The torsionless modules of an artin algebra

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We consider an artin algebra Λ with duality functor D. Usually, we will consider left Λ-modules of finite length and call them just modules. Always, morphisms will be written on the opposite side of the scalars.

A module is said to be *torsionless* provided it can be embedded into a projective module. Let $\mathcal{L} = \mathcal{L}(\Lambda)$ be the class of torsionless Λ-modules.

1. Torsionless Λ-modules and torsionless $\Lambda^{\rm op}$-modules.

Let $\mathcal{P} = \mathcal{P}(\Lambda)$ be the class of projective Λ-modules. We have $\mathcal{P}(\Lambda) \subseteq \mathcal{L}(\Lambda)$, and we denote by $\mathcal{L}(\Lambda)/\mathcal{P}(\Lambda)$ the factor category obtained from $\mathcal{L}(\Lambda)$ by factoring out the ideal of all maps which factor through a projective module.

Theorem 1. *There is a duality*

$$\eta\colon \mathcal{L}(\Lambda)/\mathcal{P}(\Lambda) \longrightarrow \mathcal{L}(\Lambda^{\rm op})/\mathcal{P}(\Lambda^{\rm op})$$

with the following property: If U is a torsionless module, and $f\colon P_1(U) \to P_0(U)$ is a projective presentation of U, then for $\eta(U)$ we can take the image of $\operatorname{Hom}(f, \Lambda)$.

Note that there also is a duality between $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\Lambda^{\rm op})$, given by the functor $\operatorname{Hom}(-, \Lambda)$. Using these two dualities, we see:

Corollary 1. *There is a canonical bijection between the isomorphism classes of the indecomposable torsionless Λ-modules and the isomorphism classes of the indecomposable torsionless $\Lambda^{\rm op}$-modules.*

Proof: $\operatorname{Hom}(-, \Lambda)$ provides a bijection between the isomorphism classes of the indecomposable projective Λ-modules and the isomorphism classes of the indecomposable projective $\Lambda^{\rm op}$-modules. For the non-projective indecomposable torsionless modules, we use the duality η.

Remark. As we have seen, there are canonical bijections between the indecomposable projective Λ-modules and $\Lambda^{\rm op}$-modules, as well between the indecomposable non-projective torsionless Λ-modules and $\Lambda^{\rm op}$-modules, both being given by categorical dualities, but these bijections do not combine to a bijection with nice categorical properties. We will exhibit suitable examples below. There, we will use the duality D in order to replace the category $\mathcal{L}(\Lambda^{\rm op})$ of torsionless $\Lambda^{\rm op}$-modules by Λ-modules, namely by the category $\mathcal{K}(\Lambda)$ of all factor modules of injective modules.

We call Λ *torsionless-finite* provided there are only finitely many isomorphism classes of indecomposable torsionless Λ-modules.

Corollary 3. *If Λ is torsionless-finite, also $\Lambda^{\rm op}$ is torsionless-finite.*

Whereas corollaries 1 and 2 are of interest only for non-commutative artin algebras, the theorem itself is also of interest for Λ commutative.

Corollary 3. *For Λ a commutative artin algebra, the category $\mathcal{L}/\mathcal{P}$ has a self-duality.*

For example, consider the factor algebra $\Lambda = k[T]/\langle T^n\rangle$ of the polynomial ring $k[T]$ in one variable, with k is a field. Since Λ is self-injective, all the modules are torsionless. Note that in this case, η coincides with the syzygy functor Ω.

Proof of theorem 1. We call an exact sequence $P_1 \to P_0 \to P_{-1}$ with projective modules P_i *strongly exact* provided it remains exact when we apply $\operatorname{Hom}(-,\Lambda)$. Let $\mathcal{E}$ be the category of strongly exact sequences $P_1 \to P_0 \to P_{-1}$ with projective modules P_i (as a full subcategory of the category of complexes).

Lemma. *The exact sequence $P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1}$, with all P_i projective and epi-mono factorization $g = ue$ is strongly exact if and only if u is a left Λ-approximation.*

Proof: Under the functor $\operatorname{Hom}(-,\Lambda)$, we obtain

$$\operatorname{Hom}(P_{-1},\Lambda) \xrightarrow{g^*} \operatorname{Hom}(P_0,\Lambda) \xrightarrow{f^*} \operatorname{Hom}(P_1,\Lambda)$$

with zero composition. Assume that u is a left Λ-approximation. Given $\alpha \in \operatorname{Hom}(P_0,\Lambda)$ with $f^*(\alpha) = 0$, we rewrite $f^*(\alpha) = \alpha f$. Now e is a cokernel of f, thus there is α' with $\alpha = \alpha' e$. Since u is a left Λ-approximation, there is α'' with $\alpha' = \alpha'' u$. It follows that $\alpha = \alpha' e = \alpha'' ue = \alpha'' g = g^*(\alpha'')$.

Conversely, assume that the sequence $(*)$ is exact, let U be the image of g, thus $e\colon P_0 \to U, u\colon U \to P_{-1}$. Consider a map $\beta\colon U \to \Lambda$. Then $f^*(\beta e) = \beta e f = 0$, thus there is $\beta' \in \operatorname{Hom}(P_{-1},\Lambda$ with $g^*(\beta') = \beta e$. But $g^*(\beta) = \beta' g = \beta' ue$ and $\beta e = \beta' ue$ implies $\beta = \beta' u$, since e is an epimorphism.

Let $\mathcal{U}$ be the full subcategory of $\mathcal{E}$ of all sequences which are direct sums of sequences of the form

$$P \to 0 \to 0, \quad P \xrightarrow{1} P \to 0, \quad 0 \to P \xrightarrow{1} P, \quad 0 \to 0 \to P.$$

Define the functor $q\colon \mathcal{E} \to \mathcal{L}$ by $q(P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1}) = \operatorname{Im} g$. Clearly, q sends $\mathcal{U}$ onto $\mathcal{P}$, thus it induces a functor

$$\overline{q}\colon \mathcal{E}/\mathcal{U} \longrightarrow \mathcal{L}/\mathcal{P}.$$

Claim: *This functor $\overline{q}$ is an equivalence.*

First of all, the functor q is dense: starting with $U \in \mathcal{L}$, let

$$P_1 \xrightarrow{f} P_0 \xrightarrow{e} U \longrightarrow 0$$

be a projective presentation of U, let $u\colon U \to P_{-1}$ be a left Λ-approximation of U, and $g = ue$.

Second, the functor q is full. This follows from the lifting properties of projective presentations and left Λ-approximations.

It remains to show that $\overline{q}$ is faithful. We will give the proof in detail (and it may look quite technical), however we should remark that all the arguments are standard; they are the usual ones dealing with homotopy categories of complexes. Looking at strongly exact sequences $P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1}$, one should observe that the image U of g has to be considered as the essential information: starting from U, one may attach to it a projective presentation (this means going from U to the left in order to obtain $P_1 \xrightarrow{f} P_0$) as well as a left Λ-approximation of U (this means going from U to the right in order to obtain P_{-1}).

In order to show that $\overline{q}$ is faithful, let us consider the following commutative diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{\quad f \quad} & P_0 & \xrightarrow{\quad g \quad} & P_{-1} \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_{-1}}\downarrow \\ P_1' & \xrightarrow{\quad f' \quad} & P_0' & \xrightarrow{\quad g' \quad} & P_{-1}' \end{array}$$

with strongly exact rows. We consider epi-mono factorizations $g = eu, g' = e'u'$ with $e\colon P_0 \to U, u\colon U \to P_{-1}, e'\colon P_0' \to U', u'\colon U' \to P_{-1}'$, thus $q(P_\bullet) = U, q(P_\bullet') = U'$. Assume that $q(h_\bullet) = ab$, where $a\colon U \to X, b\colon X \to U'$ with X projective. We have to show that $h_\bullet$ belongs to $\mathcal{U}$.

The factorizations $g = eu, g' = e'u', q(h_\bullet) = ab$ provide the following equalities:

$$eab = h_0e', \quad uh_1 = abu'.$$

Since $u\colon U \to P_{-1}$ is a left Λ-approximation and X is projective, there is $a'\colon P_{-1} \to X$ with $ua' = a$. Since $e'\colon P_0' \to U'$ is surjective and X is projective, there is $b'\colon X \to P_0'$ with $b'e' = b$.

Finally, we need $c\colon P_0 \to P_1'$ with $cf' = h_0 - eab'$. Write $f' = w'v'$ with w' epi and v' mono; in particular, v' is the kernel of g'. Note that $eab'g' = eab'e'u' = eabu' = h_0e'u' = h_0g'$, thus $(h_0 - eab')g' = h_0g' - eab'g' = h_0g' - h_0g' = 0$, thus $h_0 - eab'$ factors through the kernel v' of g', say $h_0 - eab' = c'v'$. Since P_0 is projective and w' is surjective, we find $c\colon P_0 \to P_1'$ with $cw' = c'$, thus $cf' = cw'v' = c'v' = h_0 - eab'$.

Altogether, we obtain the following commutative diagram

$$\begin{array}{ccccc}
P_1 & \xrightarrow{\ f\ } & P_0 & \xrightarrow{\ g\ } & P_{-1} \\
{\scriptstyle [1\ f]}\downarrow & & \downarrow{\scriptstyle [1\ ea]} & & \downarrow{\scriptstyle [a'\ h_1 - a'bu']} \\
P_1 \oplus P_0 & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]} & P_0 \oplus X & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]} & X \oplus P'_{-1} \\
{\scriptstyle \left[\begin{smallmatrix} h_1 - fc \\ c \end{smallmatrix}\right]}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix} h_0 - eab' \\ b' \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} bu' \\ 1 \end{smallmatrix}\right]} \\
P'_1 & \xrightarrow{\ f'\ } & P'_0 & \xrightarrow{\ g'\ } & P'_{-1}
\end{array}$$

which is the required factorization of $h_\bullet$ (here, the commutativity of the four square has to be checked; in addition, one has to verify that the vertical compositions yield the maps h_i; all these calculations are straight forward).

Now consider the functor $\operatorname{Hom}(-,\Lambda)$, it yields a duality

$$\operatorname{Hom}(-,\Lambda)\colon \mathcal{E}(\Lambda) \longrightarrow \mathcal{E}(\Lambda^{\mathrm{op}})$$

which sends $\mathcal{U}(\Lambda)$ onto $\mathcal{U}(\Lambda^{\mathrm{op}})$. Thus, we obtain a duality

$$\mathcal{E}(\Lambda)/\mathcal{U}(\Lambda) \longrightarrow \mathcal{E}(\Lambda^{\mathrm{op}})/\mathcal{U}(\Lambda^{\mathrm{op}}).$$

Combining the functors considered, we obtain the following sequence

$$\mathcal{L}(\Lambda)/\mathcal{P}(\Lambda) \xleftarrow{\ \overline{q}\ } \mathcal{E}(\Lambda)/\mathcal{U}(\Lambda) \xrightarrow{\operatorname{Hom}(-,\Lambda)} \mathcal{E}(\Lambda^{\mathrm{op}})/\mathcal{U}(\Lambda^{\mathrm{op}}) \xrightarrow{\ \overline{q}\ } \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}}),$$

this is duality, and we denote it by η.

It remains to show that η is given by the mentioned recipe. Thus, let U be a torsionless module. Take a projective presentation

$$P_1 \xrightarrow{f} P_0 \xrightarrow{e} U \to 0$$

of U, and let $m\colon U \to P_{-1}$ be a left $\mathcal{P}$-approximation of U and $g = eu$. Then

$$P_\bullet = (P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1})$$

belongs to $\mathcal{E}$ and $q(P_\bullet) = U$. The functor $\operatorname{Hom}(-,\Lambda)$ sends $P_\bullet$ to

$$\operatorname{Hom}(P_\bullet,\Lambda) = (\operatorname{Hom}(P_{-1},\Lambda) \xrightarrow{\operatorname{Hom}(g,\Lambda)} \operatorname{Hom}(P_0,\Lambda) \xrightarrow{\operatorname{Hom}(f,\Lambda)} \operatorname{Hom}(P_1,\Lambda))$$

in $\mathcal{E}(\Lambda^{\mathrm{op}})$. Finally, the equivalence

$$\mathcal{E}(\Lambda^{\mathrm{op}})/\mathcal{U}(\Lambda^{\mathrm{op}}) \xrightarrow{\ \overline{q}\ } \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}})$$

sends $\mathrm{Hom}(P_\bullet, \Lambda)$ to the image of $\mathrm{Hom}(f, \Lambda)$.

A module is said to be *co-torsionless* provided it is a factor module of an injective module. Let $\mathcal{K} = \mathcal{K}(\Lambda)$ be the class of co-torsionless Λ-modules. Of course, the duality functor D provides a bijection between the isomorphism classes of co-torsionless modules and the isomorphism classes of torsionless right modules.

If we denote by $\mathcal{Q} = \mathcal{Q}(\Lambda)$ the class of injective modules, then we see that D provides a duality

$$D\colon \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}}) \longrightarrow \mathcal{K}(\Lambda)/\mathcal{Q}(\Lambda).$$

We get the following corollaries of Theorem 1.

Corollary 4. *The categories $\mathcal{L}(\Lambda)/\mathcal{P}(\Lambda)$ and $\mathcal{K}(\Lambda)/\mathcal{Q}(\Lambda)$ are equivalent under the functor $D\eta$.*

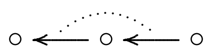
Note that $D\eta$ is is equal to $\Sigma\tau$ (restricted to $\Lambda/\mathcal{P}$), where τ is the Auslander-Reiten translation and Σ is the suspension functor (defined by $\Sigma(V) = I(V)/V$, where $I(V)$ is an injective envelope of V). Namely, in order to calculate $\tau(U)$, we start with a minimal projective presentation $f\colon P_1 \to P_0$ and take as $\tau(U)$ the kernel of

$$D\,\mathrm{Hom}(f, \Lambda)\colon D\,\mathrm{Hom}(P_1, \Lambda) \longrightarrow D\,\mathrm{Hom}(P_0, \Lambda).$$

Now the kernel inclusion $\tau(U) \subset D\,\mathrm{Hom}(P_1, \Lambda)$ is an injective envelope of $\tau(U)$; thus $\Sigma\tau(U)$ is the image of $D\,\mathrm{Hom}(f, \Lambda)$, but this image is $D\eta(U)$.

Corollary 5. *If Λ is torsionless-finite, the number of isomorphism classes of indecomposable factor modules of injective modules is equal to the number of isomorphism classes of indecomposable torsionless modules.*

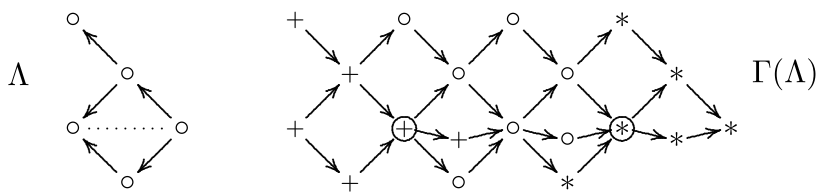
Examples: (1) The path algebra of a linearly oriented quiver of type A_3 modulo the square of its radical.

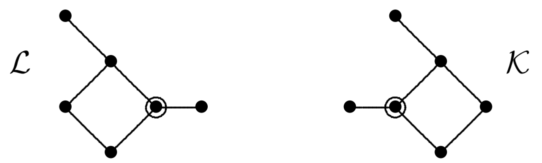


We present twice the Auslander-Reiten quiver. Left, we mark by $+$ the indecomposable torsionless modules and encircle the unique non-projective torsionless module. On the right, we mark by $*$ the indecomposable co-torsionless modules and encircle the unique non-injective co-torsionless module:

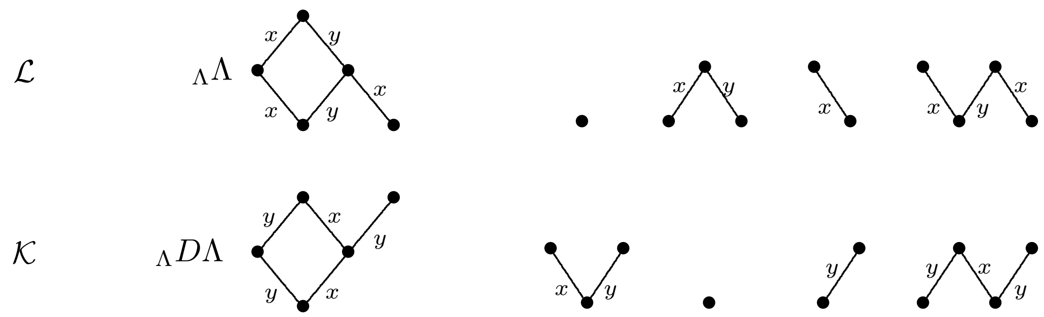
(2) Next, we look at the algebra Λ given by the following quiver with a commutative square; to the right, we present its Auslander-Reiten quiver $\Gamma(\Lambda)$ and

mark the torsionless and co-torsionless modules as in the previous example. Note that the subcategories $\mathcal{L}$ and $\mathcal{K}$ are linearizations of posets.





(3) The local algebra Λ with generators x, y and relations $x^2 = y^2$ and $xy = 0$. In order to present Λ-modules, we use the following convention: the bullets represent base vectors, the lines marked by x or y show that the multiplication by x or y, respectively, sends the upper base vector to the lower one (all other multiplications by x or y are supposed to be zero). The upper line shows all the indecomposable modules in $\mathcal{L}$, the lower one those in $\mathcal{K}$.



Let us stress the following: All the indecomposable modules in $\mathcal{L} \setminus \mathcal{P}$ as well as those in $\mathcal{K} \setminus \mathcal{Q}$ are Λ'-modules, where $\Lambda' = k[x, y]/\langle x, y\rangle^2$. Note that the category of Λ'-modules is stably equivalent to the category of Kronecker modules, thus all its regular components are homogeneous tubes. In $\mathcal{L}$ we find two indecomposable modules which belong to one tube, in $\mathcal{K}$ we find two indecomposable modules which belong to another tube. The algebra Λ' has an automorphism which exchanges these two tubes; this is an outer automorphism, and it **cannot** be lifted to an automorphism of Λ itself.

2. Torsionless-finite artin algebras have representation dimension at most 3.

Given a class $\mathcal{M}$ of modules, we denote by add $\mathcal{M}$ the modules which are (isomorphic to) direct summands of direct sums of modules in $\mathcal{M}$. We say that $\mathcal{M}$ is *finite* provided there are only finitely many isomorphism classes of indecomposable modules in $\operatorname{add}\mathcal{M}$, thus provided there exists a module M with $\operatorname{add}\mathcal{M} = \operatorname{add} M$.

Theorem 2. *Assume that Λ is torsionless-finite (thus, $\mathcal{L}$ and $\mathcal{K}$ are finite). Let K, L be modules with $\operatorname{add} K = \mathcal{K}$, and $\operatorname{add} L = \mathcal{L}$. Then the endomorphism ring of $K \oplus L$ has global dimension at most 3.*

Note that L is a generator, K a cogenerator, thus $K \oplus L$ is a generator-cogenerator. By definition, the representation dimension of Λ is the minimum of the global dimension of the endomorphism rings of generator-cogenerators. Thus, the theorem implies the following:

Corollary. *The representation dimension of a torsionless-finite artin algebra is at most 3.*

Proof of Theorem. Let $M = K \oplus L$. In order to prove that the global dimension of $\operatorname{End}(M)$ is at most 3, we have to show that for any Λ-module X, the kernel $\Omega_M(X)$ of a minimal right M-approximation of X belongs to $\operatorname{add} M$ (Auslander-Lemma, see [E] or [CP]).

Let X be a Λ-module. Let U be the trace of $\mathcal{K}$ in X (this is the sum of the images of maps $K \to X$). Since $\mathcal{K}$ is closed under direct sums and factor modules, U belongs to $\mathcal{K}$ (it is the largest submodule of X which belongs to $\mathcal{K}$). Let $p\colon V \to X$ be a right $\mathcal{L}$-approximation of X (it exists, since we assume that $\mathcal{L}$ is finite). Since $\mathcal{L}$ contains all the projective modules, it follows that p is surjective. Now we form the pullback

$$\begin{array}{ccc} V & \xrightarrow{\;p\;} & X \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ W & \xrightarrow[\;p'\;]{} & U \end{array}$$

where $u\colon U \to X$ is the inclusion map. With u also u' is injective, thus W is a submodule of $V \in \mathcal{L}$. Since $\mathcal{L}$ is closed under submodules, we see that W belongs to $\mathcal{L}$. On the other hand, the pullback gives rise to the exact sequence

$$0 \to W \xrightarrow{[p' \;\; -u']} U \oplus V \xrightarrow{\begin{bmatrix} u \\ p \end{bmatrix}} X \to 0$$

(the right exactness is due to the fact that p is surjective). By construction, the map $\begin{bmatrix} u \\ p \end{bmatrix}$ is a right M-approximation, thus $\Omega_M(X)$ is a direct summand of W and therefore in $\mathcal{L} \subseteq \operatorname{add} M$. This completes the proof.

Special cases.

(1) Let $J = \operatorname{rad}\Lambda$ and assume that $J^n = 0$. Claim: *Any indecomposable torsionless module is either projective or annihilated by* J^{n-1}. Namely, let M be an indecomposable submodule of the projective module P, write $P = \bigoplus P_i$ with P_i indecomposable. Let $u\colon M \to P$ be the inclusion and $p_i\colon P \to P_i$ the canonical projections. If we assume that $J^{n-1}M \neq 0$, then $J^{n-1}(Mup_i) \neq 0$ for some i. But then Mup_i cannot be a submodule of JP_i, since $J^n = 0$. Since JP_i is the unique maximal submodule of P_i, it follows that up_i is surjective. Since P_i is projective, we see that up_i is a split epimorphism and thus an isomorphism (since M is indecomposable). Thus we see: if M is not annihilated by J^{n-1}, then M is projective. As a consequence, we see: *If* Λ/J^{n-1} *is representation-finite, then there are only finitely many isomorphism classes of indecomposable torsionless modules.* By left-right symmetry, we also see that there are only finitely many isomorphism classes of indecomposable torsionless right modules.

This implies: *If* Λ/J^{n-1} *is representation-finite, then the representation dimension of* Λ *is at most* 3. (Auslander [A], Proposition, p.143)

(2) Following Auslander (again [A], Proposition, p.143) the special case $J^2 = 0$ should be mentioned here. It is obvious that an indecomposable torsionless module is either projective or simple, an indecomposable co-torsionless module is either injective or simple, and any simple module is either torsionless or co-torsionless. Thus M is the direct sum of all indecomposable projective, all indecomposable injective, and all simple modules. Thus, *the representation dimension of an artin algebra with radical square zero is at most* 3.

(3) Another special case of (1) is of interest: We say that Λ is minimal representation-infinite provided Λ is representation-infinite, but any proper factor algebra is representation-finite. If Λ is minimal representation-infinite, and n is minimal with $J^n = 0$, then Λ/J^{n-1} is a proper factor algebra, thus representation-finite. It follows: *The representation dimension of a minimal representation-infinite algebra is at most* 3.

(4) If Λ is hereditary, then the only torsionless modules are the projective modules, the only co-torsionless modules are the injective ones, thus both classes $\mathcal{K}$ and $\mathcal{L}$ are finite. Thus we recover Auslander's result ([A], Proposition, p. 147) that *the representation dimension of a hereditary artin algebra is at most* 3.

(5) More generally, we see that the classes $\mathcal{K}$ and $\mathcal{L}$ are finite in case Λ is stably equivalent to a hereditary artin algebra. Thus, *the representation dimension of an artin algebra which is stably equivalent to a hereditary artin algebra is at most* 3*;* (a result of Auslander-Reiten [AR], see also [X]). Here, an indecomposable torsionless module is either projective or simple (see [AR]).

(6) Right glued algebras (and similarly left glued algebras): An artin algebra Λ is said to be *right glued*, provided the functor $\operatorname{Hom}(D\Lambda, -)$ is of finite length, or equivalently, provided almost all indecomposable modules have projective dimension equal to 1. The condition that $\operatorname{Hom}(D\Lambda, -)$ is of finite length implies that

$\mathcal{K}(\Lambda)$ is finite. Also, the finiteness of the isomorphism classes of indecomposable modules of projective dimension greater than 1 implies that $\mathcal{L}(\Lambda)$ is finite. We see that *right glued algebras have representation dimension at most* 3 (a result of Coelho-Platzeck [CP]).

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