λ) and $S = P/\operatorname{rad} P$. If $[M : S]$ is bounded for the indecomposable modules M, then $\Lambda/\langle e\rangle$ is representation-infinite.*

Proof: Take a minimal subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ and let $\mathcal{I}$ be its annihilator. Let $\Lambda' = \Lambda/\mathcal{I}$, thus $\mathcal{C}$ is a minimal subcategory of $\operatorname{mod}\Lambda'$ and for every simple Λ'-module S, there is a Λ'-module Y with $[Y\colon S] \neq 0$. By Corllary 1, the numbers $[C : S]$ with C indecomposable in $\mathcal{C}$ is unbounded. If $e \notin \mathcal{I}$, then S is a simple Λ'-module and then $[C\colon S]$ with C indecomposable in $\mathcal{C}$ is unbounded. But this contradicts the assumption on S. Thus we see that $e \in \mathcal{I}$, therefore Λ' is a factor algebra of $\Lambda/\langle e\rangle$. Since Λ' is representation-infinite, also $\Lambda/\langle e\rangle$ is representation-infinite.

Corollary 3. *A representation-infinite artin algebra has indecomposable representations X such that all non-zero Jordan-Hölder multiplicities of X are arbitrarily large.*

4. Examples.

First, let us mention some ways for obtaining submodule-closed subcategories.

- Of course, we can consider the module category $\operatorname{mod}\Lambda$ itself.
- If $\mathcal{I}$ is a two-sided ideal of Λ, then the Λ-modules annihilated by $\mathcal{I}$ form a submodule-closed subcategory (this subcategory is just the category of all $\Lambda/\mathcal{I}$-modules).
- As we have mentioned in section 3, we may start with an arbitrary (not necessarily finitely generated) module M, and consider the subcategory $\mathcal{S}_M$ of all finite length modules cogenerated by M. This subcategory $\mathcal{S}_M$ is submodule-closed, and any submodule-closed subcategory of $\operatorname{mod}\Lambda$ is obtained in this way.
- The special case of dealing with $M = {}_\Lambda\Lambda$ has been studied often in representation theory; the modules in $\mathcal{S}_{{}_\Lambda\Lambda}$ are called the *torsionless* Λ-modules. Artin algebras with $\mathcal{S}_{{}_\Lambda\Lambda}$ finite have quite specific properties, for example their representation dimension is bounded by 3.

- The categories $\mathcal{A}(<\gamma)$ and $\mathcal{A}(\leq\gamma)$ of all modules X in $\mathcal{A} = \operatorname{mod}\Lambda$ with Gabriel-Roiter measure $\mu(X) < \gamma$, or $\mu(X) \leq \gamma$, respectively; here $\gamma \in \mathbb{R}$ and μ is the Gabriel-Roiter measure (or a weighted Gabriel-Roiter measure).
- In particular, the take-off subcategory of $\operatorname{mod}\Lambda$ (as introduced in [R1]) is submodule-closed (and it is infinite iff Λ is representation-infinite).
- If Λ has global dimension n, then the subcategory $\mathcal{C}$ of all modules of projective dimension at most $n-1$ is closed under cogeneration (and extensions) (this is mentioned for example in [HRS], Lemma II.1.2.).

Given such a submodule-closed subcategory $\mathcal{C}$, one may ask whether it is finite or not, and in case it is infinite, it should be of interest to look at the corresponding minimal subcategories.

Example 1 (Kerner-Takane). *Let Λ be a connected hereditary artin algebra of infinite representation type. The preprojective component of* $\operatorname{mod}\Lambda$ *is a minimal subcategory.*

Proof. Kerner-Takane ([KT], Lemma 6.3.) have shown: For every $b \in \mathbb{N}$, there is $n = n(b) \in \mathbb{N}$ with the following property: If P, P' are indecomposable projective modules, then $\tau^{-i}P'$ is cogenerated by $\tau^{-j}P$, for all $0 \leq i \leq b$ and $n \leq j$. Assume that $\mathcal{C}$ is the additive subcategory given by an infinite set of indecomposable preprojective modules. We claim that the cogeneration closure of $\mathcal{C}$ contains all the preprojective modules X. Indeed, let $X = \tau^{-b}P'$ with P' indecomposable projective. Choose a corresponding $n(b)$. Since $\mathcal{C}$ contains infinitely many isomorphism classes of indecomposable preprojective modules, there is some $C = \tau^{-j}P$ in $\mathcal{C}$ with $n \leq j$ and P indecomposable projective. According to Kerner-Takane, X is cogenerated by C.

Recall that an algebra Λ is said to be *tame concealed* provided it is the endomorphism ring of a preprojective tilting module of a tame hereditary algebra.

Example 2. *Any tame concealed algebra Λ has a unique minimal subcategory $\mathcal{C}$, namely the subcategory of all preprojective modules.*

Proof: Let k be a field and Λ a finite-dimensional k-algebra which is tame concealed. Let $\mathcal{C}$ be an infinite submodule-closed subcategory of $\operatorname{mod}\Lambda$. We want to show that $\mathcal{C}$ contains infinitely many isomorphism classes of indecomposable preprojective modules.

According to Theorem 2 (b) and (c), for any indecomposable module $C \in \mathcal{C}$, there exists an infinite inclusion sequence of indecomposable modules in $\mathcal{C}$ which starts with C. This shows that C cannot be preinjective, since an indecomposable preinjective module for a tame concealed algebra has only finitely many successors. Thus, all the modules in $\mathcal{C}$ are preprojective or regular.

Next, assume that $\mathcal{C}$ contains infinitely many indecomposable regular modules. If they are of bounded length, then the proof of Brauer-Thrall 1 presented in Appendix A of [R1] yields arbitrarily large indecomposable modules M cogenerated by these regular modules, and the modules M constructed in this way have to be preprojective. It remains to consider the case that $\mathcal{C}$ contains arbitrarily large indecomposable regular modules.

Recall that an indecomposable Λ-module H is said to be *homogeneous* provided its Auslander-Reiten translate τH is isomorphic to H. Note that if H is a homogeneous indecomposable module, then $\operatorname{Hom}(P, H) \neq 0$ for all indecomposable preprojective modules P.

We choose two indecomposable homogeneous Λ-modules H, H' which belong to different Auslander-Reiten components. Let b be an upper bound for the k-dimension of all the vector spaces $\mathrm{Ext}^1(Q, H)$ and $\mathrm{Ext}^1(Q, H')$, where Q is a submodule of an indecomposable injective Λ-module (clearly, such a bound exists).

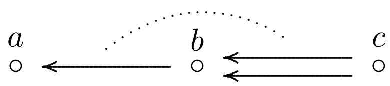
Now, let R be an indecomposable regular module in $\mathcal{C}$ of length r, and let R' be its regular socle. Let $f'\colon R' \to Q'$ be a non-zero map with Q' indecomposable injective and let $f\colon R \to Q'$ be an extension of f'. Let Q be the image of f. By construction, R' is not contained in the kernel X of f, and therefore X has no non-zero regular submodule. It follows that X is a direct sum of say t indecomposable preprojective modules X_i. At least one of H, H', say H, will belong to a different Auslander-Reiten component than R, and thus $\mathrm{Hom}(R, H) = 0 = \mathrm{Ext}^1(R, H)$. We apply $\mathrm{Hom}(-, H)$ to the exact sequence $0 \to X \to R \to Q \to 0$, and obtain

$$\mathrm{Hom}(R, H) \to \mathrm{Hom}(X, H) \to \mathrm{Ext}^1(Q, H) \to \mathrm{Ext}^1(R, H)$$

with first and last term being zero, thus the k-spaces $\mathrm{Hom}(X, H)$ and $\mathrm{Ext}^1(Q, H)$ are isomorphic. In particular, we see that the k-dimension of $\mathrm{Hom}(X, H)$ is bounded by b. Since X is the direct sum of t indecomposable preprojective modules, and $\mathrm{Hom}(P, H) \neq 0$ for any indecomposable preprojective module P, it follows that $t \leq b$. Let q be the maximal length of an indecomposable injective Λ-module, then $|X| \geq r - q$. Assume that all indecomposable direct summands X_i have length $|X_i| < \frac{1}{b}(r - q)$. Then $|X| = |\bigoplus_i X_i| < b \cdot \frac{1}{b}(r - q) = r - q$, a contradiction. This shows that at least one of the modules X_i has length $|X_i| \geq \frac{1}{b}(r - q)$. Since by assumption r is not bounded, also $\frac{1}{b}(r - q)$ is not bounded.

Thus, we have shown that $\mathcal{C}$ contains infinitely many isomorphism classes of indecomposable preprojective modules, and therefore the intersection $\mathcal{C}''$ of $\mathcal{C}$ with the preprojective component is infinite. The minimality of $\mathcal{C}$ implies that $\mathcal{C}$ contains only preprojective modules. On the other hand, as in example 1, the subcategory of all preprojective modules can be shown to be minimal.

Remark. Preprojective components are always submodule-closed, but in general an infinite preprojective component $\mathcal{P}$ does not have to be minimal. First of all, $\mathcal{P}$ may contain indecomposable injective modules, whereas this cannot happen for a minimal subcategory, as part (b) of Theorem 2 shows. But also preprojective components without indecomposable injective modules may not be minimal. For example, consider the algebra with quiver

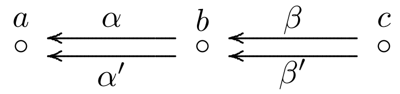


with one zero relation (thus, the indecomposable projective module P_a corresponding to the vertex a is simple, the radical of P_b is equal to P_a and the radical of P_c is the direct sum of P_b and the simple factor module of P_b). Then the preprojective component $\mathcal{P}$ contains indecomposables which are faithful, but also countable many indecomposables X with $X_a = 0$. Clearly, the subcategory of $\mathcal{P}'$ of all modules P in $\mathcal{P}$ with $P_a = 0$ is a proper subcategory which is both infinite and submodule-closed (and actually, $\mathcal{P}'$ is minimal).

Example 3. Let $\mathcal{I}$ be a twosided ideal in Λ. The category of Λ-modules annihilated by $\mathcal{I}$ is obviously submodule-closed and of course equivalent (or even equal) to the category of all $\Lambda/\mathcal{I}$-modules. If $\Lambda/\mathcal{I}$ is representation-infinite, then $\operatorname{mod}\Lambda/\mathcal{I}$ will contain a minimal subcategory. Consider for example the generalized Kronecker-algebra $K(3)$ with three arrows α, β, γ. The one-dimensional ideals of $K(3)$ correspond bijectively to the elements of the projective plane $\mathbb{P}^2$, say $a = (a_0 : a_1 : a_2) \in \mathbb{P}^2$ yields the ideal $\mathcal{I}_a = \langle a_0\alpha + a_1\beta + a_2\gamma\rangle$. Let $\mathcal{C}_a$ be the additive subcategory of $\operatorname{mod} K(3)$ of all preprojective $K(3)/\mathcal{I}_a$-modules. Then these are pairwise different subcategories (the intersection of any two of these subcategories is the subcategory of semisimple projective modules). In particular, *if the base field is infinite, there are infinitely many subcategories in* $\operatorname{mod} K(3)$ *which are minimal.* (Note that the preprojective $K(3)$-modules provide a further minimal subcategory.)

The minimal subcategories exhibited here can be distinguished by looking at the corresponding annihilators (the annihilator of a subcategory $\mathcal{C}$ is the ideal of all the elements $\lambda \in \Lambda$ which annihilate all the modules in $\mathcal{C}$). The next example will show that usually there are also different minimal subcategories which have the same annihilator. Note that a submodule-closed subcategory $\mathcal{C}$ has zero annihilator if and only if all the projective modules belong to $\mathcal{C}$.

Example 4. Here is an artin algebra Λ with different minimal categories containing all indecomposable projective modules. Consider the hereditary algebra Λ with quiver Q

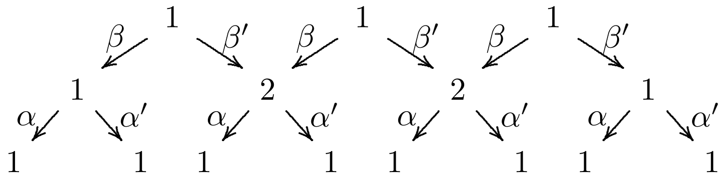


We denote by $Q(ab)$ the full subquiver of Q with vertices a, b, by $Q(bc)$ that with vertices b, c.

As we know, the preprojective component $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is a minimal subcategory. Of course, it contains all the projective Λ-modules, but it contains also, for example, the indecomposable Λ-module X with dimension vector $(3, 2, 0)$; note that the restriction of X to $Q(ab)$ is indecomposable and neither projective nor semisimple.

Second, let $\mathcal{D}$ be the full subcategory of $\operatorname{mod}\Lambda$ consisting of all the Λ-modules such that the restriction to $Q(ab)$ is projective and the restriction to $Q(bc)$ is preprojective. Clearly, $\mathcal{D}$ is submodule-closed, and it is obviously infinite: If Y is a Λ-module with $Y_a = 0$, define $\underline{Y}$ as follows: the restrictions of Y and $\underline{Y}$ to $Q(bc)$ should coincide, whereas the restriction of $\underline{Y}$ to $Q(ab)$ should be a direct sum of indecomposable projectives of length 3; in particular, $\underline{Y}_a = Y_b^2$. By $Y \mapsto \underline{Y}$ we obtain an embedding of the category of preprojective Kronecker modules into $\mathcal{D}$, which yields all the indecomposable modules in $\mathcal{D}$ but the simple projective one. It follows easily that $\mathcal{D}$ is minimal. Of course, $\mathcal{D} \neq \mathcal{C}$, and note that also $\mathcal{D}$ contains all the projective Λ-modules.

We can exhibit even a third minimal subcategory which contains all the projective Λ-modules, by looking at the full subcategory $\mathcal{E}$ of Λ-modules such that the restriction to $Q(ab)$ is the direct sum of a projective and a semisimple module, whereas the restriction to $Q(bc)$ is projective. Again, clearly $\mathcal{E}$ is submodule-closed. In order to construct an infinite family of indecomposable modules in $\mathcal{E}$, we use covering theory: The following quiver is part of the universal cover $\widehat{Q}$ of Q



and the numbers inserted form the dimension vector for a two-parameter family of indecomposable modules M. If we require in addition that the maps α and α' starting at the same vertex have equal kernels, then there is a unique isomorphism class $M = Y_3$ with this dimension vector. In a similar way, we can construct for any natural number n an indecomposable representation Y_n of $\widehat{Q}$ of length $2+5n$ (with top of length n). The kernel condition assures that the Λ-module which is covered by $M = Y_3$, or more generally, by Y_n, belongs to $\mathcal{E}$ (note that the kernel condition means that the restriction of M to any subquiver of type $\widetilde{\mathbb{D}}_4$ has socle of length 3). If $\mathcal{E}'$ is a minimal subcategory inside $\mathcal{E}$, then $\mathcal{E}'$ is different from $\mathcal{C}$ and $\mathcal{D}$.

Remark: The Λ-module covered by Y_1 is indecomposable projective and has Gabriel-Roiter measure $(1,3,7)$, this is the measure I_3 for Λ. One may show that the Λ-module covered by Y_2 has Gabriel-Roiter measure $(1,3,7,12)$ and that this is the measure I_4. For $t \geq 5$, the measures I_t are not yet known; it would be interesting to decide whether the intersection of the take-off part of $\operatorname{mod}\Lambda$ and $\mathcal{E}$ is infinite or not.

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