The Relevance and the Ubiquity of Prüfer Modules

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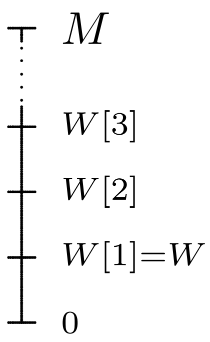
Prüfer modules

Let R be any ring. We deal with (left) R-modules.

An R-module is called a *Prüfer module* provided there exists an endomorphism ϕ of M with the following properties:

- ϕ is surjective,
- ϕ is locally nilpotent,
- ϕ the kernel W of ϕ is non-zero and of finite length.

The module W is called the *basis* of M.
Let $W[n]$ be the kernel of ϕ^n, then $M = \bigcup_n W[n]$.



$$W[n{+}1]/W[n] \simeq W$$

Example 1. Let $R = \mathbb{Z}$ and p a prime number.
Let $S = \mathbb{Z}[p^{-1}]$ the subring of $\mathbb{Q}$ generated by p^{-1}. Then

$$S/R = \lim_{\rightarrow} \mathbb{Z}/\mathbb{Z}p^n$$

is the Prüfer group for the prime p. These Prüfer groups are all the indecomposable $\mathbb{Z}$-modules which are Prüfer modules.

Example 2. Let $k[T]$ be the polynomial ring in one variable with coefficients in the field k. A $k[T]$-module is just a pair (V, f), where V is a k-space and f is a linear operator on V. If $\operatorname{char} k = 0$, then the pair $(k[T], \frac{\mathrm{d}}{\mathrm{d}\,t})$ is a Prüfer module.

Example 3. The embedding functor

$$\operatorname{mod} k[T] \longrightarrow \operatorname{mod} k\,(\circ \leftleftarrows \circ) \qquad (V, f) \mapsto V \underset{f}{\overset{1}{\leftleftarrows}} V$$

preserves Prüfer modules.

Review: Tame categories

If R is a Dedekind ring, then any indecomposable R-module of finite length is of the from $W[n]$, where W is the basis of a Prüfer module and $n \in \mathbb{N}$. In particular, this applies to $R = \mathbb{Z}$ and $R = k[T]$.

A corresponding asserting holds for the indecomposable coherent sheaves over an elliptic curve. (Atiyah)

Let Λ be a finite-dimensional k-algebra and k an algebraically closed field. If Λ is **tame,** and $d \in \mathbb{N}$, then almost all indecomposable Λ-modules of length d are of the from $W[n]$. (Crawley-Boevey)

Prüfer modules do not have to be indecomposable, for example the countable direct sum $W^{(\mathbb{N})}$ of copies of W with the shift endomorphism $(w_1, w_2, \dots) \mapsto (w_2, w_3, \dots)$ is a Prüfer module: the *trivial* Prüfer module with basis W.

Lemma. *Let M be a Prüfer module with basis W. If the endomorphism ring of W is a division ring, then either $M = W^{(\mathbb{N})}$ or else M is indecomposable.*

A module M is of *finite type* provided it is the direct sum of finitely generated modules and there are only finitely many isomorphism classes of indecomposable direct summands.

Warning. *There are Prüfer modules of finite type with indecomposable basis W and such that the embedding $W \to W[2]$ does not split.*

Our main concern will be Prüfer modules which are not of finite type. But first we consider some Prüfer modules of finite type.

The relevance and the ubiquity of Prüfer modules

Relevance	I. Degenerations of modules
	II. The second Brauer-Thrall conjecture
Ubiquity	III. The ladder construction
	IV. Self-extensions and Prüfer modules
	V. The search for pairs of embeddings

I. Degenerations of modules

Let Λ be a finite-dimensional k-algebra (k a field).
Consider first the case of k being algebraically closed. Assume that Λ is generated as a k-algebra by $a_1, a_2, \ldots, a_t$ subject to relations ρ_i. For $d \in \mathbb{N}$, let

$$\mathcal{M}(d) = \{(A_1, \ldots, A_t) \in M(d \times d, k)^t \mid \rho_i(A_1, \ldots, A_t) = 0 \text{ for all } i\},$$

the variety of d-dimensional Λ-modules: its elements are the d-dim Λ-modules with underlying vectorspace k^d (thus, up to isomorphism, all d-dim Λ-modules). The group $\mathrm{GL}(d, k)$ operates on $\mathcal{M}(d)$ by simultaneous conjugation. Elements of $\mathcal{M}(d)$ belong to the same orbit if and only if they are isomorphic.

Theorem (Zwara). *Let X, Y be d-dimensional Λ-modules. Then Y is in the orbit closure of X if and only if there exists a finitely generated Λ-module U and an exact sequence*

$$0 \to U \to X \oplus U \to Y \to 0$$

(a *Riedtmann-Zwara sequence*).

For any field k, call Y a *degeneration* of X provided there exists a finitely generated Λ-module U and an exact sequence

$$0 \to U \to X \oplus U \to Y \to 0$$

Proposition. *Y is a degeneration of X if and only if there is a Prüfer module M with basis Y such that $Y[t+1] \simeq Y[t] \oplus X$ for some t* (or, equivalently, for almost all t.)

Note that an isomorphism $Y[t+1] \simeq Y[t] \oplus X$ yields directly a Riedtmann-Zwara sequence as well as a co-Riedtmann-Zwara sequence, using the canonical exact sequences

$$0 \to Y[t] \to Y[t+1] \to Y \to 0 \ ,$$
$$0 \to Y \to Y[t+1] \to Y[t] \to 0$$

and replacing the middle term by $Y[t] \oplus X$.

II. Prüfer modules
and the second Brauer-Thrall conjecture

The representation type of a finite-dimensional (associative) k-algebra

Let Λ be a finite-dimensional k-algebra (K a field).

Krull-Remak-Schmidt Theorem. *Any finitely generated Λ-module can be written as a direct sum of indecomposable modules, and such a decomposition is unique up to isomorphism.*

A is called *representation-infinite* provided there are infinitely many isomorphism classes of indecomposable Λ-modules, otherwise *representation-finite.*

The Brauer-Thrall conjectures

Let Λ be a finite-dimensional k-algebra (K a field).

1. (Theorem of Roiter) *If Λ is not representation-finite, then there are indecomposable modules of arbitrarily large finite length.*

2. (Theorem of Bautista) *If Λ is not representation-finite and k is an infinite perfect field, then there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of length d.*

It has been conjectured by Brauer-Thrall that the second assertion holds for any infinite field.

For finite field, one may conjecture the following: there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of **endo-length** d.
(The endo-length of a module M is the length of M when M is considered as a module over its endomorphism ring).

Generic modules

An R-module M is said to be *generic* provided M is indecomposable, of infinite length, but of finite endo-length.

Theorem (Crawley-Boevey). *Let Λ be a finite-dimensional k-algebra (k a field). Let M be a generic Λ-module. Then there are infinitely many natural numbers d such that there are infinitely many indecomposable Λ-modules of endo-length d.*

Prüfer modules yield generic modules

Theorem. *If M is a Prüfer module which is not of finite type, and I is an infinite index set, then M^I has a generic direct summand.*

Outline of proof: Let $M = (M, \phi)$ and W its basis. Let $W[n]$ be the kernel of ϕ^n. We consider the infinite product $X = M^I$ and its submodule $X' = \bigcup_{n\in\mathbb{N}} W[n]^I$.
Krause has shown: $X = X' \oplus X''$ with X'' of finite endo-length.

Namely: On the one hand, X' is a pure submodule of X. On the other hand, X' is a direct sum of copies of M. But Prüfer-modules are Σ-injective, thus X' is a direct summand of X. Finally, the endo-length of X/X' is bounded by $\dim_k W$.

Next: X'' is of finite type if and only if M is of finite type.
Thus, if M is not of finite type, then X'' is endo-finite and not of finite tye, thus one of its indecomposable summands has to be of infinite length (thus generic).

Infinite type

$$? \implies ?$$

The existence of a Prüfer module
which is not of finite type

$$\implies$$

The existence of a generic module

$$\implies$$

The existence of families
of indecomposable modules of finite length

III. The ladder construction of Prüfer modules

Start with a proper inclusion $U_0 \subset U_1$ and a map $v_0 \colon U_0 \to U_1$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U_0 & \xrightarrow{w_0} & U_1 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \| & & \\
0 & \longrightarrow & U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q & = & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

We get a monomorphism $w_1 \colon U_1 \subset U_2$ and a map $v_1 \colon U_1 \to U_2$.

More precisely: from

$$0 \to U_0 \xrightarrow{w_0} U_1 \to W \to 0 \quad \text{and} \quad 0 \to K \to U_0 \xrightarrow{v_0} U_1 \to Q \to 0,$$

we obtain a module U_2 and a pair of exact sequences

$$0 \to U_1 \xrightarrow{w_1} U_2 \to W \to 0 \quad \text{and} \quad 0 \to K \to U_1 \xrightarrow{v_1} U_2 \to Q \to 0.$$

Using induction, we obtain in this way modules U_i and pairs of exact sequences

$$0 \to U_i \xrightarrow{w_i} U_{i+1} \to W \to 0 \quad \text{and} \quad 0 \to K \to U_i \xrightarrow{v_i} U_{i+1} \to Q \to 0$$

for all $i \geq 0$.

We obtain the following ladder of commutative squares:

$$\begin{array}{ccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
{\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots
\end{array}$$

We form the inductive limit $U_\infty = \bigcup_i U_i$ (along the maps w_i). Since all the squares commute, the maps v_i induce a map $U_\infty \to U_\infty$ which we denote by v_∞:

$$\begin{array}{c}
\bigcup_i U_i = U_\infty \\
\downarrow {\scriptstyle v_\infty} \\
\bigcup_i U_i = U_\infty
\end{array}$$

$$
\begin{array}{ccccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & & \bigcup_i U_i = U_\infty \\
{\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & & & \downarrow{\scriptstyle v_\infty} \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots & & \bigcup_i U_i = U_\infty
\end{array}
$$

We also may consider the factor modules U_∞/U_0 and U_∞/U_1. The map $v_\infty \colon U_\infty \to U_\infty$ maps U_0 into U_1, thus it induces a map

$$\overline{v} \colon U_\infty/U_0 \longrightarrow U_\infty/U_1.$$

Claim. *The map $\overline{v}$ is an isomorphism.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & W & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle v_{i-1}} & & \Big\downarrow{\scriptstyle v_i} & & \Big\| & & \\
0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & W & \longrightarrow & 0
\end{array}
$$

This diagram can be rewritten as

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & U_i/U_{i-1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle v_{i-1}} & & \Big\downarrow{\scriptstyle v_i} & & \Big\downarrow{\scriptstyle \overline{v}_i} & & \\
0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & U_{i+1}/U_i & \longrightarrow & 0
\end{array}
$$

with an isomorphism $\overline{v}_i\colon U_i/U_{i-1} \to U_{i+1}/U_i$. The map $\overline{v}$ is a map from a filtered module with factors U_i/U_{i-1} (where $i \geq 1$) to a filtered module with factors U_{i+1}/U_i (again with $i \geq 1$), and the maps $\overline{v_i}$ are just those induced on the factors.

It follows: The composition of maps

$$U_\infty/U_0 \xrightarrow{\quad p \quad} U_\infty/U_1 \xrightarrow{\quad \overline{v}^{-1} \quad} U_\infty/U_0$$

(p the projection map) is an epimorphism ϕ with kernel U_1/U_0. It is easy to see that ϕ is locally nilpotent.

Proposition. *The module U_∞/U_0 is a Prüfer module with respect to the endomorphism ϕ*

$$U_\infty/U_0 \xrightarrow{p} U_\infty/U_1 \xrightarrow{\overline{v}^{-1}} U_\infty/U_0,$$

its basis is $W = U_1/U_0$.

Summery. Starting with a proper inclusion $U_0 \subset U_1$ and a map $v_0 \colon U_0 \to U_1$, the ladder construction yields a Prüfer module U_∞/U_0 with bases U_1/U_0.

In case also v_0 is injective, we obtain a second Prüfer module. Namely, there is the following chessboard:

$$
\begin{array}{ccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
{\scriptstyle v_0}\Big\downarrow & & {\scriptstyle v_1}\Big\downarrow & & {\scriptstyle v_2}\Big\downarrow & & {\scriptstyle v_3}\Big\downarrow & & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & & \\
{\scriptstyle v_1}\Big\downarrow & & {\scriptstyle v_2}\Big\downarrow & & {\scriptstyle v_3}\Big\downarrow & & & & \\
U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & & & & \\
{\scriptstyle v_2}\Big\downarrow & & {\scriptstyle v_3}\Big\downarrow & & & & & & \\
U_4 & \xrightarrow{w_3} & \cdots & & & & & & \\
{\scriptstyle v_0}\Big\downarrow & & & & & & & & \\
\cdots & & & & & & & &
\end{array}
$$

We see both horizontally as well as vertically ladders: the horizontal ladders yield U_∞ and its endomorphism v_∞; the vertical ladders yield U'_∞ with an endomorphism w_∞.

IV. Self-extensions and Prüfer modules

A self-extension $0 \to W \to W[2] \to W \to 0$ is called a *ladder extension* provided there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{q} & W & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & W & \longrightarrow & W[2] & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

such that $\alpha = qv$ for some $v_0 \colon U_0 \to U_1$.

In this case we have a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{q} & W & \longrightarrow & 0 \\
 & & {\scriptstyle v_0}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & W & \longrightarrow & 0 \\
 & & {\scriptstyle q}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & W & \longrightarrow & W[2] & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

This shows that the given extension is part of the Prüfer module constructed by a ladder.

(1) *Not every self-extension of a module is a ladder extension.*

Example: Let S be a simple R-module, where R is artinian. Then no non-trivial self-extension of S is a ladder extension.

(2) *If R is hereditary, then every self-extension is a ladder extension.*

Proof: Take a projective presentation $0 \to P_1 \to P_0 \to W \to 0$. It induces the given self-extension:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{w_0} & P_0 & \xrightarrow{q} & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\
0 & \longrightarrow & W & \longrightarrow & W[2] & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

Since P_1 is projective, we obtain a required map $v_0\colon P_1 \to P_0$ with $\alpha = qv_0$.

Assume that Λ is a finite-dimensional hereditary k-algebra. (For example, Λ may be the path algebra kQ of a finite quiver Q without oriented cycles.)
Recall that the Euler characteristic

$$\sum_{i\geq 0}(-1)^i \dim \operatorname{Ext}^i(M, M')$$

yields a quadratic form q on the Grothendieck group $K_0(\Lambda)$ and q is positive definite if and only if Λ is representation-finite. (In the quiver case, this means that Q is the disjoint union of quivers of Dynkin type A_n, D_n, E_6, E_7, E_8.)

We see:
Any Λ-module M with $\operatorname{End}(M)$ a division ring and $q([M]) \leq 0$ is the basis of an indecomposable Prüfer module.
The Prüfer module is unique if and only if $q([M]) = 0$.

V. The search for pairs of embeddings

$$\begin{array}{ccc} U_0 & \xrightarrow{w_0} & U_1 \\ {\scriptstyle v_0}\Big\downarrow & & \\ U_1 & & \end{array}$$

Aim: To find pairs of embeddings $w_0, v_0 \colon U_0 \to U_1$ such that the corresponding Prüfer module U_∞/U_0 is not of finite type.

Lemma. *U_∞/U_0 is of finite type iff U_∞ is of finite type.*

Warning. Assume that Λ is tame hereditary (or concealed). If U_∞ is not of finite type, the U_1 must have a non-zero preprojective direct summand!

Proof: (1) If every indecomposable direct summand of U_1 is regular or preinjective, then only finitely many isomorphism classes of indecomposable Λ-modules are generated by U_1.

(2) Always: U_∞ is generated by U_1.

(3) Always; If a module M is not of finite type, then M has indecomposable factor modules of arbitrarily large finite length.

Take-off subcategories

Let Λ be a representation-infinite k-algebra (k a field).
Let $\operatorname{mod}\Lambda$ be the category of finitely generated Λ-modules.

A full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is said to be a *take-off* subcategory, provided the following conditions are satisfied:

(1) $\mathcal{C}$ is closed under direct sums and under submodules.
(2) $\mathcal{C}$ contains infinitely many isomorphism classes of indecomposable modules.
(3) No proper subcategory of $\mathcal{C}$ satisfies (1) and (2).

Theorem. *Take-off subcategories do exist.*

Even: Any subcategory satisfying (1) and (2) contains a take-off subcategory.

Examples:

- If Λ is a connected hereditary algebra which is representation-infinite, then the preprojective modules form a take-off subcategory.
- In general, there may be several take-off subcategories: For example, if Λ has several minimal representation-infinite factor algebras, then any such factor algebra yields a take-off subcategory of $\operatorname{mod}\Lambda$.

Remark. Observe that the existence of take-off subcategories is in sharp contrast to the usual characterization of "infinity" (a set is infinite iff it contains proper subsets of the same cardinality)!

Properties of a take-off subcategory $\mathcal{C}$

Lei $\mathcal{C}$ be a take-off subcategory of $\operatorname{mod}\Lambda$.

For any d, there are only finitely many isomorphism classes of modules of length d which belong to $\mathcal{C}$.

Thus: *$\mathcal{C}$ contains indecomposable modules of arbitrarily large finite length.*

Let $\overline{\mathcal{C}}$ be the class of all Λ-modules M such that any finitely generated submodule of M belongs to $\mathcal{C}$.

There are indecomposable modules M in $\overline{\mathcal{C}}$ of infinite length.

If M is an indecomposable modules M in $\overline{\mathcal{C}}$ of infinite length, then any indecomposable module N in $\mathcal{C}$ embeds into M — even a countable direct sum $N^{(\mathbb{N})}$ embeds into M.

We have seen that $\operatorname{mod}\Lambda$ contains take-off subcategories $\mathcal{C}$, such a subcategory $\mathcal{C}$ contains indecomposable modules M of arbitrarily large finite length, and thus indecomposable modules with arbitrarily large socle.

Conjecture. *Let U be an indecomposable Λ-module belonging to a take-off subcategory. If there is a simple module S such that S^7 embeds into U, then there are two embeddings*

$$\begin{array}{ccc} S & \xrightarrow{\ w\ } & U \\ {\scriptstyle v}\big\downarrow & & \\ U & & \end{array}$$

such that the corresponding Prüfer module is not of finite type.

Remark. The bound 7 cannot be improved, as the path algebra of the $\widetilde{E}_8$-quiver with subspace orientation shows.