The Impossible.

Some Puzzles.

Claus Michael Ringel

There are many puzzles and brainteasers which have a mathematical background, and it seems to be worthwhile to outline the relevant mathematics behind such puzzles. Math teachers should be aware of this so that they may use puzzles as an additional source for insight, but also as recreation and refreshment. After all, some mathematical notions can well be explained in this way. When working with puzzles, one learns a lot about about mental blockades and strategies to overcome them. And often one is able to vary the level of complexity quite freely, thus puzzles provide an ideal training ground.

The following text[1] will focus the attention to some puzzles which deal with the problem of impossibility. Of course, many puzzles appear to be impossible to solve, at first glance, but after a while one realizes that one has been fooled: that there is a solution, and often a rather easy one. But here we want to select puzzles which focus the attention to really impossible things, as well as some which are labeled directly in this way, such as the *impossible dovetails*, they will be considered at the end of the paper. A main theme will be to discuss the use of the words *possible — impossible* in mathematics. We will see that there are interesting puzzles which can be used very well to illustrate the necessity of mathematical proofs.

1. The 15-Puzzle.

1.1. The 15-Puzzle is a brainteaser which seems to be the oldest sliding block puzzle. Until recently, it was considered as the most successful puzzle invented by Sam Loyd (a very well-known puzzle designer and author of puzzle books), and according to some sources, it is said to be put on the market October 1865 by the international Card Company (London). Definitely, it had the greatest impact in America and Europe of any mechanical puzzle, at least before Rubik's cube. It first was called Gem Puzzle or Boss Puzzle, and later either the 15-Puzzle or the 14-15-Puzzle. In Germany, it was sold under the original

[1] This is a report on a lecture given in SEDIMA (Seminar for Didactic of Mathematics), Bielefeld, December 1999. It was part 3 of a series of annual lectures dealing with puzzles, see [R]. The lectures were always scheduled just before Christmas: In western countries, Christmas is the time when children are getting toys as gifts. And there are many puzzles which make a marvelous gift — not only for children. Unfortunately, only few of such puzzles are commercially available at any time, but actually many can be manufactured quite easily, say also in class. Thus the lectures also provided some guidance for constructions.

name *Boss-Puzzle*, in France under the name *Jeu de taquin* (teasing game) — it is already interesting to compare the different naming!

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Starting position

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Target

Here, the square blocks with the numbers 1 to 15 can be shifted horizontally and vertically. but cannot be removed from the frame. In the starting position, one can move either the block 12 downwards, or the block 14 to the right, and so on. The problem to be solved is to use a sequence of slidings in order to reach the position shown at the right: here, the blocks 14 and 15 are exchanged. Sam Loyd is said to have offered 1000 $ (at that time a very large amount of many) for a solution and he claimed that the money was deposited by a New Yorker newspaper. A hysteria broke out: A lot of incredible stories have been told [BS,T]: about a shop owner, who forgot to open his store because we was so busy trying to solve the puzzle; about a clergyman who played with the puzzle during a cold winter's night under a street light; about a publisher, who was seen going for lunch but did not return: at midnight, his colleagues found him in some back room moving pieces of a cake back and forth, simulating in this way the blocks of the 15-Puzzle. The mathematician and parliamentarian S. Günther reported that around 1880 a lot of the members of the German Reichstag, dignified men indeed, were sitting on the benches being involved with the boss puzzle, not listening to the speeches at all.

Sam Loyd often told the story that when he tried to get a patent for this puzzle, he was asked about the solution. So he told U.S. Patent Office that there does not exist a solution, that using mathematics one can prove this. The Patent Office refused to grant him a patent: they told him that *one cannot get a patent for a matter which does not work.*

1.2. Mathematical Analysis. As Loyd has pointed out, there is no solution. We want to outline the corresponding arguments. Let us look for example at a sequence of slidings, say to move the blocks 14, 11, 12, 8, one after the other, thus:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

$\overset{v_{14}}{\mapsto}$

1	2	3	4
5	6	7	8
9	10	11	12
13	15		14

$\overset{v_{11}}{\mapsto}$

1	2	3	4
5	6	7	8
9	10		12
13	15	11	14

$\overset{v_{12}}{\mapsto}$

1	2	3	4
5	6	7	8
9	10	12	
13	15	11	14

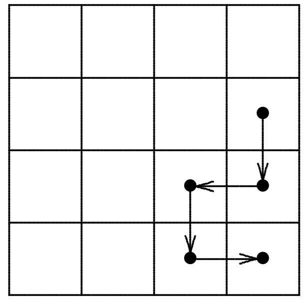
$\overset{v_{8}}{\mapsto}$

1	2	3	4
5	6	7	
9	10	12	8
13	15	11	14

Here, we have labeled the sliding of the block i as v_i, we may denote the sequence of slidings just considered by the concatenation

$$v_8 v_{12} v_{11} v_{14}$$

(note that we read this from right to left, in the same way as one considers the composition of mathematical functions). We may describe the sequence of slidings also by the following diagram of arrows:



Of course, if we look at longer sequences of slidings, the corresponding arrows may overlap!

Let us denote the empty field by the number 16.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	16

Starting position

1	2	3	4
5	6	7	16
9	10	12	8
13	15	11	14

after 4 moves

Note that the sliding v_i is nothing else then the permutation which exchanges the number i and the number 16 and fixes all the other numbers. A permutation which exchanges precisely two numbers is called a transposition; the transposition which exchanges the numbers i, j will be denoted by τ_{ij}, thus $v_i = \tau_{i,16}$.

Altogether, we see that we deal with permutations of the numbers 1 to 16, and we are interested to see which ones can be obtained by sliding blocks. Let us denote by S_{16} the set (or better group) of all permutations of the numbers 1 to 16. A permutation π is often displayed by a table with two rows, the upper row is filled with the numbers say 1 to 16 in increasing order, the lower row with the permuted numbers, thus with the sequence $\pi(1), \dots, \pi(16)$. Using this convention, the permutation $\tau_{8,16}\tau_{12,16}\tau_{11,16}\tau_{14,16}$ corresponds to the table

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 16 & 9 & 10 & 12 & 8 & 13 & 11 & 15 & 14 \end{pmatrix}$$

Let M be the set of permutations which can be achieved by sliding the blocks of the 15-Puzzle. Let us stress that this is not a subgroup, but just a subset: namely as we know, at the beginning, we can start only with $\tau_{12,16}$ or $\tau_{14,16}$, since 12 and 14 are the only neighbors of the empty field 16. But after the exchange $\tau_{14,16}$ of the numbers 14 and 16, we cannot apply $\tau_{12,16}$, since 12 and 16 are no longer neighbors. At any moment, there are at most four numbers i such that we can apply $v_i = \tau_{i,16}$, namely the numbers i which are neighbors of the empty field 16. Actually, it is not so easy to provide a general criterion which elements of S_{16} belong to M and which not, but fortunately, such a description of the set M will not be needed in our further considerations.

What is of importance is the fact that one can write any permutation π as a product of transpositions, but note that such a product factorization is not unique.

The question to be asked is the following:

Problem: *Is* $\tau_{14,15} \in M$*?*

This is the **mathematical formulation** of the 15-Puzzle.

In order to see this, let us note that the transposition $\tau_{14,15}$ obviously fixes the number 16. Let U be the set of permutations which fix the number 16, this subset U is a subgroup (we can identify it easily with the group S_{15}). Let us look at the intersection $M \cap U$. In case Is $\tau_{14,15} \in M$, then $\tau_{14,15} \in M \cap U$.

We need a little more theory. We have mentioned that any permutation π can be written as a product say of transpositions, and that such a product factorization is not unique, not even the number of transpositions to be used is fixed. But one can distinguish the even and the odd permutations: for the even permutations, the number of transposition in a factorization is always even, for the odd ones, it is odd. Namely, there is the following lemma which shows that the parity of the number of transposition used in a factorization can be characterized as follows:

Lemma. *A permutation is even if and only if the number of pairs* (i, j) *of numbers* $1 \le i, j \le n$ *with* $i < j$ *and* $\pi(i) > \pi(j)$ *is even.*

(This is not difficult to prove, a proof is usually given in any first year course on Linear Algebra, since it is used when dealing with determinants of square matrices.)

Back to the 15-Puzzle. The essential assertion is: *All the permutations in* $M \cap U$ *are even.* Proof: Consider an element in $M \cap U$, thus a product $v_{i_m} \cdots v_{i_2} v_{i_1}$, where $i_m \to i_{m-1} \to \cdots \to i_2 \to i_1 \to 16$ is a sequence of arrows which show the consecutive sliding of the blocks; thus, altogether we deal with the composition of m slidings. Consider the path of the number 16, it is moved up and down, back and forth, altogether m times (actually, 16 moves against the direct of the arrow, but this does not matter). Assume that 16 has been moved a times down and b times to the left. But then 16 has to be moved also a times up and b times to the right. Altogether we see that 16 has been moved $m = 2a + 2b$ times, and this is an even number.

On the other hand, we clearly see: *The permutation* $\tau_{14,15}$ *is odd.* That is all what we need for the proof: an odd permutation such as $\tau_{14,15}$ cannot be written as the composition of an even number of permutations, thus the permutation $\tau_{14,15}$ cannot be achieved by sliding blocks: Any permutation which belongs to U (that is, fixing 16), requires an even number of moves, thus is even.

In this way we see: It is **impossible** to exchange the position of just the numbers 14 and 15.

What we have invoked are some considerations of what is called group theory, this is a mathematical theory of central interest, with profound and sometimes quite difficult results. Fortunately, here we need only a very elementary consideration, namely the lemma quoted above: one should observe that this assertion is not at all obvious, but it is not very difficult to prove.

Some more mathematics. One can show that any even permutation which fixes 16 belongs to M, thus $M \cap U = A_{15}$ (where A_n denotes the set of even permutation in S_n). This is of interest in itself, however for the impossibility assertion considered above we only need the inclusion $M \cap U \subseteq A_{15}$ which we have established quite easily.

It may be of interest to provide as additional information the cardinality of the groups we were dealing with:

$$|S_{16}| = 16! = 20.922.789.888.000$$
$$|S_{15}| = 15! = 1.307.674.368.000$$
$$|A_{15}| = \tfrac{1}{2}\,15! = 653.837.184.000$$

1.3. Variations. A lot of variations of the 15-Puzzle have been put forward, some should be mentioned here. The 15-Puzzle asks to provide a sequence of slidings which exchange just the numbers 14 and 15 – an impossible task. However, there also was the challenge to bring any weird rearrangement of the blocks back into the starting position, a maybe time-consuming, but clearly manageable endeavor. And two sided versions were developed: if one side is in some nice order, the other side was completely disordered.

Here is a version which uses letters instead of numbers:

R	A	T	E
Y	O	U	R
M	I	N	D
P	L	A	

Starting position

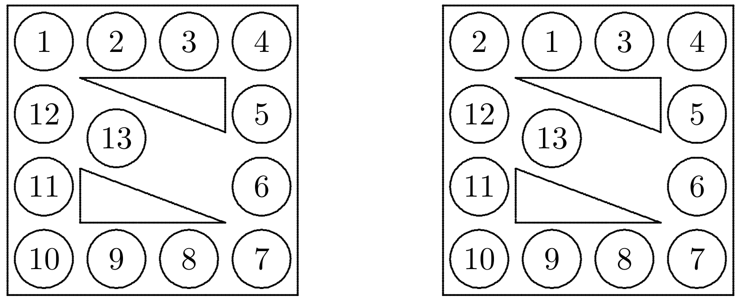
R	A	T	E
Y	O	U	R
M	I	N	D
P	A	L	

Target

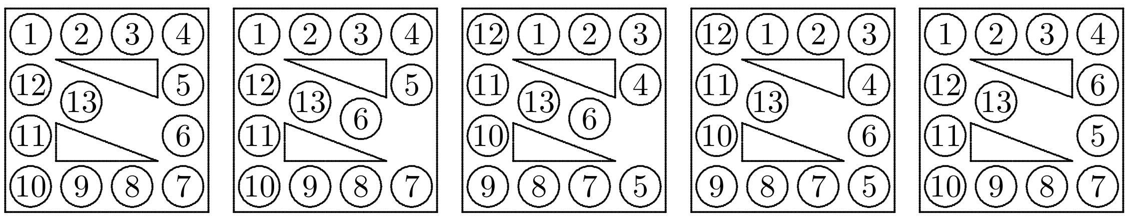
but surprisingly, this problem **can** be solved. Why? The reason is very simple: the letter A appears twice, say there is A_1 in the first row and A_2 in the last, thus we use the permutation which sends A_1 to L, L to A_2 and A_2 to A_1. This is an even permutation and thus can be realized by sliding blocks.

Another puzzle: *The Thirteen Puzzle. It* **can** *be done. A Scientific Puzzle in Permutation, Affording Amusement and Instruction.* It was produced by the Columbia Novelty

Manufacturing Co. of Boston, around 1906.



The triangles are wooden blocks glued to the ground plate. the disks are blocks which can be moved. Here one can show: all possible permutations of the disks can be achieved. For example, in order to exchange the discs labeled 5 and 6 one uses the following moves:



In a similar way, all transpositions, and therefore all permutations, can be realized.

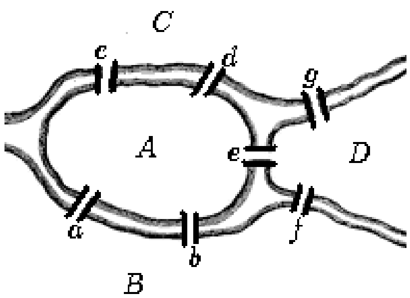
1.4. But let us return to the history of the 15-Puzzle itself. Recent investigations by J. Slocum and Dic Sonneveld [SS] have shown that Sam Loyd was not at all involved with the invention of this puzzle. The 15-Puzzle was designed by Noyes Palmer Chapman, a post man, around 1874, and he started the production in 1879. The book of Slocum and Sonneveld has the subtitle: *The Fascination True Story: How Sam Loyd's most successful hoax lasted more than a century.*

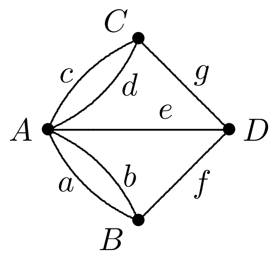
2. The necessity of proofs.

2.1. The detailed analysis of the 15-Puzzle may be too painstaking for math education in high school. But it is important that the school education provides the insight that it is the mathematics which can show that certain things are definitely impossible, without ifs and buts. Here are some suitable topics in this direction, mostly just planar problems, where one only needs paper and pencil, maybe also a pair of scissors in order to cut out pieces of paper (but one may also use pieces of a cake and move them around ...).

A. Let us start with the famous problem about the seven bridges of Königsberg. It was discussed at the time of Euler (and his solution is considered as the start of graph theory): there are seven bridges in Königsberg — is it possible to find a walk through the city passing all the bridges exactly once.

How can one show that this is **impossible**? One first singles out the decisive information by drawing a graph. As vertices one may use the different parts of the city which are separated by water, and one connects two vertices by as many edges as there are bridges connecting the corresponding parts of the city. Here, on the left, is an old plan of the river and the bridges, and, on the right, the corresponding graph; always, the parts of the city are labeled A, B, C, D, and the bridges are a, b, c, d, e, f, g.

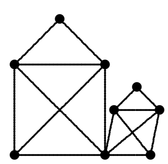




The problem now can be rephrased as follows: is there a path in the graph which runs through every edge precisely once (such a path is sometimes called a Euler path). In order to see that such a path cannot exist, it is sufficient to count the degrees of any vertex x, this is the number of edges at x. It turns out that the vertex A has degree 5, the other three vertices have degree 3, thus we deal with a graph with four vertices of odd degree. Now vertices with odd degree can only occur as the start or as the end of a Euler path, all other vertices have to have even degree.

The method of looking at the degree of vertices solves many problems. For example, one may ask whether it is possible to draw some graph without taking off the pencil, and running through every line just once, thus again we are looking for an Euler path. Let us consider the so called *House of Nicolaus*, even together with a slightly inclined outside toilet:

This is the House of Nicolaus



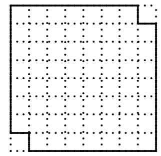
with the toilet of Nazareth

(usually one quotes the mentioned English syllables one after the other when drawing the lines: eight syllables for the eight lines). One observes that the House of Nicolaus (as well as the double house) have precisely two vertices with odd degree, namely those outside on the basis. Any Euler path has to start at one of these two vertices and will end at the other one.

There are several elementary results in graph theory which can be used in order to show that some problem cannot have a solution. For example, one knows that the complete bipartite graph K_{33} is not planar, this means that given 3 houses and 3 sources of energy, one cannot connect each of the houses with each energy source if one requires that the corresponding pipes should not cross.

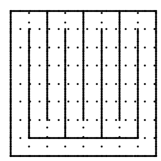
Graph-theoretical considerations can also be used in order to look for strategies for solving puzzles (see for example the book [BCG]).

B. Let us consider a chessboard, but we want to remove two opposite corner squares, thus 62 squares remain. Question: is it possible to cover these 62 squares using dominoes (any domino is supposed to cover two neighboring fields, and there should not be any overlap).



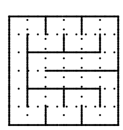
The answer is again NO. There are many diffrent proofs, but the most elegant one[2] seems to be the following: Consider the chessboard with its usual coloring of its squares, say as black and white squares. We have deleted two black squares, thus there remain 32 white squares and 30 black squares. Any domino will cover precisely one white square and one black square. That's it.

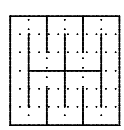
More general, one sees: Looking again at our chessboard, it is impossible to use 31 dominoes in order to cover all but two squares of the same color. But what about the corresponding problem of covering all but two given squares, a white one and a black one? Here the answer is positive, for any such pair of squares! In order to see this, one uses so called Gomory barriers [G], for example:



The Gomory barriers furnish a closed path of squares which can be covered in precisely two ways using dominoes. In particular, we see: any proper subpath of even length can be covered by dominoes in a unique way. Thus, if we delete two squares with different colors, then we get either one subpath of length 62 (this happens in case the squares are neighbors) or else two subpaths (in case the squares are not neighbors). It remains to observe that in the second case both subpaths must have even length, since they are bounded by squares of different color.

Of course, there are many possibilities to draw Gomory barriers, but this does not play any role in our argument. Our considerations can be applied to any square board of size $n \times n$ with n even. Here are two examples of Gomory barriers for $n = 6$:

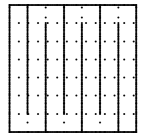




What happens if n is odd? Again we use a chessboard coloring, but now all the corner squares have the same color, say black. If we try, as before, to cover the board by dominoes, definitely one black square has to remain. One can choose from the beginning

[2] In his book *Elegance in Science* [Gl], Ian Glynn uses this problem as his starting point: by comparing different proofs he explains his vision of elegance.

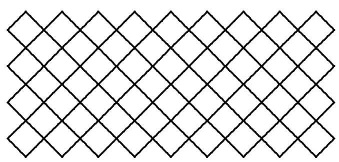
any black square which should remain! The proof uses again Gomory barriers, here is an example for the case $n = 7$.



We see a path from the lower left corner square to the lower right corner square (thus, of course, not a closed path), it starts and ends with black squares. If we delete any black square, we obtain either one subpath of even length, or two paths and again we see that these two paths have to have even length. This shows that after deletion of any black square, the remaining board can be covered by dominoes.

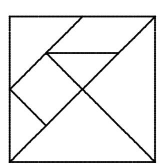
C. Coloring considerations are often helpful when we look for solutions of puzzles. For example, dealing with Piet Hein's Soma-Cube, one can use chessboard colorings in order to narrow down the possible positions of the individual pieces. Or else one can use them in order to show that some shapes which one would like to construct using the Soma pieces cannot be built (see for example [R], parts 2 and 4).

D. Here is another covering problem which can be solved using some different technique. Question: can the following area be covered by the 12 pentominoes (see for example [G])?



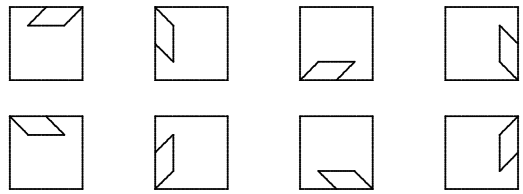
The answer is NO. For the proof, consider the various pentominoes and count how many boundary squares such a pentominoe may cover at most. It turns out that the sum of these numbers is 21, whereas the given shape has 22 boundary squares.

E. Now let us consider the Tangram puzzle. This is a very well-known puzzle and can be used in many different situations when teaching mathematics. There are seven flat pieces and the usual task is to put together the pieces in order form different shapes. Five of the seven pieces are triangles, all are isosceles right triangles, but of 3 different sizes. In addition, there are a square and a parallelogram. The pieces fit together in order to form a square:

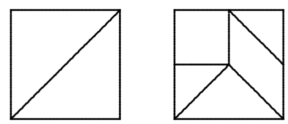


If we discuss some Tangram problems, we should mention that always all seven pieces have to be used and that they are not allowed to overlap.

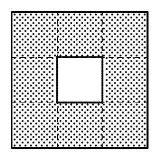
Let us discuss some problems. First, how many possibilities are there to form a square? Answer: There are precisely eight possibilities, any one is uniquely determined by the position of the parallelogram, and all are obtained from a single solution using rotations and reflections. In particular, we assert that up to symmetry (thus up to rotations and reflections), there is just one possibility. Here is the list of the possible positions of the parallelogram:



Second problem: How many possibilities are there in order to form two squares? Again, up to symmetry, there is a unique solution:



Third problem. Show that it is impossible to put together the following shape:



Many additional questions could be raised. How many different convex shapes can be formed with the Tangram pieces? Precisely 13. What is the longest line which can be covered with the Tangram pieces? The answer is $2+7\sqrt{2}+\sqrt{5}$; it may come as a surprise that here not only the square root of 2 (which is omnipresent when dealing with Tangram problems), but also the square root of 5 plays a role. For a further discussion of Tangram problems we may refer to [Sl] and [R], part 5.

2.2. One of the most important missions of mathematical education seems to be to promote the knowledge that there is a well-cut distinction between possible and impossible in mathematics. Problems which seem to have a positive solution may turn out to have no solution at all, but the important and relevant fact will be that this can be proven rigorously. To work with puzzles and brainteasers provides a lot of possibilities to discuss the necessity of proofs. This concerns a usually playful atmosphere, but still a solid and serious treatment.

So let me repeat: In my opinion, it is essential that part of the mathematical teaching in high school focus the attention to clear cut assertions and their proofs[3]. Until the

[3] But apparently, not all agree! In Germany, there was a lively debate about mathematical education when the Habilitation Thesis of Heymann was published in 1996, the title was *Allgemeinbildung und Mathematik*, general education and mathematics. Heymann argued that there should be an early differentiation (say at the age of 14) between those who want to choose a profession where one needs mathematics, and the remaining pupils. He handed out a list of topics which he felt are of general interest, it excluded explicitly quadratic equations as well as trigonometric functions, and did not mention at all the understanding of a mathematical proof.

beginning of the 20th century, the teaching of geometry in Europe and America was usually based on Euclid's treatment, and, in this way, furnished a good education in logical thinking. It is said that Abraham Lincoln (who is considered as one of the greatest US presidents) decided as a student to work through all the first six books of Euclid, in order to be prepared to become a lawyer: he wanted to understand the essence of proofs. He is quoted: *you never can make a lawyerif you do not know what "demonstrate" means.*

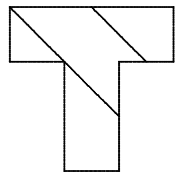
But besides geometrical themes also algebraic and number theoretical topics provide the possibility to become familiar with mathematical proofs. Classical examples always have been to show that $\sqrt{2}$ is irrational (that it cannot be written as a fraction a/b with integeres a and b) or also that there are infinitely many prime numbers.

The "modern" tendency in countries such as Germany to abolish axiomatic thinking from the mathematical curriculum is dangerous for developing an understanding of mathematical truth, and thus a proper insight into the different types of assertions. It may look curious to see that the playful use of puzzles can be used as a remedy!

2.3. Lateral Thinking. Let me close this section by mentioning two problems which both have a positive solution (even if one may have doubts at the beginning), but the solutions are of quite different nature, namely the T-puzzle and the E-puzzle. Both use only very few pieces, the T-puzzle four, the E-puzzle even only three.[4] The aim of the two puzzles is to arrange the pieces in order to show capital letters of the Latin alphabet, namely, as the names suggest, the letters T and E, respectively.

[4] Usually, an increase of the number of pieces tends to increase the difficulties of finding a solution quite drastically. Interesting puzzles with a small number of pieces are quite rare, but usually really fascinationg.

In order to find the solutions, some flexibility of the mind is necessary. Let me discuss here the solution of the T-puzzle. The difficulties which one encounters stem from the fact that the cuts which are used are really crooked[5]



For dealing with such problems, often some lateral thinking may be necessary. Or, to argue differently, to work with puzzles and brainteasers seem to be a good training of lateral thinking.

Lateral thinking means to overcome mental deadlocks by looking for alternative ways of insight, for deviating ideas. The term lateral thinking was coined by Edward de Bono [dB] in 1967. In order to illustrate the concept of lateral thinking, let me mention some questions raised by Mel Stover[6] who invented many interesting puzzles (he died in 1999).

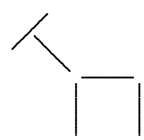
1. In the equation

$$26 - 63 = 1$$

change the position of just *one* digit to make the equation correct.

2. Tom's mother had four children. Three were girls. The girls' first names where Spring, Summer, and Autumn. What was the first name of the fourth child?

3. Form the figure of a giraffe, as shown below, with matches or toothpicks.



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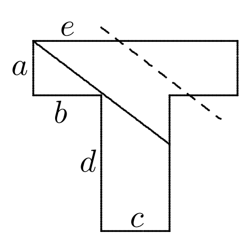
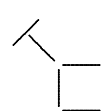
[5] Acutually, several versions of the T-puzzle have been put forward, and are sold by different companies, all have in common a diagonal cut which yields 3 pieces, one being a triangle, as well as a second cut which is parallel to the first one:



But they differ in the parameters a, b, c, d which define the T, as well as in e which yields the second cut. The most challenging choice of the parameters seem to be those where the lengths a, b, c are equal, and say $d = 2a$. Still there are two interesting possibilities for e, namely either $e = \sqrt{2}a$, so that all four pieces have width equal to a, or else to choose $e = \frac{3}{2}a$. In the latter case, the two pieces which are separated by the second cut can be rearranged in order to form a parallelogram.

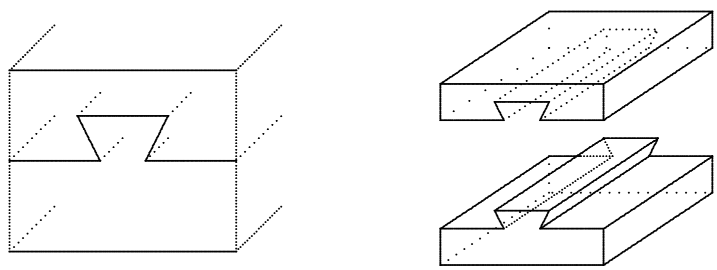
[6] see [Ga]. Here are the answers: $2^6-63=1$, Tom, and  . It should be observed that the axis of reflection used here is actually well visible when looking at the matches, it is just the neck of the giraffe. However it seems that this direction is quite seldom used for reflections.

Change the position of just *one* piece so as to leave the giraffe exactly as it was, except possibly for a rotation or reflection of the original figure.

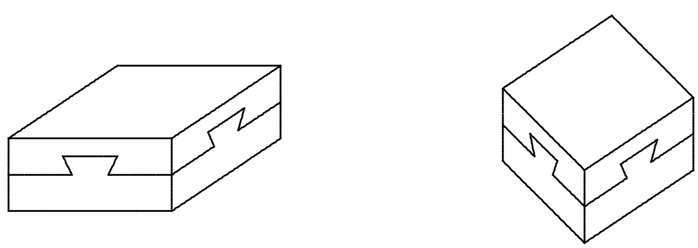
Let me come back to the E-puzzle. On a first thought, it may seem to be impossible to arrange the letter E using the three given pieces. After all, such an E consists of one vertical as well as three horizontal beams. The next section will show some examples of unorthodox strategies, and, when finishing the reading of the paper, it should be clear to the reader in which way the E-puzzle has to be solved.

3. Impossible dovetails

3.1. Dovetails are interlocking joints used in wood-works in order to catenate wooden beams. One uses trapezoidal shapes in order to provide a secure bond. Note that the two wooden pieces are joined together by sliding in the direction of the dovetail.

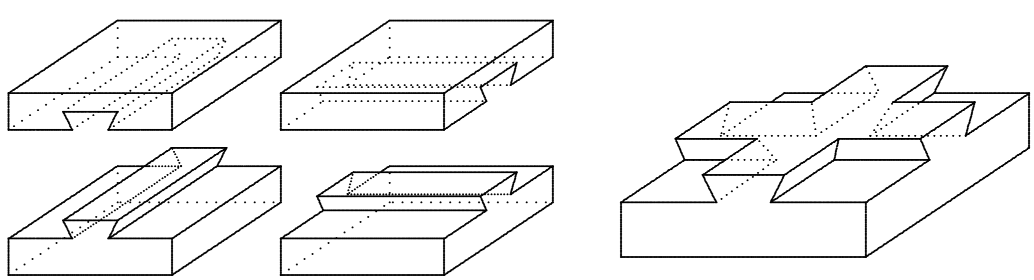


Here are two pictures showing *impossible* dovetails: an object made of two pieces of wood, with square footprint, and with identical view from all four sides: always, one sees the end of a dovetail.

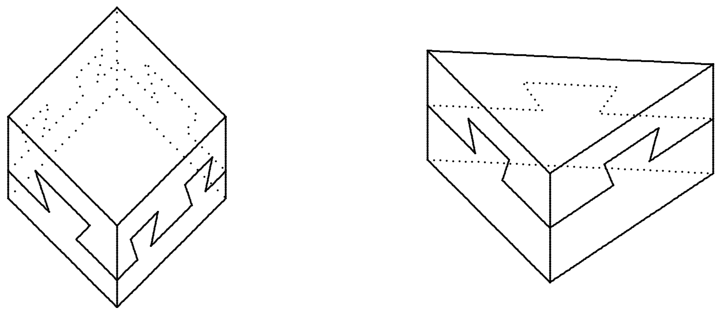


Can such a double-dovetail be realized? To be precise, it should be made from two pieces of wood, without any excavation, and made without any trick (such as using moistening and damping, or using pressure) - thus we are looking for a solution with a genuine geometrical construction.

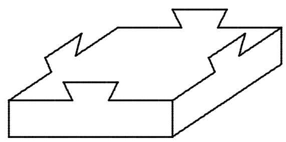
One is inclined to think that one has to deal with two orthogonal dovetails which are parallel to the side walls, but they would combine to form a cross as seen on the right:



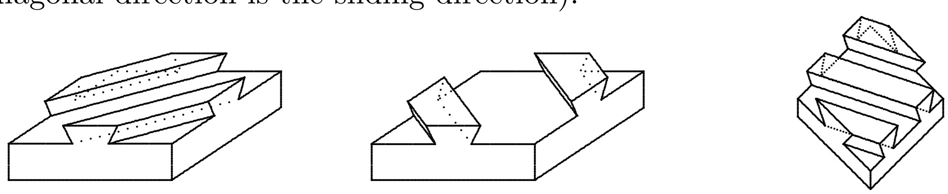
but that's impossible! Before we discuss a solution, let us present the outside view of two similar objects, for the left one the footprint is again a square, for the right one, an equilateral triangle.



Let us return to the first double dovetail. Try to look unbiased at the sidewalls:

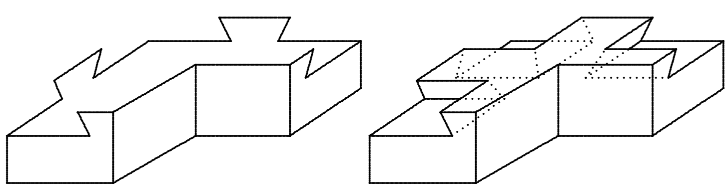


why should the dovetails be parallel to the sidewalls? There is a very simple solution (see the following pictures left and in the middle), using dovetails parallel to one diagonal (and then this diagonal direction is the sliding direction):

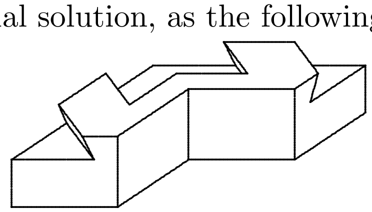


On the right, we see that in the same way we also get a solution for the second object with square footprint.

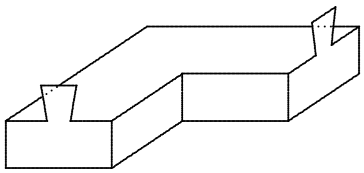
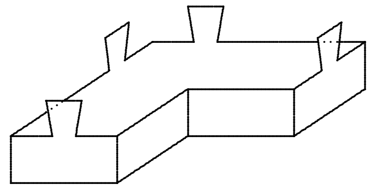
Let us discuss some variants (developed by Robert Sandfield; he made a lot of experiments with dovetails, see [Sa] and also [V]), three of them being L-shaped. The following picture on the left shows (part of) the sidewalls of the first variant. It suggest that we really deal with an "impossible" dovetail, (using two orthogonal dovetails which are parallel to the side walls, and which combine to form a cross), with the right forward corner removed in order to see what is happening inside:



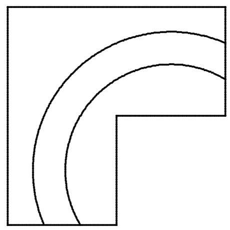
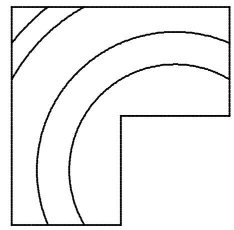
But again, there is a quite trivial solution, as the following picture shows:



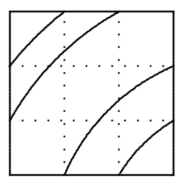
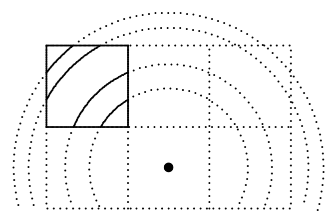
Here are two other L-shaped objects, again we show (part of) the sidewalls:

Here one needs to use circular dovetails:

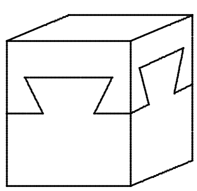
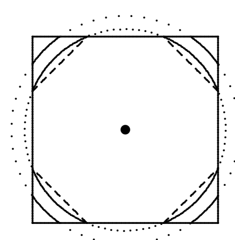



in order to allow a proper sliding, it is important that one deals with concentric circles! Let us provide some details for one of the objects:

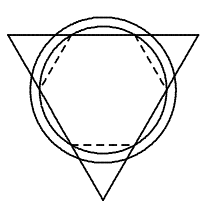
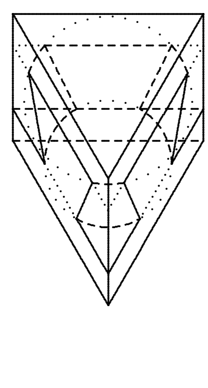



Let us return to the first “impossible dovetail” problem considered at the beginning of this section, but use the construction methods just discussed. We see that the previous picture in the middle yields a solution of the starting problem, now using circular dovetails instead of straight ones.

But there is also a deviating solution, using a central circular plug as shown below: rotation by 45° will be required in order to unplug the two wooden pieces:

But in this way, we also find a solution of the dovetail problem with triangular footprint:

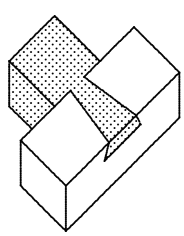
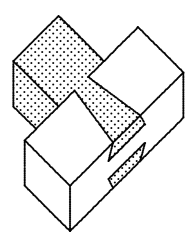
3.2. What do we learn when we are engaged in dealing with such objects?

First of all, they provide a proper training in three-dimensional vision. This again is a very important task of school education, a task which traditionally was considered as belonging to the mathematical education, but unfortunately has been reduced in Germany and many other countries in recent years. And clearly such objects have also an aesthetic flavor, one may consider them when one develops coordinate geometry, and students should be encouraged to produce corresponding drawings both by hand as well as using CAD-programs.

Second, if we analyze the problem of producing circular dovetails, one sees immediately that they are given by rotating a line L around another line L' which is skew but not perpendicular to it, so that one obtains part of a hyperboloid of one sheet (known for example from the cooling towers of power plants). Thus, we are in the realm of a part of mathematics, the quadratic surfaces, which in my opinion should belong to the general knowledge provided by secondary education.

And third: One should realize that the impossible-dovetail problems fit neatly into a set of problems of daily life which do not seem to be discussed properly otherwise: That the outside of an three-dimensional object is given and that one wants to obtain information about the unseen interior, or also a possible construction procedure. The simplest way would be using a saw and in this way to take apart the object. But even if the use of a saw would be allowed, some caution may be necessary: too many saw cuts could lead to a complete destruction of the object, so that the internal structure becomes irrecognizable. Thus, even with a saw, one may first want to discus the possible interior structure in order to guess a convenient cut. Note that lots of practitioners are confronted daily with this problem: for example, any physician. And a physician may not even use a saw at all, but definitely not too often.

3.3. Let me close this section with another two pictures of impossible dovetails. They are taken from the books *Puzzles in Wood* and *Wonders in Wood* by E.M.Wyatt [W], which provide elaborate construction manuals for wooden puzzles.

4. Further Topics.

According to the title, the aim of this report is to focus the attention to the impossible. Whereas in the last section we looked at constructions which are labeled to be impossible, but can be achieved quite easily, the first two sections put the emphasis on really impossible things, and the way to provide corresponding proofs.

But there are many other concepts of impossibility used in daily life, for example in order to stress that something seems to contradict physical, chemical or biological laws, or that a technical solution seems to be quite unlikely. There is the famous Coca-Cola bottle penetrated by a wooden arrow, or a liquor bottle with a large pear inside, and one wonders how this was achieved (one may use soft wood which is not too difficult to compress, and the pear just may grow inside the bottle). Some effects of this kind should be mentioned:

4.1. The time reversal: Look at a movie, say the destruction of a chimney, but let it run backwards. Could such a sequence of pictures be directly realized? There are a huge number of puzzles which remind on this scene: they are easy to dissolve, but difficult to reassemble, or, conversely, difficult to put apart, but easy to put together (for example bullets which vanish in a labyrinth and which one tries in vain to get out).

4.2. The 3-dimensional perception of 2-dimensional drawings. There are different ways to interprete drawings as depicting objects of the 3-dimensional world, and one may combine small pieces which are easily recognized in order to form global pictures which are impossible to realize in the actual world. It is our interpretation of two-dimensional presentations (on a sheet of paper) as depicting three-dimensional objects (often, a four-dimensional realization would provide no problem). There are many drawings of this kind by M. C. Escher and his followers (and before Escher one should name at least Otto Reutersvärd).

4.3. To find a needle in a haystack, to guess 6 numbers in advance in order to win in a lottery: On the one hand, these are tasks, which are difficult to achieve (nearly “impossible”), still every week someone wins in the lottery, thus it **is** possible.

Such features of technical problems, of statistical seldomness have not been touched here. Of course, some of them would really justify a lecture on their own. The puzzles which we have presented were chosen in order to shed light on the notion of impossibility in mathematics. It is important to learn that there are things which are really impossible, and to see proofs of the impossibility. On the other hand, we have seen that there may be related questions which actually have a solution.

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