Lattice structure of torsion classes for hereditary artin algebras.

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Abstract: Let Λ be a connected hereditary artin algebra. We show that the set of functorially finite torsion classes of Λ-modules is a lattice if and only if Λ is either representation-finite (thus a Dynkin algebra) or Λ has only two simple modules. For the case of Λ being the path algebra of a quiver, this result has recently been established by Iyama-Reiten-Thomas-Todorov and our proof follows closely some of their considerations.

Let Λ be a connected hereditary artin algebra. The modules considered here are left Λ-modules of finite length, $\operatorname{mod}\Lambda$ denotes the corresponding category. The subcategories of $\operatorname{mod}\Lambda$ we deal with are always assumed to be closed under direct sums and direct summands (in particular closed under isomorphisms). In this setting, a subcategory is a *torsion class* (the class of torsion modules for what is called a torsion pair or a torsion theory) provided it is closed under factor modules and extensions. The torsion classes form a partially ordered set with respect to inclusion, it will be denoted by $\operatorname{tors}\Lambda$. This poset clearly is a lattice (even a complete lattice). It is easy to see that a torsion class $\mathcal{C}$ in $\operatorname{mod}\Lambda$ is functorially finite if and only if it has a cover (a *cover* for $\mathcal{C}$ is a module C such that $\mathcal{C}$ is the set of modules generated by C), we denote by $\text{f-tors}\,\Lambda$ the set of functorially finite torsion classes in $\operatorname{mod}\Lambda$.

In a recent paper [IRTT], Iyama, Reiten, Thomas and Todorov have discussed the question whether also the poset $\text{f-tors}\,\Lambda$ (with the inclusion order) is a lattice.

Theorem. *The poset* $\text{f-tors}\,\Lambda$ *is a lattice if and only if* Λ *is representation-finite or* Λ *has precisely two simple modules.*

Iyama, Reiten, Thomas, Todorov have shown this in the special case when Λ is a k-algebra with k an algebraically closed field (so that Λ is Morita equivalent to the path algebra of a quiver). The aim of this note is to provide a proof in general.

Here is an outline of the essential steps of the proof. Recall that a module is called *exceptional* provided it is indecomposable and has no self-extensions. A pair of modules X, Y will be called an Ext-*pair* provided both X, Y are exceptional, $\operatorname{Hom}(X,Y) = \operatorname{Hom}(Y,X) = 0$ and $\operatorname{Ext}^1(X,Y) \neq 0$, $\operatorname{Ext}^1(Y,X) \neq 0$. We follow the strategy of [IRTT] by establishing the existence of Ext-pair for any connected hereditary artin algebra which is representation-infinite and has at least three simple modules (Proposition 5). On the other hand, we will show directly that the set of functorially finite torsion classes which contain an Ext-pair has no minimal elements (Proposition 4).

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1. Normalization.

Let $\mathcal{X}$ be a class of modules. We denote by $\operatorname{add}(\mathcal{X})$ the modules which are direct summands of direct sums of modules in $\mathcal{X}$. A module M is *generated* by $\mathcal{X}$ provided M is a factor module of a module in $\operatorname{add}(\mathcal{X})$, and M is *cogenerated* by $\mathcal{X}$ provided M is a submodule of a module in $\operatorname{add}(\mathcal{X})$. The subcategory of all modules generated by $\mathcal{X}$ is denoted by $\mathcal{G}(\mathcal{X})$. In case $\mathcal{X} = \{X\}$ or $\mathcal{X} = \operatorname{add} X$, we write $\mathcal{G}(X)$ instead of $\mathcal{G}(\mathcal{X})$, and use the same convention in similar situations. We write $\mathcal{T}(X)$ for the smallest torsion class containing the module X (it is the intersection of all torsion classes containing X, and it can be constructed as the closure of $\{X\}$ using factor modules and extensions).

Since Λ is assumed to be hereditary, we write $\operatorname{Ext}(X,Y)$ instead of $\operatorname{Ext}^1(X,Y)$.

Following Roiter [Ro], we say that a module M is *normal* provided there is no proper direct decomposition $M = M' \oplus M''$ such that M' generates M'' (this means: if $M = M' \oplus M''$ and M' generates M'', then $M'' = 0$). Of course, given a module M, there is a direct decomposition $M = M' \oplus M''$ such that M' is normal and M' generates M'' and one can show that M' is determined by M uniquely up to isomorphism, thus we call $M' = \nu(M)$ a *normalization* of M. This was shown already by Roiter [Ro], and later by Auslander-Smalø [AS]. It is also a consequence of the following Lemma which will be needed for our further considerations.

Lemma 1. (a) *Let $(f_1,\dots,f_t,g)\colon X \to X^t \oplus Y$ be an injective map for some natural number t, with all the maps f_i in the radical of* $\operatorname{End}(X)$*. Then X is cogenerated by Y.*

(b) *Let $(f_1,\dots,f_t,g)\colon X^t \oplus Y \to X$ be a surjective map for some natural number t, with all the maps f_i in the radical of* $\operatorname{End}(X)$*, then Y generates X.*

Proof. (a) Assume that the radical J of $\operatorname{End}(X)$ satisfies $J^m = 0$. Let W be the set of all compositions w of at most $m-1$ maps of the form f_i with $1 \le i \le t$ (including $w = 1_X$). We claim that $(gw)_{w\in W}\colon X \to Y^{|W|}$ is injective. Take a non-zero element x in X. Then there is $w \in W$ such that $w(x) \neq 0$ and $f_i w(x) = 0$ for $1 \le i \le t$. Since $(f_1,\dots,f_t,g)$ in injective and $w(x) \neq 0$, we have $(f_1,\dots,f_t,g)(w(x)) \neq 0$. But $f_i w(x) = 0$ for $1 \le i \le t$, thus $g(w(x)) \neq 0$. This completes the proof.

(b) This follows by duality. □

Corollary (Uniqueness of normalization). *Let M be a module. Assume that $M = M_0 \oplus M_1 = M_0' \oplus M_1'$ such that both M_0 and M_0' generate M. Then there is a module N which is a direct summand of both M_0 and M_0' which generates M.*

Proof: We may assume that M is multiplicity free. Write $M_0 \simeq N \oplus C$, $M_0' \simeq N \oplus C'$, such that C, C' have no indecomposable direct summand in common. Now, $N \oplus C$ generates $N \oplus C'$, $N \oplus C'$ generates $N \oplus C$, and $N \oplus C$ generates C. We see that $N \oplus C$ generates C, such that the maps $C \to C$ used belong to the radical of $\operatorname{End}(C)$ (since they factor through $\operatorname{add}(N \oplus C')$ and no indecomposable direct summand of C belongs to $\operatorname{add}(N \oplus C')$). Lemma 1 asserts that N generates C, thus it generates M. □

Proposition 1. *If T has no self-extensions, then T is a cover for the torsion class $\mathcal{T}(T)$. Conversely, if $\mathcal{T}$ is a torsion class with cover C, then $\nu(C)$ has no self-extensions.*

Proof. For the first assertion, one has to observe that $\mathcal{G}(T)$ is closed under extensions, thus equal to $\mathcal{T}(T)$. This is a standard result say in tilting theory. Here is the argument:

let $g'\colon T' \to M'$ and $g''\colon T'' \to M''$ be surjective maps with T', T'' in $\operatorname{add} T$. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. The induced exact sequence with respect to g'' is of the form $0 \to M' \to Y_1 \to T'' \to 0$ with a surjective map $g_1\colon Y_1 \to M$. Since Λ is hereditary and g' is surjective, there is an exact sequence $0 \to T' \to Y_2 \to T'' \to 0$ with a surjective map $g_2\colon Y_2 \to Y_1$. Since $\operatorname{Ext}(T'', T') = 0$, we see that Y_2 is isomorphic to $T' \oplus T''$, thus in $\operatorname{add} T$. And there is the surjective map $g_1 g_2\colon Y_2 \to M$.

For the converse, we may assume that C is normal and have to show that C has no self-extension. Let C_1, C_2 be indecomposable direct summands of C and assume for the contrary that there is a non-split exact sequence

$$0 \to C_1 \to M \to C_2 \to 0.$$

Now M belongs to $\mathcal{T}$, thus it is generated by C, say there is a surjective map $C' \to M$ with $C' \in \operatorname{add} C$. Write $C' = C_2^t \oplus C''$ such that C_2 is not a direct summand of C''. Consider the surjective map $C_2^t \oplus C'' \to M \to C_2$. Since the last map $M \to C_2$ is not a split epimorphism, all the maps $C_2 \to C_2$ involved belong to the radical of $\operatorname{End}(C_2)$. According to Lemma 1, C'' generates C_2. This contradicts the assumption that C is normal. $\square$

Remark. Proposition 1 provides a bijection between the isomorphism classes of normal modules without self-extensions and torsion classes with covers. This is one of the famous Ingalls-Thomas bijections, see for example [ONFR] or also [R3].

We recall that a torsion class is functorially finite if and only if it has a cover. Of course, if C is a cover of the torsion class $\mathcal{T}$, then $\mu(C)$ is a minimal cover of $\mathcal{T}$.

Proposition 2. *Let $\mathcal{T}$ be a non-zero functorially finite torsion class. Then there is an indecomposable module U in $\mathcal{T}$ such that any non-zero map $V \to U$ with $V \in \mathcal{T}$ is a split epimorphism.*

Proof. Let C be a minimal cover of $\mathcal{T}$. Since C has no self-extensions, it is a direct summand of a tilting module. In particular, the quiver of $\operatorname{End}(C)$ is directed. It follows that C has an indecomposable direct summand U such that any non-zero map $C \to U$ is a split epimorphism. Assume now that V belongs to $\mathcal{T}$ and $f\colon V \to U$ is a non-zero map. There is a surjective map $g\colon C^t \to V$ for some t. Since the composition $fg\colon C^t \to U$ is non-zero, it is split epi, thus also f is split epi. $\square$

Remark. As we have mentioned, normal modules have been considered by Roiter, but actually, he used a slightly deviating name, calling them "normally indecomposable".

2. Inclusions of functorially finite torsion classes.

If $\mathcal{X}$ is a class of modules and U is an indecomposable module, we denote by $\mathcal{X}_U$ the class of modules in $\mathcal{X}$ which have no direct summand isomorphic to U.

Proposition 3. *Assume that $\mathcal{T}$ is a torsion class and that U is an indecomposable module in $\mathcal{T}$. The following assertions are equivalent:*
(i) *The class $\mathcal{T}_U$ is a torsion class.*
(ii) *Any non-zero map $V \to U$ with $V \in \mathcal{T}$ is split epi.*

Proof. (i) $\implies$ (ii). We assume that $\mathcal{T}_U$ is a torsion class. Let $f\colon V \to U$ be a non-zero map with $V \in \mathcal{T}$. We claim that f is surjective. Note that $f(V)$ and $U/f(V)$ both belong to $\mathcal{T}$, since $\mathcal{T}$ is closed under factor modules. If f is not surjective, then $f(V)$ is a factor module of V and a proper non-zero submodule of U, whereas $U/f(V)$ is a proper non-zero factor module of U. It follows that both $f(V)$ and $U/f(V)$ belong to $\mathcal{T}_U$. Since we assume that $\mathcal{T}_U$ is a torsion class, it is closed under extensions, and therefore U belongs to $\mathcal{T}_U$, a contradiction.

Write $V = V' \oplus U^t$ for some t with V' in $\mathcal{T}_U$. If f is not split epi, then Lemma 1 (b) asserts that V' generates U. But we assume that $\mathcal{T}_U$ is a torsion class, thus closed under direct sums and factor modules. Therefore, if V' generates U, then U has to belong to $\mathcal{T}_U$, again a contradiction. Altogether we have shown that f is split epi.

(ii) $\implies$ (i). We assume now that any non-zero map $V \to U$ with $V \in \mathcal{T}$ is a split epimorphism, and we have to show that $\mathcal{T}_U$ is a torsion class. In order to see that $\mathcal{T}_U$ is closed under factor modules, let V belong to $\mathcal{T}_U$ and let W be a factor module of V. Assume that U is a direct summand of W, thus U is a factor module of V. The projection $p\colon V \to U$ is a non-zero map, thus by assumption p is a split epimorphism. But this implies that U is a direct summand of V, whereas V belongs to $\mathcal{T}_U$. This shows that W belongs to $\mathcal{T}_U$.

In order to show that $\mathcal{T}_U$ is closed under extensions, consider a module M with a submodule V such that both V and M/V belong to $\mathcal{T}_U$. Since $\mathcal{T}$ is closed under extension, M belongs to $\mathcal{T}$. Assume that U is a direct summand of M, say $M = U \oplus M'$. If $V \subseteq M'$, then $M/V = U \oplus M'/V$ shows that U is a direct summand of M/V in contrast to our assumption that M/U belongs to $\mathcal{T}_U$. Thus $V \not\subseteq M'$. It follows that V is not contained in the kernel of the canonical projection $q\colon M \to M/M' \simeq U$, thus the restriction of q to V is a non-zero map $V \to U$. The condition (ii) asserts that this map $V \to U$ is split epi, therefore V does not belong to $\mathcal{T}_U$, a contradiction. This shows that M belongs to $\mathcal{T}_U$. $\square$

Proposition 4. *Let $\mathcal{E}$ be a class of indecomposable modules with the following property: If E belongs to $\mathcal{E}$, there is E' in $\mathcal{E}$ with* $\operatorname{Ext}(E, E') \neq 0$. *Then the set of functorially finite torsion classes $\mathcal{T}$ which contain $\mathcal{E}$ has no minimal elements.*

Proof. Let $\mathcal{T}$ be a functorially finite torsion class which contains $\mathcal{E}$. According to Proposition 2, there is an indecomposable module U in $\mathcal{T}$ such that any non-zero map $V \to U$ with $V \in \mathcal{T}$ is a split epimorphism. According to Proposition 3, the class $\mathcal{T}_U$ is a torsion class. Since $\mathcal{T}$ is functorially finite, also $\mathcal{T}_U$ is functorially finite.

We claim that $\mathcal{E}$ is contained in $\mathcal{T}_U$. Thus, let E belong to $\mathcal{E}$. Since E is indecomposable, we have to show that E is not isomorphic to U. By assumption, there is E' in $\mathcal{E}$ with $\operatorname{Ext}(E, E') \neq 0$. Thus, there is a non-split exact sequence $0 \to E' \to M \to E \to 0$. Since E, E' both belong to $\mathcal{E} \subseteq \mathcal{T}$ and $\mathcal{T}$ is closed under extensions, M belongs to $\mathcal{T}$. Since the given map $M \to E$ is not split epi, it follows that E is not isomorphic to U. Thus $\mathcal{E} \subseteq \mathcal{T}_U$. Since $\mathcal{T}_U$ is properly contained in $\mathcal{T}$, we see that $\mathcal{T}$ is not minimal in the set of functorially finite torsion classes which contain $\mathcal{E}$. $\square$

3. Construction of Ext-pairs.

The aim of this section is to show the following proposition.

Proposition 5. *A connected hereditary artin algebra which is representation-infinite and has at least three simple modules has* Ext*-pairs.*

Given a finite dimensional artin algebra R, we denote by $Q(R)$ its Ext-*quiver:* its vertices are the isomorphism classes $[S]$ of the simple R-modules S, and given two simple R-modules S, S', there is an arrow $[S] \to [S']$ provided $\operatorname{Ext}(S, S') \neq 0$. If R is hereditary, then clearly $Q(R)$ is directed. If necessary, we endow $Q(R)$ with a valuation as follows: Given an arrow $S \to S'$, consider $\operatorname{Ext}(S, S')$ as a left $\operatorname{End}(S)^{\rm op}$-module or as a left $\operatorname{End}(S')$-module and put

$$v([S],[S']) = (\dim_{\operatorname{End}(S)^{\rm op}} \operatorname{Ext}(S,S'))(\dim_{\operatorname{End}(S')} \operatorname{Ext}(S,S'))$$

(note that in contrast to [DR], we only will need the product of the two dimensions, not the pair). Given a vertex i of $Q(R)$, we denote by $S(i), P(i), I(i)$ a simple, projective or injective module corresponding to the vertex i, respectively.

The valuation of any arrow can be interpreted as follows (τ is the Auslander-Reiten translation).

Lemma 2. *If* $Q(\Lambda) = (1 \to 2)$*, then the arrow* $1 \to 2$ *has valuation at least* 2 *if and only if* $I(2)$ *is not projective if and only if* $P(1)$ *is not injective. If the arrow* $1 \to 2$ *has valuation at least* 3*, then* $\tau S(1)$ *is neither projective, nor a neighbor of* $P(1)$ *in the Auslander-Reiten quiver, consequently* $\operatorname{Hom}(P(1), \tau^2 S(1)) \neq 0$*, thus* $\operatorname{Ext}(\tau S(1), P(1)) \neq 0$. □

In the proof of Proposition 5, we will have to construct some exceptional modules. Two general results will be needed.

Lemma 3. *Let* e *be an idempotent of the artin algebra* Λ *and* $\langle e \rangle$ *the twosided ideal generated by* e*. Let* M *be a* Λ*-module with* $eM = 0$*. Then* M *is exceptional as a* Λ*-module if and only if* M *is exceptional when considered as a* $\Lambda/\langle e \rangle$*-module.*

Proof. Of course, if $0 \to M \to M' \to M \to 0$ is an exact sequence in $\operatorname{mod}\Lambda$, then $eM' = 0$, thus it is an exact sequence in $\operatorname{mod}\Lambda/\langle e \rangle$. □

A Λ-module M is said to be sincere provided there is no non-zero idempotent $e \in \Lambda$ with $eM = 0$.

Lemma 4. *Any connected artin algebra* Λ *has sincere exceptional modules.*

(Let us add that sincere exceptional modules are even faithful, see for example Corollary 2.3 of [R2].)

Proof, using induction on the number n of vertices of $Q(\Lambda)$. If $n = 1$, then any simple Λ-module is a sincere exceptional module.

Now assume that $n \geq 2$. Up to duality, we can assume that there exists a simple injective module S such that the full subquiver Q' of $Q(\Lambda)$ whose vertices are the isomorphism classes $[S']$ of the simple modules S' which are not isomorphic to S is connected. Let Λ'

be the restriction of Λ to Q'. By induction, there is a sincere exceptional Λ'-module M'. We form the universal extension M of M' by S, thus there is an exact sequence

$$0 \to M' \to M \to S^t \to 0$$

such that S is not a direct summand of M and $\mathrm{Ext}(S, M) = 0$. It is well-known (and easy to see) that M is indecomposable and has no self-extensions. $\square$

The proof of Proposition 5 requires to look at four special cases.

Case 1. The algebra Λ is tame.

We use the structure of the Auslander-Reiten quiver of Λ as presented in [DR]. Since we assume that Λ has at least 3 vertices, there is a tube of rank $r \geq 2$. The simple regular modules in this component form an Ext-cycle of cardinality r, say $X_1, \ldots, X_r$. There is a unique indecomposable module Y with a filtration $Y = Y_0 \supset Y_1 \supset \cdots \supset Y_{r-1} = 0$ such that $Y_{i-1}/Y_i = X_i$ for $1 \leq i \leq r-1$. Clearly, the pair Y, X_r is an Ext-pair.

Case 2. The quiver $Q = Q(\Lambda)$ is not a tree.

Deleting, if necessary, vertices, we may assume that the underlying graph of Q is a cycle. Let w be a path from a source i to a sink j of smallest length, let Q' be the subquiver of Q given by the vertices and the arrows which occur in w. Not every vertex of Q belongs to Q', since otherwise Q is obtained from Q' by adding just arrows, thus by adding a unique arrow, namely an arrow $i \to j$. But then this arrow is also a path from a sink to a source, and it has length 1. By the minimality of w, we see that also w has length 1 and therefore Q has just the two vertices i, j. But then Q can have only one arrow, thus is a tree. This is a contradiction.

Let Q'' be the full subquiver given by all vertices of Q which do not belong to Q'. Of course, Q'' is connected (it is a quiver of type $\mathbb{A}$). According to Lemma 4, there is an exceptional module X with support Q' and an exceptional module Y with support Q''. Since Q', Q'' have no vertex in common, we see that $\mathrm{Hom}(X, Y) = 0 = \mathrm{Hom}(Y, X)$.

There is an arrow $i \to j''$ with j'' a vertex of Q''. This arrow shows that $\mathrm{Ext}(X, Y) \neq 0$. Similarly, there is an arrow $i'' \to j$ with i'' a vertex of Q''. This arrow shows that $\mathrm{Ext}(Y, X) \neq 0$.

We consider now algebras Λ with Ext-quiver $1 \to 2 \to 3$. We denote by Λ' the restriction of Λ to the subquiver with vertices $1, 2$, and by Λ'' the restriction of Λ to the subquiver with vertices $2, 3$. Given a representation M, let M_3 be the sum of all submodules of M which are isomorphic to $S(3)$, then M/M_3 is a Λ'-module.

Lemma 5. *Let X, Y be Λ-modules. If $X_3 = 0$ and $\mathrm{Ext}(Y/Y_3, X) \neq 0$, then also $\mathrm{Ext}(Y, X) \neq 0$.*

Proof: The exact sequence $0 \to Y_3 \to Y \to Y/Y_3 \to 0$ yields an exact sequence

$$\mathrm{Hom}(Y_3, X) \to \mathrm{Ext}(Y/Y_3, X) \to \mathrm{Ext}(Y, X)$$

The first term is zero, since Y_3 is a sum of copies of $S(3)$ and $X_3 = 0$. Thus, the map $\mathrm{Ext}(Y/Y_3, X) \to \mathrm{Ext}(Y, X)$ is injective.

Case 3. $Q(\Lambda) = (1 \to 2 \to 3)$**, and** $v(1, 2) \geq 2,\ v(2, 3) \geq 2$.

Let $X = S(2)$ and let Y be the universal extension of X using the modules (1) and $S(3)$ (thus, we form the universal extension from above using copies of $S(1)$ and the universal extension from below using copies of $S(3)$. Clearly, Y is exceptional. Since the socle of Y consists of copies of $S(3)$, we have $\operatorname{Hom}(S(2), Y) = 0$. Since the top of Y consists of copies of $S(1)$, we have $\operatorname{Hom}(Y, S(2)) = 0$.

Since $v(1,2) \geq 2$, the module Y/Y_3 is not a projective Λ'-module. As a consequence, $\operatorname{Ext}(Y/Y_3, S(2)) \neq 0$. Lemma 5 shows that also $\operatorname{Ext}(Y, S(2)) \neq 0$. By duality, we similarly see that $\operatorname{Ext}(S(2), Y) \neq 0$.

Case 4. $Q(\Lambda) = (1 \to 2 \to 3)$**, and** $v(1,2) \geq 3,\ v(2,3) = 1$.

Let $X = P(1)/P(1)_3$ (thus X is the projective Λ'-module with top $S(1)$). Let $Y = \tau X$, where $\tau = D\operatorname{Tr}$ is the Auslander-Reiten translation in $\operatorname{mod}\Lambda$. Of course, both modules X, Y are exceptional. Since $Y = \tau X$, we know already that $\operatorname{Ext}(X, Y) \neq 0$.

We claim that $Y/Y_3 = \tau' S(1)$, where τ' is the Auslander-Reiten translation of Λ'. Since $P(1)_3 = S(3)^a$ for some $a \geq 1$, a minimal projective presentation of X has the form

$$(*) \qquad 0 \to S(3)^a \to P(1) \to X \to 0,$$

thus the defining exact sequences for $Y = \tau X$ is of the form

$$0 \to Y \to I(3)^a \to S(1) \to 0.$$

In order to obtain $\tau' S(1)$, we start with a minimal projective presentation

$$(**) \qquad 0 \to S(2)^a \to P'(1) \to S(1) \to 0,$$

where $P'(1)$ is the projective cover of $S(1)$ as a Λ'-module (actually, $P'(1) = X$). Since $\nu(2,3) = 1$, the number a in $(*)$ and $(**)$ is the same. The defining exact sequences for $Y = \tau X$ and $\tau' S(1)$ are part of the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \tau'S(1) & \longrightarrow & I(2)^a & \longrightarrow & S(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & I(3)^a & \longrightarrow & S(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & S(3)^a & = & S(3)^a & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The left column shows that $Y/Y_3 = \tau' S(1)$.

As we have mentioned in Lemma 2, $v(1,2) \geq 3$ implies that $\operatorname{Ext}(\tau' S(1), P'(1)) \neq 0$. According to Lemma 5, we see that $\operatorname{Ext}(Y, X) \neq 0$.

Finally, let us show that X, Y are orthogonal. Since $Y = \tau X$ and X is exceptional, we see that $\mathrm{Hom}(X, Y) = 0$. On the other hand, any homomorphism $Y \to X$ vanishes on Y_3, since X has no composition factor $S(3)$. Now Y/Y_3 is indecomposable and not projective as a Λ'-module, whereas X is a projective Λ'-module, thus $\mathrm{Hom}(Y, X) = \mathrm{Hom}(Y/Y_3, X) = 0$.

Remark. Concerning the cases 3 and 4, there is an alternative proof which uses dimension vectors and the Euler form on the Grothendieck group $K_0(\Lambda)$. But for this approach, one needs to deal with the valuation of $Q(\Lambda)$ as in [DR], attaching to any arrow $i \to j$ a pair (a, b) of positive numbers instead of the single number $v(i, j) = ab$.

Proof of Proposition 5. Let Λ be connected, hereditary, representation-infinite, with at least 3 simple modules. Case 2 shows that we can assume that $Q(\Lambda)$ is a tree.

Assume that there is a subquiver Q' such that at least two of the arrows have valuation at least 2, choose such a Q' of minimal length. We want to construct an Ext-pair for the restriction of Λ to Q'. Using reflection functors (see [DR]), we can assume that Q' has orientation $1 \to 2 \to \cdots \to n{-}1 \to n$. If $n = 3$, then this is case 3. Thus assume $n \geq 4$. The minimality of Q' asserts that $\nu(i, i+1) = 1$ for $2 \leq i \leq n-2$. If we denote by Λ' the restriction of Λ to Q', then Λ' has a full exact abelian subcategory $\mathcal{U}$ which is equivalent to the module category of an algebra as discussed in case 3 (namely the subcategory of all Λ'-modules which do not have submodules of the form $S(i)$ with $2 \leq i \leq n-2$ and no factor modules of the form $S(i)$ with $3 \leq i \leq n-1$). Since $\mathcal{U}$ has Ext-pairs, also $\mathrm{mod}\,\Lambda$ has Ext-pairs.

Thus, we can assume that at most one arrow $i \to j$ has valuation greater than 2. If $v(i, j) \geq 3$, then we take a connected subquiver Q' with 3 vertices containing this arrow $i \to j$. If necessary, we use again reflection functors in order to change the orientation so that we are in case 4.

Thus we are left with the representation-infinite algebras Λ with the following properties: $Q(\Lambda)$ is a tree, there is no arrow with valuation greater than 2 and at most one arrow with valuation equal to 2. It is easy to see that $Q(\Lambda)$ contains a subquiver Q' such that the restriction of Λ to Q' is tame, thus we can use case 1. $\square$

Proof of Theorem.

Let Λ be connected and hereditary. If Λ is representation-finite, then $\mathrm{tors}\,\Lambda = \text{f-tors}\,\Lambda$, thus $\text{f-tors}\,\Lambda$ is a lattice. If Λ has precisely two simple modules, then $\text{f-tors}\,\Lambda$ can be described easily (see the proof of Proposition 2.2 in [IRTT] which works in general), it obviously is a lattice.

On the other hand, if Λ is representation-infinite and has at least three simple modules, then Proposition 5 asserts that Λ has an Ext-pair, say X, Y. Since X, Y are exceptional modules, Proposition 1 shows that $\mathcal{T}(X) = \mathcal{G}(X)$ and $\mathcal{T}(Y) = \mathcal{G}(Y)$ both belong to $\text{f-tors}\,\Lambda$. The join of $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ in $\mathrm{tors}\,\Lambda$ is $\mathcal{T}(X, Y)$. According to Proposition 4, $\mathcal{T}(X, Y)$ cannot belong to $\text{f-tors}\,\Lambda$. $\square$

4. References

[AS] M. Auslander, S. O. Smalø: Preprojective modules over artin algebras, J. Algebra 66 (1980) 61–122.

[DR] V. Dlab, C. M. Ringel: Indecomposable representations of graphs and algebras. Mem. Amer. Math. Soc. 173 (1976).

[IRTT] O. Iyama, I. Reiten, H. Thomas, G. Todorov: Lattice structure of torsion classes for path algebras of quivers. Bull. London Math. Soc. 47 (2015) 4, 639–650.

[ONFR] M. A. A. Obaid, S. K. Nauman, W. M. Fakieh and C. M. Ringel: The Ingalls-Thomas bijections. IEJA 20 (2016), 28-44.

[R1] C. M. Ringel: Representations of k-species and bimodules. J. Algebra 41 (1976), 269–302.

[R2] C. M. Ringel: Exceptional objects in hereditary categories. Proceedings Constantza Conference. An. St. Univ. Ovidius Constantza Vol. 4 (1996), f. 2, 150-158.

[R3] C. M. Ringel: The Catalan combinatorics of the hereditary artin algebras. to appear in: Recent Developments in Representation Theory, Contemp. Math., 673, Amer. Math. Soc., Providence, RI, 2016.

[Ro] A. V. Roiter: Unboundedness of the dimension of the indecomposable representations of an algebra which has infinitely many indecomposable representations. Izv. Akad. Nauk SSSR. Ser. Mat. 32 (1968), 1275-1282

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