Universität Bielefeld
Fakultät für Mathematik

**Skriptum zur Vorlesung**

# Partielle Differentialgleichungen 4

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# Contents

# 1 Maximum principles

**1.1 Definition** (Abstract formulation of the Harnack inequality)**.** Let $L$ be an operator acting on functions from $\mathbb{R}^d$ to $\mathbb{R}$. Let $\Omega \subset \mathbb{R}^d$ be open. We say that $L$ *satisfies the Harnack inequality in* $\Omega$, if for every $\Omega_0 \subset \Omega$ open and bounded there is a constant $c \geq 1$ such that for every function $u\colon \mathbb{R}^d \to \mathbb{R}$ with $u \geq 0$ in $\mathbb{R}^d$ and $Lu = 0$ in $\Omega$ the following holds:

$$\sup_{\Omega_0} u \leq c \cdot \inf_{\Omega_0} u.$$

**Remark.**

- *„$Lu = 0$ in $\Omega$" needs to make sense, i.e. a notion if what a solution is need with some regularity of $u$.*

- *In the case $L = -\Delta$, $\Omega = B_R(x_0)$, $\Omega_0 = B_r(x_0)$ for some $x_o \in \mathbb{R}^d$ and $0 < r < R$ one can compute $c$ explicitely. This calculation is easy if one uses the Poisson kernel representation (c.f. Harnack 1981 for d = 2). In the case of local operators (such as $-\Delta$), one only needs $u\colon \Omega \to \mathbb{R}$.*

- *Def. 1.1 can be used also for nonlocal operators: For $s \in (0,1)$ and $v \in C_c^\infty(\mathbb{R}^d)$ define*

$$(-\Delta)^s v(x) = \frac{c(d,s)}{2} \int_{\mathbb{R}^d} (v(x+h) - 2v(x) + v(x-h)) |h|^{-d-2s} \,\mathrm{d}h.$$

*Note that $\widehat{-\Delta u(\xi)} = |\xi|^{2s} \hat{u}(\xi)$ and*

$$c(d,s) = \int_{\mathbb{R}^d} \frac{1-\cos(h_1)}{|h|^{d+2s}} \,\mathrm{d}h \overset{!}{=} \frac{2^{2s-1}}{\pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d+2s}{2}\right)}{|\Gamma(-s)|} \asymp s(1-s).$$

**1.2 Theorem.** *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Let $L$ be an operator acting on functions from $\mathbb{R}^d$ to $\mathbb{R}$ satisfying that for every $c \in \mathbb{R}$ and every function $v$ with $L(v) = 0$ in $\Omega$ one knows $L(c - v) = 0$ in $\Omega$. (Think of $L$ being a linear differential operator without zero order terms.) Let $L$ satisfy the Harnack inequality in $\Omega$ and let $u\colon \mathbb{R}^d \to \mathbb{R}$ satisfy $Lu = 0$ in $\Omega$ and be continuous in $\Omega$.*
*Then either $u$ is constant in $\Omega$ or does not attain its overall maximum in $\Omega$.*

*Proof.* Assume $u$ attains its overall maximum in $\Omega$, i.e. there is $x_0 \in \mathbb{R}^d$ with $u(x_o) \geq u(x)$ for every $x \in \mathbb{R}^d$. Set $M = u(x_0)$, $A = \{x \in \Omega \,|\, u(x) = M\}$, $v(x) = M - u(x)$ for $x \in \mathbb{R}^d$. Then $v \geq 0$ in $\mathbb{R}^d$. Moreover $Lv = 0$ in $\Omega$. Choose $\varrho > 0$ sufficient small enough that $B_\varrho(x_0) \subset \Omega$. Then by Harnack inequality $v = 0$ on $B_\varrho(x_0)$, because $\sup_{B_\varrho(x_0)} v \leq c \cdot \inf_{B_\varrho(x_0)} v = 0$. Thus $B_\varrho(x_0) \subset A$. So $A$ is open in $\Omega$. But $A$ is also closed in $\Omega$. Hence $A \in \{\Omega, \emptyset\}$ and thus $u$ is continuous in $\Omega$.

□

Let us collect some well known results about local elliptic operators. Consider $a_{ij}$, $b_i$, $c \in C(\Omega)$ and $a_{ji} = a_{ij}$ for $i, j \in \{1, \ldots, d\}$ and some $\Omega \subset \mathbb{R}^d$ open. In the sequel we study the operator

$$Lu = -a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu \quad \text{for } u \in C^2(\Omega).$$

**1.3 Definition.** $L$ is called an *elliptic operator* if, at every point $x \in \Omega$, the matrix $(a_{ij}(x))_{i,j=1,\ldots,d}$ is positive definit, i.e.

$$\forall\, \xi \in \mathbb{R}^d\colon \sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\xi_j \geq \lambda\,|\xi|^2$$

where $\lambda$ could depend on $x$. $L$ is called *uniformly elliptic* if there is $\lambda > 0$ such that

$$\forall\, x \in \Omega\,\forall\, \xi \in \mathbb{R}^d\colon \sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\xi_j \geq \lambda\,|\xi|^2.$$

**1.4 Proposition.** *Assume $L$ is elliptic with $c \geq 0$ in $\Omega$. Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy $Lu < 0$ in $\Omega$. If $\max_{\overline{\Omega}} u \geq 0$, then $u$ attains its maximum in $\partial\Omega$.*

*Proof.* Assume $x_0 \in \Omega$ satisfies $u(x_0) \geq u(x)$ for every $x \in \Omega$ and $u(x_0) \geq 0$. We know $\nabla u(x_0) = 0$ and $D^2 u(x_0) \leq 0$. Since $(a_{ij}(x_0)) \geq 0$ and $a_{ij} = a_{ji}$, by diagonalisation

$$\sum_{i,j=1}^{d} a_{ij}(x_0)\,\partial_i\partial_j u(x_0) \leq 0.$$

This contradicts $Lu < 0$ in $\Omega$.

□

**1.5 Theorem.** *Assume $L$ is uniformly elliptic with $c \geq 0$ in $\Omega$. Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy $Lu \leq 0$. If $\max_{\overline{\Omega}} u \geq 0$, then $u$ attains its maximum on $\partial\Omega$.*

*Proof.* Let $\varepsilon > 0$ and $\alpha > 0$. Set $v(x) = u(x) + \varepsilon e^{\alpha x_1}$. Then

$$Lv = Lu + \varepsilon e^{\alpha x_1}\left(-a_{11}\alpha^2 + b_1\alpha + c\right).$$

Choose $\alpha > 0$ sufficient large such that

$$-a_{11}(x)\alpha^2 + b_1(x)\alpha + c(x) < 0 \quad \text{for every } x \in \Omega.$$

Thus $Lv < 0$ in $\Omega$. Proposition 1.4 implies

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} v^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon \sup_{x \in \partial\Omega} e^{\alpha x_1}.$$

Choose $\varepsilon$ small, we proved the assertion.

□

**Remark** (on Theorem 1.5). *If $c \equiv 0$ then one can drop the assumption* $\max_{\overline{\Omega}} u \geq 0$.

**Example.** Let $u(x) = \sin(x)\sin(y)$. Then

- $\Delta u - 2u = 0$ in $\Omega = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0 < x < \pi, 0 < y < \pi\right\}$.
- $(-2\sin(x)\sin(y)) - 2\sin(x)\sin(y) = 0$.

**1.6 Theorem** (Strong maximum principle, Hopf maximum principle)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and connected and $L$ be uniformely elliptic in $\Omega$ with $c \geq 0$. Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy $Lu \leq 0$ in $\Omega$. Assume* $\max_{\overline{\Omega}} u \geq 0$. *Then $u$ is constant or does not attain its maximum in $\Omega$.*

If $c \equiv 0$ then one can drop the assumption $\max_{\overline{\Omega}} u \geq 0$. The proof is based on the following auxiliary result, the so called *Hopf lemma*:

**1.7 Lemma** (Hopf Lemma)**.** *Let $B \subset \mathbb{R}^d$ be a ball and $x_0 \in \partial B$. Assume $L$ is uniformly elliptic in $B$ with $c \geq 0$. Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy $Lu \leq 0$ in $B$, $u(x) < u(x_0)$ for all $x \in B$ and $u(x_0) > 0$. Then for every $\xi \in \mathbb{R}^d$ with $\langle \xi, \nu(x_0)\rangle > 0$, where $\nu(x_0)$ is the outer normal to $B$ at $x_0$, we have*

$$\liminf_{t \to 0} \frac{u(x_0) - u(x_0 - t\xi)}{t} > 0.$$

*Proof of Theorem 1.6.* Set $M = \max_{\overline{\Omega}} u$ and $A = \{x \in \Omega \,|\, u(x) = M\}$. $A$ is closed, since $u$ is continuous. We aim to show $A \in \{\emptyset, \Omega\}$. Assume $A \neq \emptyset$, $A \neq \Omega$. Note that $\Omega \backslash A \subset \Omega$

is open and $\partial(\Omega \setminus A) \neq \emptyset$. It is possible to find a ball $B \subset \Omega \setminus A$ such that $\partial B \cap A \neq \emptyset$. Fix $x_0 \in \partial B \cap A$. We know $Lu \leq 0$ in $B$, $u(x) < u(x_0)$ for $x \in B$ and $u(x_0) = M \geq 0$. Lemma 1.7 implies $\frac{\partial u}{\partial \xi}(x_0) > 0$ for some $\xi \in \mathbb{R}^d$ which contradicts $\nabla u(x_0) = 0$.

□

**Remark.** *Lemma 1.7 and Theorem 1.6 are interesting, i.e. no nice proofs are known, for nonlocal operators such as $(-\Delta)^s$ for $s \in (0,1)$.*

**Remark.** *Theorem 1.6 is almost trivial for a probabilist. $Lu \leq 0$ in $\Omega$ means*

$$\forall x \in \Omega \colon u(x) \leq \mathbb{E}^x\left(u\left(X_{\tau_\Omega}\right)\right),$$

*where the stochastic process $X = (X_t)_{t \geq 0}$ is generated by $L$ and $\tau_\Omega = \inf\{t \geq 0 \,|\, X_t \notin \Omega\}$.*

**1.8 Corollary** (Comparison principle)**.** *Let $\Omega$ and $L$ be as in Theorem 1.6. Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy $Lu \leq 0$ in $\Omega$ and $u \leq 0$ on $\partial\Omega$. Then either $u \equiv 0$ or $u < 0$ in $\Omega$. In particular $u \leq 0$ on $\overline{\Omega}$.*

*Proof.* Theorem 1.6. □

**Remark.** *Corollary 1.8 uses the assumption $c \geq 0$ in $\Omega$, but this assumption can be dropped without affecting the assertion.*
*Let $x_0 \in \Omega$ with $u(x_0) \geq u(x)$ for all $x \in \Omega$. Then for all $x \in \Omega$*

$$-a_{ij}\partial_{ij}u(x) + b_i(x)\partial_i u(x) + c^+(x)u(x) \underbrace{-c^-(x)u(x) + c^-(x)u(x)}_{=0} \leq Lu(x) \leq 0.$$

*Now we can apply Theorem 1.6 to $\tilde{L} = -a_{ij}\partial_i\partial_j + b_i\partial_i + c^+$.*

Set

$$\Lambda = \max_{i,j} \max_{\overline{\Omega}} |a_{ij}(x)| + \max_i \max_{x \in \overline{\Omega}} |b_i(x)|.$$

**1.9 Theorem.** *Assume $\Omega \subset \mathbb{R}^d$ is open, bounded and connected. Assume $L$ is uniformly elliptic in $\Omega$ with $c \geq 0$. Assume $f \in C\left(\overline{\Omega}\right)$, $\varphi \in C(\partial\Omega)$ and $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ such that*

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

*Then*

$$|u(x)| \leq \max_{\partial\Omega} |\varphi| + c_0 \max_{\overline{\Omega}} |f|,$$

*where* $c_0 = c_0(\lambda, \Lambda, \operatorname{diam}(\Omega)) \geq 1$.

*Proof.* We search for a function $v\colon \Omega \to \mathbb{R}$ satisfying $-v \leq u \leq v$ in $\Omega$, i.e. we want $u - v \leq 0$ in $\Omega$ and $-u - v \leq 0$ in $\Omega$. This will follow from the comparison principle, Corollary 1.8, once we know

$$\begin{aligned} u - v &\leq 0 \quad \text{on } \partial\Omega, \\ L(u - v) &\leq 0 \quad \text{in } \Omega, \\ -u - v &\leq 0 \quad \text{on } \partial\Omega, \\ L(-u - v) &\leq 0 \quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

W.l.o.g. we assume $\Omega \subset \left\{x \in \mathbb{R}^d \,\middle|\, 0 < x_1 < l\right\}$ for some $l > 0$. Set $\bar{f} = \max_{\overline{\Omega}} |f|$, $\bar{\varphi} = \max_{\overline{\Omega}} |\varphi|$. For $\alpha > 0$ set

$$v(x) = \bar{\varphi} + \left(e^{\alpha l} - e^{\alpha x_1}\right) \bar{f} \geq 0.$$

We need to check (1.1),

$$\partial_i \partial_j v = \partial_i \left(-\bar{f} \alpha e^{\alpha x_1} \partial_{j1}\right) = -\bar{f} \alpha^2 \partial_{j1} \partial_{i1} e^{\alpha x_1},$$

$$a_{ij} \partial_i \partial_j v = -a_{11} \alpha^2 \bar{f} e^{\alpha x_1},$$

$$\begin{aligned} Lv(x) &= a_{11} \alpha^2 \bar{f} e^{\alpha x_1} - b_1 \bar{f} \alpha e^{\alpha x_1} + c(x) v(x) \\ &\geq e^{\alpha x_1} \bar{f} \left(\lambda \alpha^2 - \alpha b_1\right) \geq e^{\alpha x_1} \bar{f} \left(\alpha^2 \lambda - \alpha \Lambda\right) \\ &\geq \bar{f} \quad \text{for } \alpha \geq \alpha_0(\lambda, \Lambda). \end{aligned}$$

Thus (1.1) holds. Similary one proves the assertion on $-u - v$. Application of Corollary 1.8 leads to

$$|u(x)| \leq \bar{\varphi} + \left(e^{\alpha l} - e^{\alpha x_1}\right) \bar{f} \leq \bar{\varphi} + c_0 f.$$

□

Next we will prove the Alexndrov-Bakelmann-Pucci maximum principle for linear partial differential equations. We assume $\Omega \subset \mathbb{R}^d$ to be open, bounded and connected. The main observation is

**1.10 Proposition.** *Let* $u \in C^2(\Omega) \cap C(\overline{\Omega})$*. Then*

$$\sup_{\Omega} u \leq \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{|B_1|^{\frac{1}{d}}} \left( \int_{G^+(u,\Omega)} \left|\det D^2 u\right| \right)^{\frac{1}{d}},$$

*where* $G^+(u,\Omega)$ *is the contact set of* $u$ *in* $\Omega$*.*

**1.11 Definition.** Let $u \in C(\Omega)$ and $y \in \Omega$. We define

$$\mathcal{X}(y) = \mathcal{X}(y;u,\Omega) = \left\{ p \in \mathbb{R}^d \,\middle|\, \forall x \in \Omega \colon u(x) \leq u(y) + \langle p, x-y \rangle \right\},$$

$$\mathcal{X}(u,\Omega) = \bigcup_{y \in \Omega} \mathcal{X}(y;u,\Omega)$$

and the *upper contact set*

$$G^+(u,\Omega) = \left\{ y \in \Omega \,\middle|\, \exists p \in \mathbb{R}^d \, \forall x \in \Omega \colon u(x) \leq u(y) + \langle p, x-y \rangle \right\}.$$

**Remark.** *If* $u$ *is differentiable at* $x \in \Omega$ *and* $p \in \mathcal{X}(x)$*, then* $p = \nabla u(x)$*. If* $u \in C^1(\Omega)$ *then*

$$\begin{aligned} \mathcal{X}(u,\Omega) &= \nabla u\left(G^+(u,\Omega)\right) \\ &= \text{Image of } G^+(u,\Omega) \text{ under the function } \nabla u. \end{aligned}$$

**Example.** $\Omega = B_R(x_0) \subset \mathbb{R}^d$, $a > 0$, $u \colon \Omega \to \mathbb{R}^d$ defined by

$$u(x) = a\left(1 - \frac{|x-x_0|}{R}\right) = a - a\frac{|x-x_0|}{R}.$$

Then $G^+(u,\Omega) = \Omega$ and

$$\mathcal{X}(y) = \begin{cases} -\frac{a}{R}, & \text{if } y \neq x_0, \\ B_{\frac{a}{R}}(0), & \text{if } y = x_0. \end{cases}$$

**1.12 Lemma.** *Let* $u \in C^2(\Omega)$*. Then the Hessian* $\left(D^2 u\right)(x)$ *at* $z$ *is negative definit for every* $z \in G^+(u,\Omega)$*.*

*Proof.* Let $z \in G^+(u,\Omega)$. Set $w(x) = u(x) - u(z) - \langle \nabla u(z), x-z \rangle$. Then $w(z) = 0$ and $w(x) \leq 0$ for every $x \in \Omega$. Thus $w$ attains its maximum at $z$, hence $D^2 w(z) \leq 0$. $\square$

**1.13 Lemma.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$|\mathcal{X}(u,\Omega)| \leq \int_{G^+(u,\Omega)} \left|\det D^2 u\right|.$$

*Proof.* Note: If $J$ denotes Jacobian, then

$$\begin{aligned} J_{\nabla u}(x) = D^2 u(x) \quad & \text{for } x \in \Omega, \\ J_{\nabla u}(x) \leq 0 \quad & \text{for } x \in G^+(u,\Omega). \end{aligned}$$

For $\varepsilon > 0$ define $\mathcal{X}_\varepsilon \colon \Omega \to \mathbb{R}^d$, $\mathcal{X}_\varepsilon(x) = \mathcal{X}(x) - \varepsilon x$. Then for $x \in G^+(u,\Omega)$

$$J_{\mathcal{X}_\varepsilon}(x) = \left(D^2 u - \varepsilon I_{d\times d}\right)(x) < 0.$$

By a change of variables we obtain

$$|\mathcal{X}_\varepsilon(\Omega)| = \int_{\mathcal{X}_\varepsilon(\Omega)} 1 = \int_{G^+(u,\Omega)} \left|\det\left(D^2 u - \varepsilon I_{d\times d}\right)(x)\right| \,\mathrm{d}x.$$

Lemma of Fatou.

□

Let us prove Proposition 1.10.

*Proof of Proposition 1.10.* W.l.o.g. we assume $u \leq 0$ on $\partial\Omega$. If not, we consinder $\tilde{u} = u - max_{\partial\Omega} u$. Let $k \colon \Omega \to \mathbb{R}^d$ be such that its graph is a „tent" with apex in $(x_0, u(x_0))$, where $x_0 \in \Omega$ is chosen such that $u$ attains its maximum at $x_0$, and basis $\partial\Omega$. Set $R = \operatorname{diam}(\Omega)$. Consider another similar function $\tilde{k}$, this time describing a tent with apex in $(x_0, u(x_0))$ and basis $\partial B(x_0, R)$. We have

$$\begin{aligned} \mathcal{X}(k,\Omega) &\subset \mathcal{X}(u,\Omega), \\ \mathcal{X}\left(\tilde{k},\Omega\right) &\subset \mathcal{X}(k,\Omega) \end{aligned}$$

and hence $\left|\mathcal{X}\left(\tilde{k},\Omega\right)\right| \leq |\mathcal{X}(u,\Omega)|$. We obtain

$$\left(\frac{u(x_0)}{\operatorname{diam}(\Omega)}\right)^d |B_1| = \left|\mathcal{X}\left(\tilde{k},\Omega\right)\right| \leq \int_{G^+(u,\Omega)} \left|\det D^2 u\right|.$$

□

Assume $a_{ij} \in C(\overline{\Omega})$, $a_{ij} = a_{ji}$, with the property that for each $x \in \Omega$ the matrix $A(x) = (a_{ij}(x)) > 0$. Set $D(x) = \det A(x)$ and $D^*(x) = (\det(A(x)))^{\frac{1}{d}}$.

**1.14 Lemma.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $(a_{ij})$ satisfy the conditions above.. Then*

$$\sup_{\Omega} u \le \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{|B_1|^{\frac{1}{d}}} \left\| \frac{a_{ij}\partial_i\partial_j u}{D^*} \right\|_{L^d(G^+(u,\Omega))}.$$

*Proof.* Set $H(x) = D^2u(x)$. Observe

$$(\det(H)\det(A))^{\frac{1}{d}} \le \frac{\operatorname{Tr}(AH)}{d}.$$

For $x \in G^+(u, \Omega)$

$$\begin{aligned} \left|\det D^2u(x)\right| &= \det H(x) \le \frac{1}{\det(A(x))} \left( \frac{\operatorname{Tr}(A(x)H(x))}{d} \right)^d \\ &= \left(\frac{1}{d}\right)^d \left( \frac{a_{ij}(x)\partial_i\partial_j u(x)}{D^*(x)} \right)^d \end{aligned}$$

□

**1.15 Theorem.** *Let $L$ be a uniformly elliptic operator in a bounded domain. Assume $c \ge 0$ and $\frac{b_i}{D^*} \in L^d(\Omega)$. Assume $f\colon \Omega \to \mathbb{R}$ satisfies $\frac{f}{D^*} \in L^d(\Omega)$. If $u \in C^2(\Omega) \cap C(\Omega)$ satisfies*

$$Lu \le f \quad \text{in } \Omega,$$

*then*

$$\sup_{\Omega} u \le \max_{\partial\Omega} u + c_0 \left\| \frac{f}{D^*} \right\|_{L^d(\Omega)}$$

*where $c_0$ is some positive constant depending on $d$,* $\operatorname{diam}(\Omega)$ *and* $\left\| \frac{b_i}{D^*} \right\|_{L^d(\Omega)}$.

*Proof.* For $c = b_i = 0$, the proof follows from Lemma 1.14. In the general case, one needs more refined tools like the co-area formula.

□

**Remark.**
- *If $L$ is uniformly elliptic, then $D^* \ge \lambda$ for some $\lambda > 0$.*
- *Comparison results follow as usually.*

# 2 Fully nonlinear differential equations

This chapter is based on Fully Nonlinear Elliptic Equations by Luis A. Caffarelli and Xavier Cabré.
History: Crandall, Ishii, P.L. Lions, Jensen around 1980.

## 2.1 Definitions and setup

In the sequel $\Omega$ denotes a bounded domain. We want to study equations of the form

$$F\left(x, D^2 u\left(x\right)\right) = 0$$

where $F\colon \Omega \times S \to \mathbb{R}$ and $S$ is the set of $d \times d$ symmetric matrices. For $M \in S$ we use $\|M\|^2 = \max_{|\xi|=1} \langle M\xi, M\xi\rangle$.

**2.1 Definition.** We say $F\colon \Omega \times S \to \mathbb{R}$ is *uniformly elliptic in* $\Omega$, if for some positive numbers $\lambda \leq \Lambda$ and all $M \in S$, $x \in \Omega$, $N \geq 0$, $N \in S$ we have

$$\lambda\|N\| \leq F\left(x, M+N\right) - F\left(x, M\right) \leq \Lambda\|N\|.$$

**Remark.**

- *$\lambda$ and $\Lambda$ sometimes are called „ellipticity constants".*
- *If $F$ is uniformly elliptic in $\Omega$, then trivially $F\left(x, M\right) \leq F\left(x, M+N\right)$ for all $x \in \Omega$, $M, N \in S$ and $N \geq 0$. In this sense $F$ is monotone with respect to the second argument.*
- *If $F$ is uniformly elliptic in $\Omega$, then it is Lipschitz continuous w.r.t. the second argument uniformly w.r.t. the first argument.*

*Proof.* For $M \in S$ set $\|M\|_F = \sqrt{\operatorname{Tr}\left(M^2\right)}$. Fact: Let $N \in S$. Then there are $N^+$, $N^-$ with $N = N^+ - N^-$, $N^+N^- = 0$ and $N^+, N^- \geq 0$. Thus

$$\|N^+ + N^-\|_F = \sqrt{\operatorname{tr}\left(N^+ + N^-\right)^2} = \sqrt{\operatorname{tr}\left(N^+ - N^-\right)^2} = \|N^+ - N^-\|_F = \|N\|_F.$$

Let $M, N \in S$, $x \in \Omega$. Then

$$\begin{aligned} |F(x, M+N) - F(x, M)| &\leq \Lambda \left(\|N^+\| + \|N^-\|\right) \leq \Lambda \operatorname{Tr}\left(N^+ + N^-\right) \\ &\leq \Lambda\sqrt{d}\|N^+ + N^-\|_F = \Lambda\sqrt{d}\|N\|_F \\ &\leq d\Lambda\|N\|. \end{aligned}$$

□

Exercise: $F\left(x, D^2 u\right) = a_{ij}(x)\,\partial_i\partial_j u(x)$.

**2.2 Definition.** Let $f \in C(\Omega)$. We consider

$$F\left(x, D^2 u(x)\right) = f(x) \tag{2.1}$$

(i) A function $u \in C(\Omega)$ is a *viscosity subsolution* of equation (2.1) at a point $x \in \Omega$, if for every function $\varphi \in C^2(\Omega)$ with the property that $u - \varphi$ has a local maximum at $x$:

$$F\left(x, D^2\varphi(x)\right) \geq f(x).$$

If $u \in C^2(\Omega)$ is viscosity subsolution of equation (2.1) at every $x \in \Omega$, then we say that $u$ is a viscosity subsolution of (2.1) in $\Omega$.

(ii) A function $u \in C(\Omega)$ is a *viscosity supersolution* of equation (2.1) at a point $x \in \Omega$, if for every function $\varphi \in C^2(\Omega)$ with the property that $u - \varphi$ has a local minimum at $x$:

$$F\left(x, D^2\varphi(x)\right) \leq f(x).$$

If $u \in C^2(\Omega)$ is a viscosity supersolution of equation (2.1) at every $x \in \Omega$, then we say that $u$ is a viscosity supersolution of (2.1) in $\Omega$.

(iii) A function $u \in C(\Omega)$ is a *viscosity solution*, if it is a viscosity subsolution and a viscosity supersolution.

**Example.** Assume $u \in C^2(\Omega)$. Then the two following properties are equivalent:

(i) $F\left(x, D^2(u(x))\right) \geq f(x)$ for every $x \in \Omega$.

(ii) $u$ is a viscosity subsolution of (2.1) in $\Omega$.

*Proof.* (i) $\Rightarrow$ (ii): Pick $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ with $(u-\varphi)(x_0) \geq (u-\varphi)(x)$ for every $x \in \Omega$. Thus $D^2(u-\varphi)(x_0) \leq 0$.

$$F\left(x, D^2\varphi\left(x_0\right)\right) \geq F\left(x, D^2 u\left(x_0\right)\right) \geq f\left(x_0\right)$$

by monotonicity of $F$ in the second argument.
(ii) $\Rightarrow$ (i): Follows by choosing $\varphi = u$. Also compare

$$\int_B \nabla u \nabla \varphi = 0 \quad \forall \varphi \in C_c^\infty(B), u \in C^2(B),$$

then

$$\int_B (\Delta u)\varphi = 0 \quad \forall \varphi \in C_c^\infty(B)$$

and hence

$$\Delta u = 0 \quad \text{in } B.$$

□

It does make sense to use the following notation: $u$ is a viscosity subsolution to (2.1) at $x \in \Omega$ if and only if $F\left(x, D^2 u(x)\right) \geq f(x)$. But watch out!
If $u$ is a viscosity subsolution of $F\left(x, D^2 v(x)\right) = 0$ in $\Omega$, then in general, $-u$ is not a viscosity supersolution to $-F\left(x, D^2 v(x)\right) = 0$ in $\Omega$.

*Proof.* $u$ is a viscosity subsolution of $F(\cdot) = 0$ in $\Omega$ if and only if $F\left(x, D^2\varphi(x)\right) \geq 0$ for all $\varphi \in C^2(\Omega)$ with $u - \varphi$ has a local maximum at $x \in \Omega$. But this is *not* equivalent to $-F\left(x, D^2\varphi(x)\right) \leq 0$ for all $\varphi \in C^2(\Omega)$ with $u - \varphi$ has a local minimum at $x \in \Omega$, which is true if and only if $u$ is a viscosity supersolution to $-F(\cdot) = 0$ in $\Omega$.

□

What, in fact, is true?
If $u$ is a viscosity subsolution of (2.1) and $v = -u$, then $v$ is a viscosity supersolution to

$$G\left(x, D^2 v(x)\right) = -f \quad \text{in } \Omega,$$

where $G(x, M) := -F(x, -M)$. If $f$ is uniformly elliptic, so is $G$.
Let us look at a simple example: $\Omega = (-1, 1)$, $u(x) = |x|$,

$$u'' = 0 \quad \text{in } \Omega.$$

Guess: $u$ is a viscosity subsolution. Presumably it is not a viscosity supersolution.
Need: $\varphi''(x) \geq 0$ if $x \in \Omega$, $\varphi \in C^2(\Omega)$, $u - \varphi$ has a local maximum at $x$.

**2.3 Theorem.** *The following statements are equivalent:*

*(i)* *$u$ is a viscosity subsolution of (2.1) in $\Omega$.*

*(ii)* *For every choice of $x \in \Omega$, $\mathcal{U} \subset \Omega$ open with $x \in \mathcal{U}$, $\varphi \in C^2(\mathcal{U})$ with $u \leq \varphi$ in $\mathcal{U}$ and $u(x) = \varphi(x)$:*

$$F\left(x, D^2\varphi(x)\right) \geq f(x).$$

*(iii)* *Same as (ii) with $\varphi \in C^2(\mathcal{U})$ being replaced by a paraboloid $P$ that touches $u$ at $x$ from above.*

Recall: We call $P\colon \mathbb{R}^d \to \mathbb{R}$ a paraboloid, if it has the form

$$P(x) = l_0 + l(x) \pm \frac{M}{2}|x|^2,$$

where $l_0 \in \mathbb{R}$, $l$ is a linear fnction and $M > 0$.

*Proof.* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hols true trivially (since we reduce the set of test functions). We prove (iii) $\Rightarrow$ (i): Let $\varphi \in C^2(\Omega)$, $x_0 \in \Omega$ such that $u - \varphi$ has a local maximum at $x_0$. For $\varepsilon > 0$, set

$$P_\varepsilon(x) = u(x_0) + D\varphi(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^t D^2\varphi(x_0)(x - x_0) + \frac{\varepsilon}{2}|x - x_0|^2.$$

Note: $P_\varepsilon(x_0) = u(x_0)$ and $P(x) \geq u(x)$ locally at $x_0$ by Taylor's formula and

$$u(x_0) - \varphi(x_0) \geq u(x) - \varphi(x) \;\Leftrightarrow\; u(x_0) - u(x) \geq \varphi(x_0) - \varphi(x).$$

Now: $F\left(x_0, D^2\varphi(x_0) + \varepsilon I\right) \geq f(x_0)$ because of (iii). Since $F$ is continuous in the second argument, we conclude

$$F\left(x_0, D^2\varphi(x_0)\right) \geq f(x_0).$$

□

Now let us finish the example: Set $u(x) = |x|$ for $x \in (-1, 1)$.

(1) $u$ is a viscosity subsolution to $u'' = 0$ in $\Omega$.

*Proof.* Need to check $\varphi''(x) \geq 0$ for all quadratic functions that touch $u$ at $x$ from above $\Rightarrow$ There is none!

Alternative: Work with 2.3 (ii).

□

(2) $u$ is not a viscosity supersolution to $u'' = 0$ in $\Omega$.

*Proof.* Need to check $\varphi''(x) \leq 0$ for all quadratic functions that touch $u$ from below. False at $x = 0$, choose $\varphi(x) = x^2$.

□

**2.4 Definition.** A *paraboloid* is a polynomial in the variables $x_1, \ldots, x_d$ of order 2, i.e. a function $P$ of the form

$$P(x) = L(x) + x^2 Ax,$$

where $L$ is an affine function and $A$ is symmetric.

**Example.** $P(x, y) = x^2 + y^2 = |(x, y)|^2$,
$P(x, y) = x^2 + (2y)^2$,
$P(x, y) = x^2 - y^2$.

**Remark.** *In proposition 2.3 we can weaken the assumption $u \leq \varphi$ by $u < \varphi$. 2.3 (ii) follows from this.*

*Proof.* We have $\varphi$ with $\varphi(x) = u(x)$, $u(y) \leq (\cdot)$ in $\mathcal{U}$. Now define

$$\varphi'(y) = \varphi(y) + \varepsilon |y - x|^4 1_{B_\varrho}(y - x)$$

for some $\varepsilon > 0$, $\varrho > 0$. Then for $y \in B_\varrho(x) \setminus \{x\}$ we have

$$\varphi'(x) = \varphi(x), \quad \varphi'(y) > \varphi(y).$$

Here we need to make $\varphi' \in C^2(\mathcal{U})$ by smoothing $\varphi'$ outside $B_\varrho(x)$.

□

**Example.** $u(x) = |x|$. Want $-u'' \leq 0$, $u$ is viscosity subsolution in $(-1, 1)$.

**2.5 Theorem.** *If $u$ and $v$ are viscosity subsolutions of (2.1) in $\Omega$, so is $w = \max(u, v)$.*

*Proof.* Let $x_0 \in \Omega$, $\mathcal{U} \subset \Omega$ open, $\varphi \in C^2(\Omega)$ such that

$$(w - \varphi)(x_0) \geq (w - \varphi)(x) \quad \text{for } x \in \mathcal{U}.$$

W.l.o.g. we assume $w(x_0) = u(x_0)$. Then $u(x_0) - \varphi(x_0) = w(x_0) - \varphi(x_0) \geq u(x) - \varphi(x)$ for $x \in \mathcal{U}$. So $u - \varphi$ has a local maximum at $x_0$. Thus

$$F\left(x_0, D^2\varphi(x_0)\right) \geq f(x_0).$$

□

**2.6 Theorem.** *Let $\mathcal{F}$ be a family of viscosity subsolutions (supersolutions) to (2.1) in $\Omega$. Set*

$$w(x) = \sup_{u \in \mathcal{F}} u(x) \quad \Big( w(x) = \inf_{u \in \mathcal{F}} u(x) \Big)$$

*and assume that $w$ is lower-semicontinuous (upper-semicontinuous). Then $w$ is a viscosity subsolution (supersolution) in $\Omega$.*

*Proof.* Let $x_0 \in V \Subset U \subset \Omega$ for $V, U$ open. Let $\varphi \in C^2(U)$ such that

$$(w - \varphi)(x_0) > (w - \varphi)(x) \quad \text{for } x \in U.$$

Consider $(u_n)_n \subset \mathcal{F}$ such that

$$u_n(x_0) > w(x_0) - \frac{1}{n} \quad \text{for } n \in \mathbb{N}.$$

Let $(x_n)_n$ be any sequence in $\overline{V}$ such that $u_n - \varphi$ attains its maximum over $\overline{V}$ at $x_n$, i.e.

$$u_n(x_n) - \varphi(x_n) \geq u_n(x) - \varphi(x) \quad \text{for } x \in V. \tag{2.2}$$

For some $\bar{x} \in \overline{V}$ and a subsequence $(x_{n_k})_k$ we have $x_{n_k} \to \bar{x}$ for $k \to \infty$.

$$\begin{aligned}(2.2) \quad \Rightarrow \quad & w(x_n) - \varphi(x_n) \geq u_n(x_n) - \varphi(x_n) \geq u_n(x_0) - \varphi(x_0) \\ & > w(x_0) - \varphi(x_0) - \frac{1}{n}.\end{aligned}$$

Taking $\liminf$ we obtain

$$w(\bar{x}) - \varphi(\bar{x}) \geq w(x_0) - \varphi(x_0)$$

and therefore $\bar{x} = x_0$. From (2.2) we know $F\left(x_n, D^2\varphi(x_n)\right) \geq f(x_n)$. Thus

$$F\left(x_0, D^2\varphi(x_0)\right) \geq f(x_0),$$

where we applied the continuity of $F$.

□

**2.7 Theorem.** *Let $(F_k)$ be a sequence of uniformly elliptic operators with elliptic con-*

*stants $\lambda < \Lambda$. Let $(u_k)$ be a sequence of $C^2(\Omega)$-functions such that for each $k$*

$$F_k\left(x, D^2 u_k(x)\right) \geq f(x)$$

*in the viscosity sense in $\Omega$. Assume $F_k$ converges uniformly on compact subsets of $\Omega \times S$ to $F$ and that $u_k$ converges uniformly on compact subsets of $\Omega$ to $u$. Then*

$$F\left(x, D^2 u(x)\right) \geq f(x)$$

*in the viscosity sense in $\Omega$.*

The most important example is

**Example.** $A\colon \Omega \to S$, $A(x) = \left(a_{ij}(x)\right)_{ij}$ symmetric with: There are $0 < \lambda < \Lambda$ with

$$\forall\, x \in \Omega\colon\ \lambda\, |\xi|^2 \leq \sum a_{ij}(x)\, \xi_i \xi_j \leq \Lambda\, |\xi|^2 .$$

Define $F(x, M) = \operatorname{tr}(A(x)\, M)$. Then $F$ is uniformly elliptic in $\Omega$.

*Proof.* We need to prove for $N, M \in S$, $N \geq 0$, $x \in \Omega$

$$\lambda \|N\| \leq \underbrace{F(x, M+N) - F(M)}_{=\operatorname{tr}(A(x)N)} \leq \Lambda \|N\|.$$

□

Key observation:

**2.8 Lemma.** *If $A$ and $N$ are symmetric, $N \geq 0$, then*

$$\lambda_{min}(A) \operatorname{tr}(N) \leq \operatorname{tr}(AN) \leq \lambda_{max}(A) \operatorname{tr}(N) .$$

*Proof.* Choose a matrix $B$ such that $B^{-1}AB = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$. Then for $\tilde{N} = B^{-1}NB$ we know

$$\operatorname{tr}\left(\tilde{N}\right) = \operatorname{tr}\left(B^{-1}NB\right) = \operatorname{tr}\left(BB^{-1}N\right) = \operatorname{tr}(N) \geq 0.$$

Now:

$$\begin{aligned}
\operatorname{tr}(AN) &= \operatorname{tr}\left(B^{-1}ANB\right) = \operatorname{tr}\left(B^{-1}ABB^{-1}NB\right) \\
&= \operatorname{tr}\left(\operatorname{diag}(\lambda_1,\ldots,\lambda_d)\,\tilde{N}\right) \\
&= \begin{pmatrix} \lambda_1 \tilde{n_{11}} & & \sim \\ & \ddots & \\ \sim & & \lambda_d \tilde{n_{dd}} \end{pmatrix} \\
&= \sum_{i=1}^{d} \lambda_i \tilde{n_{ii}} \begin{cases} \leq & \lambda_{\max} \sum_{i=1}^{d} \tilde{n_{ii}} \\ \geq & \lambda_{\min} \sum_{i=1}^{d} \tilde{n_{ii}} \end{cases}
\end{aligned}$$

($\tilde{n_{ii}} \geq 0$ for all $i$)

□

## 2.2 Extremal Operators and the class $S$

Very often, in PDE theory, one considers elliptic differential operators of second order of the form $a_{ij}\partial_i\partial_j$ for coefficient functions $a_{ij}\colon \Omega \to \mathbb{R}$ satisfying

$$\lambda\,|\xi|^2 \leq \sum a_{ij}(x)\,\xi_i\xi_j \leq \Lambda\,|\xi|^2$$

for some $0 < \lambda \leq \Lambda$ and all $x \in \Omega$. Several results (i.e. the constants therein) depend on nothing but $\lambda$, $\Lambda$ and $d$. Now, we introduce the class of „all solutions to all elliptic equations" for fixed $\lambda \leq \Lambda$.
For $M \in S$ set

$$\begin{aligned}
\mathcal{M}^-(M) &= \mathcal{M}(M,\lambda,\Lambda) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i, \\
\mathcal{M}^+(M) &= \mathcal{M}(M,\lambda,\Lambda) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,
\end{aligned}$$

where $e_i$ are the eigenvalues of $M$. We watn to study and use the extremal operators

$$\mathcal{M}^-\left(D^2u\right) \quad \text{and} \quad \mathcal{M}^+\left(D^2u\right).$$

The key is that for several results concerning solutions to $F\left(x, D^2u(x)\right) = f(x)$ the only thing that matters are properties of functions $u$ satisfying

$$\begin{aligned}\mathcal{M}^{+}\left(D^{2}u\left(x\right)\right) &\geq f\left(x\right) \quad \text{and}\\ \mathcal{M}^{-}\left(D^{2}u\left(x\right)\right) &\leq f\left(x\right)\end{aligned}$$

in the viscosity sense.
Let us give an alternative to how one can compute $\mathcal{M}^{+}\left(M\right)$ and $\mathcal{M}^{-}\left(M\right)$. Denote by $\mathcal{A}\left(\lambda,\Lambda\right)$ the set of all symmetric $d\times d$-matrices with eigenvalues in $[\lambda,\Lambda]$. Let $M\in S$ have eigenvalues $e_1,\ldots,e_d$. Then

$$\begin{aligned}\min_{A\in\mathcal{A}(\lambda,\Lambda)}\left(\operatorname{tr}\left(AM\right)\right) &= \lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i=\mathcal{M}^{-}\left(M\right),\\ \max_{A\in\mathcal{A}(\lambda,\Lambda)}\left(\operatorname{tr}\left(AM\right)\right) &= \Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i=\mathcal{M}^{+}\left(M\right).\end{aligned}$$

**2.9 Lemma.** *Let $M,N\in S$. The operators $\mathcal{M}^{+}$, $\mathcal{M}^{-}$ have the following properties:*

*(1)* $\mathcal{M}^{-}\left(M\right)\leq\mathcal{M}^{+}\left(M\right)$.

*(2) Given* $\lambda'\leq\lambda\leq\Lambda\leq\Lambda'$, *then* $\mathcal{M}^{-}\left(M,\lambda',\Lambda'\right)\leq\mathcal{M}^{-}\left(M,\lambda,\Lambda\right)$ *and* $\mathcal{M}^{+}\left(M,\lambda',\Lambda'\right)\geq\mathcal{M}^{+}\left(M,\lambda,\Lambda\right)$.

*(3)* $\mathcal{M}^{-}\left(M,\lambda,\Lambda\right)=-\mathcal{M}^{+}\left(-M,\lambda,\Lambda\right)$.

*(4)* $\mathcal{M}^{\pm}\left(\alpha M\right)=\alpha\mathcal{M}^{\pm}\left(M\right)$ *for* $\alpha\geq 0$.

*(5)* $\mathcal{M}^{+}\left(M\right)+\mathcal{M}^{-}\left(N\right)\leq\mathcal{M}^{+}\left(M+N\right)\leq\mathcal{M}^{+}\left(M\right)+\mathcal{M}^{+}\left(N\right)$.

*(6)* $\mathcal{M}^{-}\left(M\right)+\mathcal{M}^{-}\left(N\right)\leq\mathcal{M}^{-}\left(M+N\right)\leq\mathcal{M}^{-}\left(M\right)+\mathcal{M}^{+}\left(N\right)$.

*(7)* $N\geq 0$ *implies* $\lambda\|N\|\leq\mathcal{M}^{-}\left(N\right)\leq\mathcal{M}^{+}\left(N\right)\leq d\Lambda\|N\|$.

*(8)* $\mathcal{M}^{-}$ *and* $\mathcal{M}^{+}$ *as operators from* $S\to\mathbb{R}$ *are uniformly elliptic with ellipticity constants* $\lambda$ *and* $d\Lambda$.

Recall:

**2.10 Theorem.** $F\colon\Omega\times S\to\mathbb{R}$ *is uniformly elliptic in* $\Omega$ *if and only if for every* $x\in\Omega$, *every* $M,N\in S$

$$F\left(x,M+N\right)\leq F\left(x,M\right)+\Lambda\|N^{+}\|-\lambda\|N^{-}\|$$

*where* $N^{+},N^{-}\geq 0$, $N^{+}N^{-}=0$ *and* $N=N^{+}-N^{-}$ *(this composition is unique).*

**2.11 Definition.** Given $f \in C(\Omega)$ and $0 < \lambda \leq \Lambda$, we denote by $\underline{S}(\lambda, \Lambda, f)$ or $\underline{S}(f)$ the set of continuous functions $u \colon \Omega \to \mathbb{R}$ such that

$$\mathcal{M}^+\left(D^2 u, \lambda, \Lambda\right) \geq f$$

in the viscosity sense in $\Omega$.
$\overline{S}(\lambda, \Lambda, f)$ or $\overline{S}$ denotes the set of all $u \in C(\Omega)$ such that

$$\mathcal{M}^-\left(D^2 u, \lambda, \Lambda\right) \leq f$$

in the viscosity sense in $\Omega$.
We define $S(\lambda, \Lambda, f)$ or $S(f)$ by

$$S(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f)$$

and furthermore

$$S^*(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, -|f|) \cap \overline{S}(\lambda, \Lambda, |f|).$$

Note:

$$\begin{aligned} S(\lambda, \Lambda, f) &\subset S^*(\lambda, \Lambda, f), \\ S(\lambda, \Lambda, 0) &= S^*(\lambda, \Lambda, 0). \end{aligned}$$

**2.12 Lemma.** *(1) Given $\lambda' \leq \lambda \leq \Lambda \leq \Lambda'$ implies $\underline{S}(\lambda, \Lambda, f) \subset \underline{S}(\lambda', \Lambda', f)$. The same statement holds for $S$, $\overline{S}$ and $S^*$.*

*(2) $u \in \underline{S}(\lambda, \Lambda, f)$ implies $-u \in \overline{S}(\lambda, \Lambda, -f)$.*

*(3) $\alpha > 0$, $r > 0$, $u \in \underline{S}(\lambda, \Lambda, f)$, $v(y) := \alpha u\left(\frac{y}{r}\right)$ for $y \in r\Omega$ implies $v \in \underline{S}\left(\lambda, \Lambda, \frac{\alpha}{r^2} f\left(\frac{y}{r}\right)\right)$.*

*(4) $u \in \underline{S}(\lambda, \Lambda, f)$, $\varphi \in C^2(\Omega)$, $\mathcal{M}^+\left(D^2\varphi(x)\right) \leq g(x)$ for all $x \in \Omega$ implies $u - \varphi \in \underline{S}(\lambda, \Lambda, f - g)$.*

*Proof.* (3): Exercise.
(4): Let $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ be such that $\psi$ touches $u - \varphi$ at $x_0$ from above. Then $\psi + \varphi$ touches $u$ at $x_0$ from above. Thus

$$\begin{aligned} f(x_0) &\leq \mathcal{M}^+\left(D^2(\psi + \varphi)(x_0)\right) = \mathcal{M}^+\left(D^2\psi(x_0) + D^2\varphi(x_0)\right) \\ &\leq \mathcal{M}^+\left(D^2\psi(x_0)\right) + \mathcal{M}^+\left(D^2\varphi(x_0)\right). \end{aligned}$$

So we know

$$\mathcal{M}^+\left(D^2\psi\left(x_0\right)\right) \geq f\left(x_0\right) - \mathcal{M}^+\left(D^2\varphi\left(x_0\right)\right) \geq f\left(x_0\right) - g\left(x_0\right).$$

□

Note that proposition 2.6 and 2.7 imply furthermore

- $u, v \in \underline{S}(f) \quad \Rightarrow \quad \max(u, v) \in \underline{S}(f)$.
- $u \in \underline{S}(f) \quad \Rightarrow \quad u^+ \in \underline{S}(f)$.
- $\underline{S}(f)$, $\overline{S}(f)$ and $S(f)$ are closed under uniform limit on compact sets.

We still have to discuss the relation between solutions of $F\left(x, D^2u\right) = f$ and the corresponding extremal operators.

**2.13 Theorem.** *Let $F\colon \Omega \times S \to \mathbb{R}$ be uniformly elliptic with ellipticity constants $\lambda \leq \Lambda$. Assume $F\left(x, D^2u\left(x\right)\right) \geq f\left(x\right)$ for $x \in \Omega$ in the viscosity sense. Then*

$$u \in \underline{S}\left(\frac{\lambda}{d}, \Lambda, f - F\left(\cdot, 0\right)\right). \tag{2.3}$$

*In fact we know more: For every $\varphi \in C^2\left(\Omega\right)$ we know that*

$$u - \varphi \in \underline{S}\left(\frac{\lambda}{d}, \Lambda, f - F\left(\cdot, D^2\varphi\right)\right). \tag{2.4}$$

*Proof.* Note that (2.4) implies (2.3) by choosing $\varphi = 0$. Let $\varphi \in C^2\left(\Omega\right)$, $x_0 \in \Omega$ and $\psi \in C^2\left(\Omega\right)$ such that $\psi$ touches $u - \varphi$ at $x_0$ from above. Thus $\psi + \varphi$ touches $u$ at $x_0$ from above. Hence

$$\begin{aligned} f\left(x_0\right) &\leq F\left(x_0, D^2\varphi\left(x_0\right) + D^2\psi\left(x_0\right)\right) \\ &\overset{2.10}{\leq} F\left(x_0, D^2\varphi\left(x_0\right)\right) + \Lambda\left\|\left[D^2\psi\left(x_0\right)\right]^+\right\| - \lambda\left\|\left[D^2\psi\left(x_0\right)\right]^-\right\| \\ &\leq F\left(x_0, D^2\varphi\left(x_0\right)\right) + \Lambda\sum_{e_i>0} e_i + \frac{\lambda}{d}\sum_{e_i<0} e_i, \end{aligned}$$

where $e_i$ are the eigenvalues of $D^2\psi\left(x_0\right)$. Finally

$$\mathcal{M}^+\left(D^2\psi\left(x_0\right), \frac{\lambda}{d}, \Lambda\right) \geq f\left(x_0\right) - F\left(x_0, D^2\varphi\left(x_0\right)\right).$$

□

**2.14 Theorem.** *Let $F\colon \Omega\times S \to \mathbb{R}$ be uniformly elliptic with ellipticity constants $\lambda \leq \Lambda$. Assume $F\left(x, D^2u\left(x\right)\right) \leq f\left(x\right)$ for $x \in \Omega$ in the viscosity sense. Then*

$$u \in \overline{S}\left(\frac{\lambda}{d}, \Lambda, f - F\left(\cdot, 0\right)\right). \tag{2.5}$$

*In fact we know more: For every $\varphi \in C^2\left(\Omega\right)$ we know that*

$$u - \varphi \in \overline{S}\left(\frac{\lambda}{d}, \Lambda, f - F\left(\cdot, D^2\varphi\right)\right). \tag{2.6}$$

*Proof.* Analoguesly to the proof above. □

# 3 Regularity Estimates

Before we formulate and prove the most important results, the Harnack inequality and Hölder regularity estimates, let us note that one can prove a result which corresponds to the ABP maximum principle for (sub-) solutions to equations of the form

$$a_{ij}\partial_i\partial_j u = f.$$

Let us quickly summarize the results and the setup.
A function $L\colon \mathbb{R}^d \to \mathbb{R}$ is affine if it is of the form $L(x) = l_0 + l(x)$, where $l_0 \in \mathbb{R}^d$ and $l$ is a linear function. Let $w\colon A \to \mathbb{R}$, $A \subset \mathbb{R}^d$, $x_0 \in A$. Suppose $L$ is an affine functions that touches $w$ by below at $x_0$ in $A$. Then we say that $L$ (rather the graph of $L$) is a *supp. hyperplane* for $w$ at $x_0 \in A$.
Let $A \subset \mathbb{R}^d$ be open and convex. Let $v \in C(A)$. Then the *convex envelope of $v$ in $A$* is defined by

$$\begin{aligned}\Gamma_v(x) &= \sup\{w(x) \,|\, w \le v \text{ in } A,\ w \text{ convex in } A\}\\ &= \sup\{L(x) \,|\, L \le v \text{ in } A,\ L \text{ is affine}\}.\end{aligned}$$

Then $\Gamma_v$ is a convex function $\Gamma_v\colon A \to \mathbb{R}$. The set

$$\{v = \Gamma_v\} = \{x \in A \,|\, v(x) = \Gamma_v(x)\}$$

is called the *(lower) contact set*. Elements in this set are sometimes called *contact points*.

**3.1 Theorem.** *Let $u \in \overline{S}(\lambda, \Lambda, f)$ in a ball $B_R \subset \mathbb{R}^d$ where $f \in C\left(\overline{B_R}\right)$. Assume $u \in C\left(\overline{B_R}\right)$ and $u \ge 0$ on $\partial B_R$. Then*

$$\sup_{B_R} u^- \le CR\left(\int_{B_R\cap\{u=\Gamma_u\}} \left(f^+\right)^d\right)^{\frac{1}{d}},$$

*where $\Gamma_u$ is the convex envelope of $-u^-$ in the set $B_{2R}$, i.e. of the function*

$$\begin{cases} -u^-(x), & \text{if } x \in B_R, \\ 0, & \text{if } x \in B_{2R} \setminus B_R, \end{cases}$$

*and $C = C(d, \lambda, \Lambda)$ with $C \geq 1$.*

**3.2 Corollary.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $f \in C(\Omega)$. Assume $u \in \overline{S}(\lambda, \Lambda, f)$ in $\Omega$ and $u \in C\left(\overline{\Omega}\right)$, $u \geq 0$ on $\partial\Omega$. Then*

$$\sup_{\Omega} u^- \leq C \operatorname{diam}(\Omega) \left\| f^+ \right\|_{L^d(\Omega \cap \{u = \Gamma_u\})},$$

*where $\Gamma_u$ is the convex envelope of $-u^-$ (extended) in a huge ball $B_R$ with $R = \operatorname{diam}(\Omega)$ and $B_R \supset \Omega$.*

We can now deduce a maximum principle for viscosity solutions. It is implied by Corollary 3.2 but it could be proved separately avoiding the ABP result.

**3.3 Corollary.** *Assume $u \in C\left(\overline{\Omega}\right)$ where $\Omega \subset \mathbb{R}^d$ is a bounded domain. Then*

*(i) If $u \in \underline{S}(\lambda, \Lambda, 0)$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in $\Omega$.*

*(ii) If $u \in \overline{S}(\lambda, \Lambda, 0)$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in $\Omega$.*

Recall Lebesgue's differentiation theorem. First note that for $f$ continuous and $x \in \mathbb{R}^d$

$$f(x) = \fint_{B_r(x)} f(y)\, \mathrm{d}y \quad \text{for } r \to 0.$$

**3.4 Definition.** We say a family $\mathcal{F}$ of measurable subsets of $\mathbb{R}^d$ is *regular at a point* $x \in \mathbb{R}^d$, if

(i) the elements in $\mathcal{F}$ are bounded and of positive measure,

(ii) there is a sequence of sets $S_n \in \mathcal{F}$ with $|S_n| \to 0$ as $n \to \infty$ and

(iii) there is $C > 0$ such that for every $S \in \mathcal{F}$ there is a ball $B$ centered at $x$ with $S \subset B$ and $|S| \geq C|B|$.

**Example.**
- $\mathcal{F} = \left\{ B_{\frac{1}{n}}(x) \,\middle|\, n \in \mathbb{N} \right\}$ is regular at $x$.
- $\mathcal{F} = \left\{ \text{rectangles } R \subset \mathbb{R}^2 \,\middle|\, |R| \leq 1,\ x \in \mathbb{R} \right\}$ is *not* regular at $x$.

From now on we work a lot with dyadic cubes $Q$, which are sets of the form

$$\Pi_{i=1}^{d}\left(m_i 2^{-k},(m_i+1)\,2^{-k}\right)\quad \text{for } k\in\mathbb{Z},\ m_i\in\mathbb{Z}.$$

Open dyadic cubes have the following important property: Two cubes are either disjoint or one contains the other.

**3.5 Theorem** (Lebesgue's differentiation theorem). *Assume $f\in L^1_{loc}\left(\mathbb{R}^d\right)$. Assume $x\in\mathbb{R}^d$ is a Lebesgue-point of $f$ and $\mathcal{F}$ is a family of subsets of $\mathbb{R}^d$ which is regular at $x$. Then*

$$\lim_{\substack{S\in\mathcal{F},\\ |S|\to 0}} \fint_S f(y)\,\mathrm{d}y = f(x).$$

*Almost all points in $\mathbb{R}^d$ are Lebesgue-points of $f$.*

**3.6 Theorem** (Calderon-Zygmund decomposition). *Let $Q_0\subset\mathbb{R}^d$ be a dyadic cube and $f\in L^1(Q_0)$. Assume that for some $\lambda>0$*

$$\fint_{Q_0}|f(x)|\,\mathrm{d}x\le\lambda.$$

*There is a countable family $\mathcal{F}_\lambda$ of dyadic cubes $Q_1,Q_2,\dots$ such that*

*(i) $Q_j\cap Q_k=\emptyset$ if $j\ne k$.*

*(ii) For $j\in\mathbb{N}$ we have*

$$\lambda<\fint_{Q_j}|f(x)|\,\mathrm{d}x\le 2^d\lambda \tag{3.1}$$

*(iii) $|f(x)|\le\lambda$ for almost all $x\in Q_0\setminus\bigcup_{j=1}^{\infty}Q_j$.*

*Proof.* Note $Q_0$ does not satisfy (3.1). We decompose $Q_0$ in $2^d$ dyadic subcudes with side length $\frac{l(Q_0)}{2}$, where $l(Q_0)$ denotes the side length of a cube $Q$. We collect all those subcudes $Q$ for which (3.1) holds. They are pairwise disjoint and satisfy

$$\lambda<\fint_Q|f(x)|\,\mathrm{d}x\le\frac{|Q_0|}{|Q|}\fint_{Q_0}|f(x)|\,\mathrm{d}x\le 2^d\fint_{Q_0}|f|\le 2^d\lambda.$$

For each of those subcudes $Q$ which are *not* collected (because they violate (3.1)) we apply the decomposition process again. This is how we build $\mathcal{F}_\lambda$.

For every $z \in Q_0 \setminus \bigcup_{\mathcal{F}_\lambda} Q$ there is a sequence of cubes $Q_1', Q_2', \ldots$ such that $z \in \overline{Q_j'}$ and $⨍_{Q_j'} |f(x)| \, \mathrm{d}x \leq \lambda$. If $z$ is a Lebesgue-point of $f$, then we deduce from Theorem 3.5 that $|f(z)| \leq \lambda$.

$\square$

**3.7 Corollary.** *Assume* $0 < \delta < 1$, $A \subset B \subset Q_1$ *measurable sets. Assume*

*(i)* $|A| \leq \delta$,

*(ii) for each dyadic cube* $Q$ *with* $\frac{|A \cap Q|}{|Q|} > \delta$ *one has* $\tilde{Q} \subset B$, *where* $\tilde{Q}$ *is a predecessor (father) cube of* $Q$.

*Then* $|A| \leq \delta |B|$.

*Proof.* (Think of Theorem 3.6 with $f(x) = \chi_A(x)$.) We know $\frac{|Q_1 \cap A|}{|A|} = |A| \leq \delta$. We subdivide $Q_1$ into $2^d$ subcubes. We collect all subcubes $Q$ for which

$$\frac{|Q \cap A|}{|Q|} > \delta. \tag{3.2}$$

We further decompose those cubes $Q$ which do not satisfy (3.2). In this way we create a family $\mathcal{F}_\delta$ such that (3.2) holds for all $Q \in \mathcal{F}_\delta$.
For $x \in Q_1 \setminus \bigcup_{\mathcal{F}_\delta} Q$ we know there is a sequence of closed cubes $\overline{Q_i}$ such that

$$|Q_i| \to 0, \quad \frac{|Q_i \cap A|}{|Q_i|} \leq \delta < 1.$$

Note $\frac{|Q \cap A|}{|Q|} = ⨍_Q \chi_A(x) \, \mathrm{d}x$. By (3.5) we conclude that for almost every $x \in Q_1 \setminus \bigcup_{\mathcal{F}_\delta} Q$

$$\chi_A(x) \leq \delta < 1.$$

Thus $\left| A \cap \left( Q_1 \setminus \bigcup_{\mathcal{F}_\delta} Q \right) \right| = 0$. Now we consider the family of predecessors $\tilde{Q}_i$. We relabel this fmaily such that $\left\{ \tilde{Q}_i \right\}$ is a pairwise disjoint family. We know $A \subset \bigcup_{i \in \mathbb{N}} Q_i \subset \bigcup_{i \in \mathbb{N}} \tilde{Q}_i$ and

$$\frac{\left| \tilde{Q}_i \cap A \right|}{\left| \tilde{Q}_i \right|} \leq 1 \quad \forall \, i \in \mathbb{N}.$$

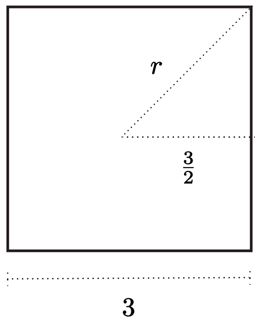
Since $\frac{|Q_i \cap A|}{|Q_i|} > \delta$ and because of condition (ii) we know $\tilde{Q}_i \subset B$ for all $i \in \mathbb{N}$. Hence we know $A \subset \bigcup_{i\in\mathbb{N}} \tilde{Q}_i \subset B$. We conclude

$$\begin{aligned} |A| &\leq \sum_{i\in\mathbb{N}} \left|\tilde{Q}_i \cap A\right| \leq \delta \sum_{i\in\mathbb{N}} \left|\tilde{Q}_i\right| \leq \delta \left|\bigcup_{i\in\mathbb{N}} \tilde{Q}_i\right| \\ &\leq \delta\, |B|\,. \end{aligned}$$

□

Recall:

$$B_{\frac{1}{4}} \subset B_{\frac{1}{2}} \subset Q_1 \subset Q_3 \subset B_{\frac{3}{2}\sqrt{d}} \subset B_{2\sqrt{d}}.$$




$$r^2 = \left(\tfrac{3}{2}\right)^2 \cdot d \quad \Rightarrow \quad r = \tfrac{3}{2}\sqrt{d}.$$

**3.8 Proposition.** *Assume* $0 < \delta \leq \Lambda$. *There exists a function* $\varphi \in C^\infty\left(\mathbb{R}^d\right)$, *two numbers* $C \geq 1$, $M \geq 1$ *and* $\psi \in C\left(\mathbb{R}^d\right)$ *with* $0 \leq \psi \leq 1$, $\operatorname{supp}(\psi) \subset \overline{Q_1}$ *such that*

*(i)* $\varphi \geq 0$ *in* $\mathbb{R}^d \setminus B_{2\sqrt{d}}$ *and* $\varphi \geq -M$ *in* $\mathbb{R}^d$.

*(ii)* $\varphi \leq -2$ *in* $Q_3$.

*(iii)* $\mathcal{M}^+\left(D^2\varphi, \lambda, \Lambda\right) \leq C\psi$ *in* $\mathbb{R}^d$.

*Proof.* Let $\gamma > 0$ be fixed, to be determined later. Set $\varphi(x) = M_1 - M_2 |x|^{-\gamma}$ for $x \in \mathbb{R}^d \setminus B_{\frac{1}{4}}$. Choose $M_1 > 0$, $M_2 > 0$ such that

$$\varphi\restriction_{\partial B_{2\sqrt{d}}} \equiv 0, \quad \varphi\restriction_{\partial B_{\frac{3}{2}\sqrt{d}}} = -2.$$

By this choice, conditions (i) and (ii) are satisfied. We consider an arbitrary smooth extension of $\varphi$ to $\mathbb{R}^d$.

We need to verify (iii).

$$\begin{aligned}
\partial_i \varphi &= M_2 \gamma \, |x|^{-\gamma-1} \frac{x_i}{|x|} = M_2 \gamma \, |x|^{-\gamma-2} \, x_i, \\
\partial_i \partial_i \varphi &= M_2 \gamma \left[ (-\gamma - 2) \, |x|^{-\gamma-4} \, (x_i)^2 + |x|^{-\gamma-2} \right], \\
\partial_j \partial_i \varphi &= M_2 \gamma \left[ -\gamma - 2 \, |x|^{-\gamma-3} \, \frac{x_j x_i}{|x|} \right].
\end{aligned}$$

Note that for $x \in \mathbb{R}^d$ with $|x| = r$

$$\mathcal{M}^+ \left( D^2 \varphi \,(x) \right) = \mathcal{M}^+ \left( D^2 \varphi \,(r, 0, \ldots, 0) \right).$$

Pick $x = (r, 0, \ldots, 0)$ for some $r > 0$. Then

$$\begin{aligned}
\partial_i \partial_j \varphi \,(x) &= 0 \quad \text{if } i \neq j, \\
\partial_1 \partial_1 \varphi \,(x) &= -M_2 \gamma \,(\gamma + 1), \\
\partial_i \partial_i \varphi \,(x) &= M_2 \gamma r^{-\gamma-2} \quad \text{for } i \neq 1.
\end{aligned}$$

Using the definition of $\mathcal{M}^+$ (note that $D^2 \varphi \,(x)$ is diagonal) we obtain for $|x| \geq \frac{1}{4}$ ($x \in \mathbb{R}^d \setminus B_{\frac{1}{4}}$)

$$\begin{aligned}
\mathcal{M}^+ \left( D^2 \varphi \,(x), \lambda, \Lambda \right) &= \Lambda \gamma \,(d-1) \, M_2 \, |x|^{-\gamma-2} - \lambda \gamma \,(\gamma+1) \, M_2 \, |x|^{-\gamma-2} \\
&= M_2 \gamma \, |x|^{-\gamma-2} \underbrace{\left[ \Lambda \,(d-1) - \lambda \,(\gamma+1) \right]}_{\leq 0}.
\end{aligned}$$

Choose $\gamma = \max \left\{ 1, \frac{\Lambda(d-1)}{\lambda} - 1 \right\}$. Note that for $|x| \leq \frac{1}{4}$ with some constant $C = C\,(d, \lambda, \Lambda) \geq 1$ we know

$$\mathcal{M}^+ \left( D^2 \varphi \,(x) \right) \leq C.$$

Choose $\psi$ to be a smooth function with $\psi \equiv 1$ on $B_{\frac{1}{4}}$ and $\psi \equiv 0$ on $\mathbb{R}^d \setminus B_{\frac{1}{2}}$.

□

The previous technical tools are the major tools needed for the proof of two fundamental results, the Harnack inequality and apriori bounds in Hölder spaces. Recall:

$$S^* \,(\lambda, \Lambda, f) = \underline{S} \,(\lambda, \Lambda, -|f|) \cap \overline{S} \,(\lambda, \Lambda, |f|) \supset S \,(\lambda, \Lambda, f).$$

**3.9 Theorem** (Harnack inequality)**.** *Assume* $0 < \lambda \leq \Lambda$. *There is* $C \geq 1$ *such that for every* $f \in C\,(Q_1)$, $f$ *bounded and every* $u \in S^* \,(\lambda, \Lambda, f)$ *in* $Q_1$ *with* $u \geq 0$ *in* $Q_1$ *the* *following holds:*

$$\sup_{Q_{\frac{1}{2}}} u \le C \left( \inf_{Q_{\frac{1}{2}}} u + \|f\|_{L^d(Q_1)} \right).$$

**3.10 Corollary** (Interior Hölder estimates). *Assume* $0 < \delta \le \Lambda$, $f \in C(Q_1)$, $f$ *bounded,* $u \in S^*(\lambda, \Lambda, f)$ *in* $Q_1$.

*(i) There is* $\mu \in (0,1)$ *independent of* $f$ *and* $u$*, such that*

$$\operatorname*{osc}_{Q_{\frac{1}{2}}} u \le \mu \operatorname*{osc}_{Q_1} u + 2\|f\|_{L^d(Q_1)}.$$

*(ii) There are* $\alpha \in (0,1)$, $C \ge 1$, *both independent of* $f$ *and* $u$*, such that* $u \in C^\alpha\left(\overline{Q_{\frac{1}{2}}}\right)$ *and*

$$\|u\|_{C^\alpha\left(\overline{Q_{\frac{1}{2}}}\right)} \le C \left( \sup_{Q_1} |u| + \|f\|_{L^d(Q_1)} \right).$$

**Remark.** *Using a covering argument, one can replace cubes by appropriate balls in 3.10.*

*Proof.* (ii) follows from (i) by using a standard iteration technique (cp. Lemma 8.23 in [Gilbarg/Trudinger]) Let us prove (i). For $r > 0$ set $\omega(r) = \operatorname{osc}_{Q_r}(u)$, $m(r) = \inf_{Q_r} u$ and $M(r) = \sup_{Q_r} u$. Note that $u - m(1) \ge 0$ in $Q_1$ and $u - m(1) \in S^*(\lambda, \Lambda, f)$. Same for $M(1) - u$.

$$\sup_{Q_{\frac{1}{2}}} u \le C \left( \inf_{Q_{\frac{1}{2}}} u + \|f\|_{L^d(Q_1)} \right).$$

We apply the Harnack inequality to $M(1) - u$ and $u - m(1)$.

$$M(1) - m\left(\frac{1}{2}\right) \le C\left(M(1) - M\left(\frac{1}{2}\right) + \|f\|_{L^d(Q_1)}\right),$$
$$M\left(\frac{1}{2}\right) - m(1) \le C\left(m\left(\frac{1}{2}\right) - m(1) + \|f\|_{L^d(Q_1)}\right).$$

Addition of these 2 inequalities gives

$$\underbrace{M(1) - m(1) + M\left(\frac{1}{2}\right) - m\left(\frac{1}{2}\right)}_{\omega(1)+\omega\left(\frac{1}{2}\right)} \le C\left[\omega(2) - \omega\left(\frac{1}{2}\right) + 2\|f\|_{L^d(Q_1)}\right],$$

$$\omega\left(\frac{1}{2}\right)(C+1) \le \omega(1)(C-1) + 2C\|f\|_{L^d(Q_1)}$$

and we obtain

$$\omega\left(\frac{1}{2}\right) \le \underbrace{\frac{C-1}{C+1}}_{=:\mu}\omega(1) + \frac{2C}{C+1}\|f\|_{L^d(Q_1)}.$$

□

The following Lemma is sufficient for a proof of Theorem 3.9:

**3.11 Lemma.** *There exists $\varepsilon_0 \in (0,1)$ and $C \ge 1$ such that for $f \in C\left(Q_{4\sqrt{d}}\right)$, $f$ bounded and $u \in S^*(\lambda,\Lambda,f)$ in $Q_{4\sqrt{d}}$, $u \in C\left(\overline{Q_{4\sqrt{d}}}\right)$, $u \ge 0$ on $Q_{4\sqrt{d}}$ with $\inf_{Q_{\frac{1}{4}}} u \le 1$ and $\|f\|_{L^d\left(Q_{4\sqrt{d}}\right)} \le \varepsilon_0$. Then*

$$\sup_{Q_{\frac{1}{4}}} u \le C.$$

How / Why does Lemma 3.11 imply Theorem 3.9? Assume $u \in S^*(\lambda,\Lambda,f)$ in $Q_{4\sqrt{d}}$, $u \in C\left(\overline{Q_{4\sqrt{d}}}\right)$ and $u \ge 0$ in $Q_{4\sqrt{d}}$. Set for $\delta > 0$, $u_\delta = \frac{u}{A_\delta}$ with $A_\delta = \inf_{Q_{\frac{1}{4}}} u + \delta + \varepsilon_0^{-1}\|f\|_{L^d\left(Q_{4\sqrt{d}}\right)}$, where $\varepsilon_0$ is as in Lemma 3.11. Then $u_\delta \in S^*\left(\lambda,\Lambda,\frac{f}{A_\delta}\right)$ and $\|f\|_{L^d\left(Q_{4\sqrt{d}}\right)} \le \varepsilon_0$. Note $\inf_{Q_{\frac{1}{4}}} u_\delta \le 1$. Lemma 3.11 now implies

$$\begin{aligned}\sup_{Q_{\frac{1}{4}}} u_\delta \le CA_\delta &= C\left(\inf_{Q_{\frac{1}{4}}} u + \delta + \varepsilon_0^{-1}\|f\|_{L^d\left(Q_{4\sqrt{d}}\right)}\right)\\ &\le C\varepsilon_0^{-1}\left(\inf_{Q_{\frac{1}{4}}} u + \delta + \|f\|_{L^d\left(Q_{4\sqrt{d}}\right)}\right).\end{aligned}$$

Now let $\delta \to 0$.
What we have shown: Lemma 3.11 implies

$$\begin{aligned}&u \in S^*(\lambda,\Lambda,f) \quad \text{in } Q_{4\sqrt{d}},\\ &u \in C\left(\overline{Q_{4\sqrt{d}}}\right),\\ &u \ge 0 \quad \text{in } Q_{4\sqrt{d}}\\ \Rightarrow \sup_{Q_{\frac{1}{4}}} u &\le C\left(\inf_{Q_{\frac{1}{4}}} u + \|f\|_{L^d\left(Q_{4\sqrt{d}}\right)}\right).\end{aligned}$$

**3.12 Lemma** (Key Lemma)**.** *There exists $\varepsilon_0 \in (0,1)$, $\mu \in (0,1)$ and $M > 1$ such that for $u \in \overline{S}(\lambda,\Lambda,|f|)$ in $Q_{4\sqrt{d}}$, $u \in C\left(\overline{Q_{4\sqrt{d}}}\right)$ and $f\colon Q_{4\sqrt{d}} \to \mathbb{R}$ the properties hold:*

*(i)* $u \ge 0$ *in* $Q_{4\sqrt{d}}$.

*(ii)* $\inf_{Q_3} u \leq 1$.

*(iii)* $\|f\|_{L^d(Q_{4\sqrt{d}})} \leq \varepsilon_0$ *implies* $|\{u \leq M\} \cap Q_1| > \mu$.

*Proof.* (Using Theorem 3.1 and Proposition 3.8.) Let $\varphi$ be as in 3.8. Set $w = u + \varphi$. We want to apply Theorem 3.1 to $w$ for $R = 2\sqrt{d}$. Note $w \in \overline{S}(\lambda, \Lambda, |f| + C_1\psi)$ in $B_{2\sqrt{d}}$, because $\mathcal{M}^+ \left(D^2\varphi\right) \leq C_1\psi$ and $B_{2\sqrt{d}} \subset Q_{4\sqrt{d}}$. Furthermore $w \in C\left(\overline{B_{2\sqrt{d}}}\right)$, $w \geq 0$ on $\partial B_{2\sqrt{d}}$.

$$\inf_{Q_3} w \leq -1.$$

With 3.1 we have

$$\begin{aligned} 1 \leq \sup_{B_{2\sqrt{d}}} w^- &\leq C_2 \left( \int_{\{w=\Gamma_w\}\cap B_{2\sqrt{d}}} \left(|f| + C_1\psi\right)^d \right)^{\frac{1}{d}} \\ &\leq C_3 \|f\|_{L^d(Q_{4\sqrt{d}})} + C_3 \left|\{w = \Gamma_w\} \cap Q_1\right|^{\frac{1}{d}}, \end{aligned}$$

since $0 \leq \varphi \leq 1$ and $\operatorname{supp}(\psi) \subset Q_1$. ($\varepsilon_0 \leq \frac{1}{2C_3}$) Now

$$\frac{1}{2} \leq C_3 \left|\{w = \Gamma_w\} \cap Q_1\right|^{\frac{1}{d}}.$$

Note that $w(x) = \Gamma_w(x)$ implies $w(x) \leq 0$ and hence $(u(x) \leq -\varphi(x)) = (u(x) \leq -M)$. Finally

$$\frac{1}{2} \leq C_3 \left|\{u \leq M\} \cap Q_1\right|^{\frac{1}{d}}$$

and hence

$$\left|\{u \leq M\} \cap Q_1\right| \geq \left(\frac{1}{2C_3}\right)^d =: P.$$

□

**3.13 Lemma.** *Let $u$ be as in Lemma 3.12. Then*

$$\left|\left\{u > M^k\right\} \cap Q_1\right| \leq (1-\mu)^k \quad \text{for } k \in \mathbb{N} \tag{3.3}$$

*with $M$ and $\mu$ as in Lemma 3.12. Hence for every $t > 0$*

$$\left|\{u > t\} \cap Q_1\right| \leq Ct^{-\varepsilon} \quad \text{for some } C \geq 1,\ \varepsilon \in (0,1). \tag{3.4}$$

**Remark.** *Estimate (3.4) follows from (3.3) by very simple arguments. Most important: (3.4) is already sufficient for the derivation of interior Hölder estimates. In the proof of Corollary 3.10 we used the (full) Harnack inequality, but we could have used (3.4)!*

*Estimate (3.4) is also know as the $L^\varepsilon$-*lemma. *Why?*
$f\colon (X,m) \to \mathbb{R}$ *measurable,* $(X,m)$ *measure space. Then* $t \mapsto m_f(t) = m(\{x \in X \mid |f(x)| > t\})$ *is non-negative, non-increasing and right-continuous.*

*(i) By Fubini*

$$\int_X |f(x)|\, m(\mathrm{d}x) = \int_0^\infty m_f(t)\, \mathrm{d}t.$$

*(ii)*

$$\|f\|_{L^p(X)} = p\int_0^\infty m_f(t)\, t^{p-1}\, \mathrm{d}t.$$

*(iii)*

$$\|f\|_{L^\infty(X)} = \sup\{t > 0 \mid m_f(t) > 0\} = \inf\{t > 0 \mid m_f(t) < 0\}.$$

*Weak $L^p$-spaces:* $\exists\, C > 1\ \forall\, t > 0$

$$\begin{aligned} &L^p_w(X) = \left\{f\colon X \to \mathbb{R} \textit{ measurable} \,\middle|\, m_f(t)\} \le Ct^{-p}\right\}\\ &L^p(X) \hookrightarrow L^p_w(X) \ \textit{ continuous}\\ &L^p_w(X) \hookrightarrow L^{p-\varepsilon}(X) \ \textit{ continuous, if } m(X) < \infty. \end{aligned}$$

*Note that* $\|f\|^*_{L^p_w(X)} = \sup_{t>0} t\,|\{x \in X \mid f(x) > t\}|^{\frac{1}{p}}$ *is a quasi-norm.*

*Proof of 3.13.* **Step 1.** For $k = 0$ estimate (3.3) holds true (Lemma 3.12). We prove (3.3) by induction. Suppose (3.3) holds for $k-1$. Set $A = \left\{u > M^k\right\} \cap Q$, $B = \left\{u > M^{k-1}\right\} \cap Q_1$. <u>Aim:</u>

$$|A| \le (1-\mu)\,|B|.$$

**Step 2.** Idea: We use 3.7 with $\delta = 1-\mu$. We know: $A \subset B \subset Q_1$ and $|A| \le |\{u > M\} \cap Q_1| \le 1-\mu = \delta$. Let $Q = Q_{2^{-i}}(x_0)$ be a dyadic cube with

$$\frac{|A \cap Q|}{|Q|} > \delta \quad \Leftrightarrow \quad |A \cap Q| > (1-\mu)\,|Q|. \tag{3.5}$$

If we can show $\tilde{Q} \subset B$, then we can apply Corollary 3.7 and we are done.
Suppose $\tilde{Q} \not\subset B$. Define $\tilde{u}\colon Q_1 \to \mathbb{R}$ by

$$\tilde{u}(y) = \frac{u(\varphi(y))}{M^{k-1}},$$

where $\varphi\colon Q_1 \to Q$, $\varphi(y) = x_0 + 2^{-i}y$.

Claim: $\tilde{u}$ satisfies all assumptions of Lemma 3.12. (We prove this claim in the third step.) Then Lemma 3.12 implies (with the same $\mu$!)

$$\mu \leq |\{\tilde{u} \leq M\} \cap Q_1| = 2^{id} \left| \left\{ u \leq M^k \right\} \cap Q \right|$$
$$\Rightarrow \frac{|Q \setminus A|}{|Q|} > \mu,$$

which is a contradiction to (3.5).

**Step 3.** We prove the claim from step 2 assuming $\tilde{Q} \not\subset B$. Let $\tilde{x} \in \tilde{Q}$ with $u(\tilde{x}) \leq M^{k-1}$. Set $\tilde{f}(y) = \frac{f(\varphi(y))}{2^{2i} M^{k-1}}$. Then $\tilde{u} \in \overline{S}\left(\lambda, \Lambda, \tilde{f}\right)$ in $Q_{4\sqrt{d}}$. Note $x \in \tilde{Q}$ implies $\varphi^{-1}(x) = y \in Q_3$. Since $\tilde{u} \geq 0$, $\inf_{Q_3} \tilde{u} \leq \frac{u(\tilde{x})}{M^{k-1}} \leq 1$.

$$\|f\|_{L^d(Q_{4\sqrt{d}})} = \frac{2^i}{2^{2i} M^{k-1}} \|f\|_{L^d(Q_{4\sqrt{d}})} \leq \|f\|_{L^d(Q_{4\sqrt{d}})} \leq \varepsilon_0.$$

□

→ Lemma 3.13: $u \in \overline{S}(\lambda, \Lambda, |f|)$, $u \geq 0 \Rightarrow |\{u > t\} \cap Q_1| \leq C t^{-\varepsilon}$ for all $t > 0$. ($C \geq 1$, $\varepsilon \in (0,1)$ independent of $u$).

**3.14 Lemma** (Chasing the sup)**.** *Assume $u \in \underline{S}(\lambda, \Lambda, -|f|)$ in $Q_{4\sqrt{d}}$ with*

*(A) $\forall\, t > 0$: $|\{u > t\} \cap Q_1| \leq C_1 t^{\varepsilon}$ for some $C_1 \geq 1$, $\varepsilon \in (0,1)$.*

*(B) $\|f\|_{L^d(Q_{4\sqrt{d}})} \leq \varepsilon_0$ for some $\varepsilon_0 \in (0,1)$.*

*There exists $M_0 > 1$ and $\sigma > 0$, both independent of $u$ and $f$, such that for $\nu = \frac{M_0}{M_0 - \frac{1}{2}}$ the following holds:*
*If $x_0 \in Q_{\frac{1}{2}} \Leftrightarrow |x_0|_\infty \leq \frac{1}{4}$ and $u(x_0) \geq \nu^{j-i} M_0$ for some $j \in \mathbb{N}$, then*

$$\sup_{Q^j} u \geq v^j M_0,$$

*where $Q^j := Q_{l_j}(x_0) \subset Q_1$ with $l_j = \sigma M_0^{-\frac{\varepsilon}{d}} \nu^{-\frac{\varepsilon j}{d}}$.*

*Proof of Lemma 3.11.* Note that under the assumptions of Lemma 3.11 we can apply 3.13 and 3.14. Set $l_j = \sigma M_0^{-\frac{\varepsilon}{d}} \nu^{-\frac{\varepsilon j}{d}}$ for $j \in \mathbb{N}$, where $\sigma$, $M_0$, $\varepsilon$ are taken from Lemma 3.12, 3.13 and 3.14. Choose $j_0 \in \mathbb{N}$ so large such that $\sum_{j \geq j_0} l_j \leq \frac{1}{4}$.
Claim: $\sup_{Q_{\frac{1}{4}}} u \leq \nu^{j_0 - 1} M_0$.
Suppose that the claim is false. Then there is $x_{j_0} \in Q_{\frac{1}{4}}$ with $u(x_{j_0}) \geq \nu^{j_0-1} M_0 \overset{3.14}{\Rightarrow}$ $\exists\, x_{j_0+1}$: $u(x_{j_0+1}) \geq \nu^{j_0} M_0$, $|x_{j_0+1} - x_{j_0}| \leq \frac{l_{j_0}}{2}$. We repeat this process and find points

$x_j$ $(j \geq j_0)$ such that $|x_{j+1} - x_j| \leq \frac{l_j}{2}$ and $u(x_j) \geq \nu^j M_0$. This is ok as long as $x_j \in Q_{\frac{1}{2}}$ for all $j \geq j_0$. No problem:

$$\begin{aligned} |x_j|_\infty &\leq |x_{j_0}| + \sum_{k=j_0}^{j-1} |x_{k+1} - x_k| \\ &\leq \frac{1}{8} + \sum_{k \geq j_0} \frac{l_k}{2} \leq \frac{1}{4}, \end{aligned}$$

because of our choice of $j_0$. Contradiction, since $u(x_j) \nearrow \infty$ for $j \to \infty$, but $u \in C\left(\overline{Q_{4\sqrt{d}}}\right)$ bounded.

□

As an immediate consequence from Corollary 3.10 we have

**3.15 Proposition.** *Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence of uniformly elliptic operators with ellipticity constants $\lambda$, $\Lambda$. Let $\{u_k\}_{k\in\mathbb{N}} \subset C(\Omega)$ be a sequence of viscosity solutions to*

$$F_k\left(x, D^2 u_k\right) = f(x) \quad \text{in } \Omega,$$

*where $f \in C(\Omega)$. Assume $\{F_k\}$ converges uniformly on compact subsets of $\Omega \times S$ and that $\{u_k\}$ is uniformly bounded in compact subsets of $\Omega$. Then there exists $u \in C(\Omega)$ and a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ which converges uniformly on compact subsets of $\Omega$. Moreover,*

$$F\left(x, D^2 u\right) = f(x) \quad \text{in } \Omega$$

*in the viscosity sense.*

*Proof.* Arzela-Ascoli for $\{u_k\}$ and compatibility of concept of viscosity solutions with limit operations.

□

## 3.1 Regularity up to the boundary

Concrete Problem: Let $B = B_1((0,1))$. Let $g\colon \partial B \to \mathbb{R}$ with $g(0) = 0$ and $g \in C^\beta(\partial\Omega)$, in particular $|g(x)| \leq C|x|^\beta$ for $|x|$ small. Let $u\colon B \to \mathbb{R}$ be a solution to

$$\begin{aligned} \Delta u &= 0 \quad \text{in } B, \\ u &= g \quad \text{on } \partial B. \end{aligned}$$

Question: Can we deduce regularity of $u$ up to the boundary and if so, how much?
Important: Only those techniques which work for fully nonlinear equations are admissible here. In particular, not allowed: Poisson formula, Green function, complex variable function theory, difference quotient technique.
Comparison:

$$\begin{aligned} \Delta u \leq 0 \quad & \text{in } B, \\ u \leq 0 \quad & \text{on } \partial B, \\ \Rightarrow u \leq 0 \quad & \text{in } B. \end{aligned}$$

Aim: To show:

$$|u(x)| \leq C_1 |x|^\gamma \quad \text{for some } \gamma \in (0, \beta].$$

This means $u$ is Hölder continuous in the neighborhood of 0 of order $\gamma$. (Recall $u(0) = g(0) = 0$.)

$$\tilde{u} := u - C_1 |x|^\gamma.$$

If $\tilde{u}$ solved

$$\begin{aligned} -\Delta \tilde{u} \leq 0 \quad & \text{in } B, \\ \tilde{u} \leq 0 \quad & \text{on } \partial B, \end{aligned}$$

we would be done.

$$\begin{aligned} \varphi(x) &= |x|^\gamma \quad \text{for } \gamma \in (0,1), \\ \partial_i \varphi(x) &= \gamma |x|^{\gamma-1} \frac{x_i}{|x|}, \\ \partial_i \partial_i \varphi(x) &= \gamma(\gamma-2)|x|^{\gamma-4}(x_i)^2 + \gamma |x|^{\gamma-2}, \\ \Delta \gamma(x) &= \gamma |x|^{\gamma-2}[(\gamma-2)+d] \stackrel{d=2}{=} \gamma. \end{aligned}$$

Let $b(x) = C_2 x_n^{\frac{\beta}{2}}$.

$$\begin{aligned}
\partial_n b(x) &= C_2 \frac{\beta}{2} x_n^{\frac{\beta-2}{2}}, \\
\partial_n \partial_n b(x) &= C_2 \frac{\beta(\beta-2)}{2} x_n^{\frac{\beta-4}{2}}, \\
\Rightarrow \Delta b(x) &= \sum_{i=1}^{n} \partial_i \partial_i b(x) = \frac{C_2}{4} \beta(\beta-2) x_n^{\frac{\beta-4}{2}} \leq 0
\end{aligned}$$

since $\beta - 2 < 0$. For $x \in \partial B_1((0,1))$ we have

$$\begin{aligned}
x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + (x_n - 1)^2 &= 1 \\
\Leftrightarrow |x|^2 &= 1 + x_n^2 - (x_n - 1)^2 \\
&= 2x_n.
\end{aligned}$$

For $x \in \partial B$ the following holds

$$x_n = \frac{|x|^2}{2} \quad \Leftrightarrow \quad |x|^2 = 2x_n.$$

We wanted:

$$b(x) \leq C_1 |x|^{\gamma}$$

for some $\gamma \in (0, \beta]$.

$$C_2 x_n^{\frac{\beta}{2}} = C_2 \left( \frac{|x|^2}{2} \right)^{\frac{\beta}{2}} = C_3 |x|^{\beta}.$$

We have proved:

$$u(x) \leq C |x|^{\frac{\beta}{2}}$$

for all $x \in B$ and some $C \geq 1$. Replacing $u$ by $-u$ one finally obtains

$$|u(x)| \leq C |x|^{\frac{\beta}{2}} \quad \text{for all } x \in B \text{ and some } C \geq 1.$$

**3.16 Proposition** (Boundary regularity)**.** *Assume* $0 < \beta < 1$, $g \in C^{\beta}(\partial B)$, *where* $B \subset \mathbb{R}^d$ *is a ball with radius* 1. *Let* $u \in S(\lambda, \Lambda, 0)$ *in* $B$. *Fix* $x_0 \in \partial B$. *Then* $u$ *is* $C^{\frac{\beta}{2}}$*-continuous in a neighborhood of* $x_0$ *and satisfies*

$$\sup_{x \in B} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\frac{\beta}{2}}} \leq 2^{\frac{\beta}{2}} \sup_{x \in \partial B} \frac{|g(x) - g(x_0)|}{|x - x_0|^{\beta}}.$$

*Proof.* W.l.o.g. we assume $B = B((0,0,\dots,0,1))$ and $x_0 = 0$, $g(0) = 0$. Set $K = \sup_{x\in\partial B} \frac{|g(x)|}{|x|^\beta}$. Thus $|g(x)| \le K|x|^\beta$ for $x \in \partial B$. Note: For $x \in \partial B$:

$$x_1^2 + \dots + x_{d-1}^2 + (x_d - 1)^2 = 1 \quad \Leftrightarrow \quad |x|^2 = 1 + x_d^2 - (x_d - 1)^2 = 2x_d.$$

Therefore, for $x \in \partial B$

$$u(x) = g(x) \le K|x|^\beta = K\left(|x|^2\right)^{\frac{\beta}{2}} \le K2^{\frac{\beta}{2}} x_d^{\frac{\beta}{2}}.$$

We want to use $b(x) = K2^{\frac{\beta}{2}} x_d^{\frac{\beta}{2}}$ as a barrier. Set $h(x) = x_d^{\frac{\beta}{2}}$. Then $\partial_i\partial_j h = 0$ if $i \ne d$, $j \ne d$. Moreover

$$\partial_d\partial_d h(x) = \frac{\beta}{2}\frac{\beta-2}{2} x_d^{\frac{\beta-4}{2}}.$$

$$\mathcal{M}^+\left(D^2h\right) = \lambda\frac{\beta}{2}\frac{\beta-2}{2} x_d^{\frac{\beta-4}{2}} < 0.$$

Comparison: $u - b \in \underline{S}(\lambda,\Lambda,0)$. Note $u - b \le 0$ on $\partial B \Rightarrow u \le b$ in $B$, i.e. $u(x) \le K2^{\frac{\beta}{2}} x_d^{\frac{\beta}{2}} \le K2^{\frac{\beta}{2}}|x|^{\frac{\beta}{2}}$ for $x \in B$. In the same way: $-u(x) \le K2^{\frac{\beta}{2}}|x|^{\frac{\beta}{2}}$ for $x \in B$. Hence

$$|u(x)| \le 2^{\frac{\beta}{2}} K|x|^{\frac{\beta}{2}} \quad \text{for } x \in B.$$

□

Interior estimates as in Corollary 3.10 and boundary estimates as in Proposition 3.16 together imply the following:

**3.17 Theorem** (Global regularity)**.** *Assume* $0 < \beta < 1$, $g \in C^\beta(\partial B)$, *where* $B \subset \mathbb{R}^d$ *is a ball with radius* 1. *There are* $C \ge 1$ *and* $\gamma \in \left(0, \frac{\beta}{2}\right)$ *such that for every* $u \in C\left(\overline{B}\right)$ *with* $u = g$ *on* $\partial B$ *and* $u \in S(\lambda,\Lambda,0)$ *in* $B$, *on obtains*

$$u \in C^\gamma\left(\overline{B}\right) \quad \textit{with} \quad \|u\|_{C^\gamma(\overline{B})} \le C\|g\|_{C^\beta(\partial B)}.$$

*Proof.* Comparison $\Rightarrow$

$$\inf_{\partial B} g \le u \le \sup_{\partial B} g \quad \text{in } B.$$

Aim: Estimate for $\frac{|u(x)-u(y)|}{|x-y|^\gamma}$ from above for all $x, y \in B$. Fix $x, y \in B$, set $d_x = \operatorname{dist}(x,\partial B)$, $d_y = \operatorname{dist}(y,\partial B)$. W.l.o.g. $d_y \le d_x$. Let $x_0, y_0 \in \partial B$ be those points such that $|x - x_0| = d_x$, $|y - y_0| = d_y$.
Case 1: $d_x \ge 2|x-y|$. In this case $y \in \overline{B_{\frac{d_x}{2}}(x)} \subset B_{d_x}(x) \subset B$. We apply Corollary 3.10

to $u - u(x_0)$ in the ball $B_{d_x}(x) \Rightarrow \|u\|_{C^\alpha\left(\overline{B_{\frac{R}{2}}}\right)} \le CR^{-\alpha}\|u\|_{L^\infty(B_R)}$. Set $\gamma = \min\left(\alpha, \frac{\beta}{2}\right)$.

$$
\begin{aligned}
(d_x)^\gamma \frac{|u(x) - u(y)|}{|x-y|^\gamma} &\le d_x^\gamma \frac{|u(x) - u(x_0) - (u(y) - u(x_0))|}{|x-y|^\gamma} \qquad [\tilde{u} := u(\cdot) - u(x_0)] \\
&\le d_x^\alpha \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^\alpha} \le C\|u - u(x_0)\|_{L^\infty(B_{d_x}(x))}\,. \qquad (3.6)
\end{aligned}
$$

Proposition 3.16 implies

$$
\|u - u(x_0)\|_{L^\infty(B_{d_x}(x))} \le C d_x^{\frac{\beta}{2}} \|g\|_{C^\beta(\partial B)}\,. \qquad (3.7)
$$

Recall $\gamma \le \frac{\beta}{2}$, $d_y \le 1$. (3.6) and (3.7) imply

$$
\begin{aligned}
\frac{|u(x) - u(y)|}{|x-y|^\gamma} &\le C d_x^{\frac{\beta}{2}-\gamma}\|g\|_{C^\beta(\partial B)} \\
&\le C\|g\|_{C^\beta(\partial B)}.
\end{aligned}
$$

<u>Case 2</u>: $d_y \le d_x \le 2|x-y|$. Proposition 3.16 and conditions on $g$ give

$$
\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\
&\le C\left(d_x^{\frac{\beta}{2}} + |x_0 - y_0|^{\frac{\beta}{2}} + d_y^{\frac{\beta}{2}}\right)\|g\|_{C^\beta(\partial B)}.
\end{aligned}
$$

Note:

$$
|x_0 - y_0| \le d_x + |x-y| + d_y \le 5|x-y|\,.
$$

Therefore

$$
\begin{aligned}
|u(x) - u(y)| &\le C\left(2^{\frac{\beta}{2}}|x-y|^{\frac{\beta}{2}} + 5^{\frac{\beta}{2}}|x-y|^{\frac{\beta}{2}} + 2^{\frac{\beta}{2}}|x-y|^{\frac{\beta}{2}}\right)\|g\|_{C^\beta(\partial B)} \\
&\le C'|x-y|^{\frac{\beta}{2}}\|g\|_{C^\beta(\partial B)} \le C''|x-y|^\gamma\|g\|_{C^\beta(\partial B)}.
\end{aligned}
$$

□

**Remark.** *If $g\colon \partial B \to \mathbb{R}$ is only continuous with* some *control on the modulus of continuity, one can still prove global regularity with some modulus of continuity for $u$ up to the boundary. This is sufficient for questions of compactness.*

<u>Exercise</u>: $u \in C([-1,1])$, i.e. $u \in C\left(\overline{\Omega}\right)$ for $\Omega = (-1,1)$. Let $h \in (-1,1)$. Set $u_h(x) = \frac{u(x+h)-u(x)}{|h|^{\frac{1}{2}}}$ for $x \in \Omega_h$, where $\Omega_h = \{x \in \Omega \,|\, x+h \in \Omega\}$. Assume there is

$K \geq 1$ such that for all $h \in (-1, 1)$

$$\|u_h\|_{C^{\frac{1}{4}}(\Omega_h)} \leq K.$$

Prove $\|u\|_{C^{\frac{3}{4}}(\overline{\Omega})} \leq CK$ with some generic constant $C \geq 1$.

# 4 Homogenization

## 4.1 Introduction to Homogenization (Non-Divergence elliptic equations)

(1) Intro generically homogenization

(2) Periodic fully nonlinear non-divergence equations

(3) Random linear equations

(4) Random fully nonlinear equations

(5) Furhter topics

References: Engquist, Souganidis: Asymptotic and numerical homogenization. Defranceschi: An introduction to Homogenization and $G$-convergence.

#### What is homogenization?

- The act of replacing an exact microscopic model with macroscopic model and significantly less complexity.
- Typically (true uniformly elliptic PDE) micro $\Rightarrow$ elliptic PDE fine structure, marco model $\Rightarrow$ elliptic PDE same class, translation invariant (e.g. constant coefficients)

Micro:

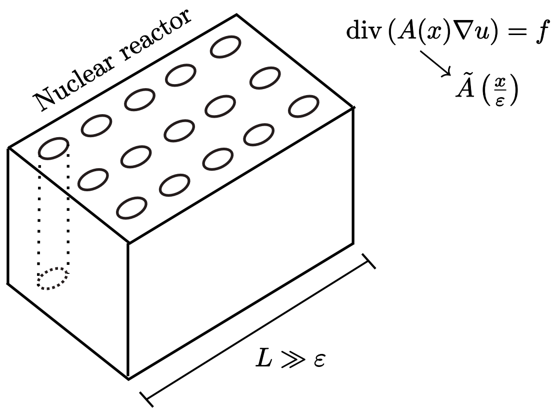
$$\begin{aligned} a_{ij}\left(\frac{x}{\varepsilon}\right) u^{\varepsilon}_{x_i x_j}(x) = f(x) &\quad \text{in } \Omega, \\ u^{\varepsilon} = g &\quad \text{on } \partial\Omega. \end{aligned}$$

Macro:

$$\begin{aligned} \bar{a}_{ij} \bar{u}_{x_i x_j}(x) = \bar{f}(x) &\quad \text{in } \Omega, \\ u^{\varepsilon} = g &\quad \text{on } \partial\Omega. \end{aligned}$$

**Why? (Origins of Homogenization)**

- Neutron transport / thermal effects (Babuska 1976)




- Flow in porous media

$$\operatorname{div}\left(A\left(x\right)\nabla u\right) = f$$

- Diffusion process (random or periodic)

Why?

- Computational reduction:
  - $10^5$ or more cells in each dimension, in each cell
  - $10^2$ (or more) gridpoints in each cell for numerics

  Therefore: exact simulation $\gg$ solve constant coefficient equation

- Model understanding
  - Elliptic micro $\longrightarrow$ Elliptic macro? $\longrightarrow$ Is there large scale canonical behavior? Model stable by change scale.
  - Elliptic behaviour? E.g. leanr something about material properties of model ($\bar{a}$)

Homogenization - Oscillations and regularity of $u^\varepsilon \Rightarrow$ Convergnce to effective limit
Simple example:

$$\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f$$

or

$$a_{ij}\left(\frac{x}{\varepsilon}\right)u_{x_i x_j}\left(x\right) = f\left(x\right)$$

Why oscillations?

$$u^{\varepsilon}(x) - a_{ij}\left(\frac{x}{\varepsilon}\right) u^{\varepsilon}_{x_i x_j}(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{in } \mathbb{R}^d,$$

where $a_{ij}$ $\mathbb{Z}^d$ periodic, $f$ is $\mathbb{Z}^d$ periodic.
Unique (viscosity) solution:

$$w^{\varepsilon}(x) = u^{\varepsilon}(x + \varepsilon z)$$

$$w^{\varepsilon}(x) - a_{ij}\left(\frac{x + \varepsilon z}{\varepsilon}\right) w^{\varepsilon}_{x_i x_j}(x) = f\left(\frac{x + \varepsilon z}{\varepsilon}\right),$$

where $z \in \mathbb{Z}^d$. $w^{\varepsilon}$, $u^{\varepsilon}$ solve the same equation (uniqueness) $\Rightarrow w^{\varepsilon} = u^{\varepsilon}$ $\varepsilon$-scale periodicty.

**Regularity?**

- (Not helpful) Schauder-estimates?

$$[u^{\varepsilon}]_{C^{\alpha}} \leq c\left[a_{ij}\left(\frac{\cdot}{\varepsilon}\right)\right]_{C^{\alpha}} + \cdots$$

- a-priori Hölder. Need estiamtes uniformly in $\varepsilon$.

$$[u^{\varepsilon}]_{C^{\alpha}} \leq c(\lambda, \Lambda)\, \|u^{\varepsilon}\|_{L^{\infty}} + \|f\|_{L^d}$$

  $u^{\varepsilon}$ Hölder independent of $\varepsilon$

$$|u^{\varepsilon}(x) - u^{\varepsilon}(x + y)| \leq c\,|y|^{\alpha} \leq c\varepsilon^{\alpha}$$

  By periodicity only need $|y| \leq \varepsilon \Rightarrow u^{\varepsilon} \to$ const. as $\varepsilon \to 0$.

  What constant?

  Unique $\bar{u}$? Relate to $\bar{f}$? $\longrightarrow$ Equation for $\bar{u}$?

$$u^{\varepsilon} - a_{ij}\left(\frac{x}{\varepsilon}\right) u^{\varepsilon}_{x_i x_j} = f\left(\frac{x}{\varepsilon}\right)$$

**1-d example completely**

$$\begin{cases} -\left(a\left(\frac{x}{\varepsilon}\right) u_{\varepsilon}'\right)' = f(x) & \text{in } (0,1), \\ u^{\varepsilon}(0) = u^{\varepsilon}(1) = 0 \end{cases} \xrightarrow{?} \begin{cases} -(\bar{a}\bar{u}')' = f & \text{in } (0,1), \\ \bar{u}(0) = \bar{u}(1) = 0 \end{cases}$$

$a$ is periodic, $0 < \lambda \leq a \leq \Lambda$.

Shouldn't we just recover

$$\text{constant } \bar{a} = m(a) = \int_0^1 a \, \mathrm{d}y \ ?$$

$a\left(\frac{\cdot}{\varepsilon}\right) \xrightarrow[L^\infty \text{ weak } *]{\varepsilon \to 0} m(a)$ means $\int a\left(\frac{\cdot}{\varepsilon}\right) g \, \mathrm{d}s \to \int m(a) g \, \mathrm{d}s$ for all $g \in L^1$.
Exact formula, $F' = f$

$$\begin{aligned}
a\left(\frac{x}{\varepsilon}\right) u_\varepsilon' &= F + c_1^\varepsilon \\
u_\varepsilon(x) &= \int_0^x \frac{-1}{a\left(\frac{s}{\varepsilon}\right)} \left(F(s) + c_1^\varepsilon\right) \, \mathrm{d}s + c_2^\varepsilon \\
u^\varepsilon &\xrightarrow{\varepsilon \to 0} \int_0^x m\left(\frac{-1}{a}\right) F(s) \, \mathrm{d}s + \bar{c}_1 \int_0^x m\left(\frac{-1}{a}\right) \mathrm{d}s
\end{aligned}$$

Use $x = 1$ $c_1^\varepsilon$ limit

$$0 = u^\varepsilon(1) = \underbrace{\int_0^1 \frac{-1}{a\left(\frac{s}{\varepsilon}\right)} F(s) \, \mathrm{d}s}_{\to \int_0^1 m\left(\frac{-1}{a}\right) F(s) \, \mathrm{d}s} + \int_0^1 \frac{1}{a\left(\frac{s}{\varepsilon}\right)} c_1^\varepsilon \, \mathrm{d}s$$

$$\Rightarrow c_1^\varepsilon \int_0^1 \frac{1}{a\left(\frac{s}{\varepsilon}\right)} \mathrm{d}s \to -\int_0^1 m\left(\frac{-1}{a}\right) F(\cdot) \, \mathrm{d}s$$

$$c_1^\varepsilon \to \bar{c}_1 = -\int_0^1 F \, \mathrm{d}s$$

$$\frac{1}{m\left(\frac{1}{a}\right)} \left(\bar{u}'\right) = F(x) + \bar{c}_1 \Rightarrow \begin{cases} \left(\frac{1}{m\left(\frac{1}{a}\right)} \bar{u}'\right)' = f(x) \\ \bar{u}(0) = \bar{u}(1) = 0 \end{cases}$$

but $\bar{a} = m\left(a^{-1}\right)^{-1} \neq m(a)$

**Viscosity solutions**

$$\begin{cases} a_{ij}(x) u_{x_i x_j}(x) = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

We need:

- Definitions of upper semicontinuous (USC) subsolutions, lower semicontinuous (LSC) supersolutions

- Comparison between USC subs. & LSC supers.
- Hölder regularity ($\lambda, \Lambda$ dependent)
- Half relaxed limits, e.g.

$$u_{\varepsilon}^{*}(x) = \lim_{\delta \to 0} \sup_{\varepsilon \leq \delta} \sup_{|x-y|<\varepsilon} u_{\varepsilon}(y)$$

Viscosity solutions stable w.r.t. $(\cdot)^{*}$!

Lectures 2,3,4

- Recap of intro
- Background viscosity solutions
- carefully periodic case $a_{ij}\left(\frac{x}{\varepsilon}\right) u_{x_i x_j}^{\varepsilon}(x) = f\left(\frac{x}{\varepsilon}\right)$

Recap:

- Informally

$$\begin{aligned} a_{ij}\left(\frac{x}{\varepsilon}\right) u_{x_i x_j}^{\varepsilon}(x) &= f(x) \quad \text{in } \Omega, \\ u^{\varepsilon} &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

$\to$ Where? Why? What expect?

(i) Expect $\varepsilon$-scale oscillations of $u^{\varepsilon}$ (e.g. special case $u^{\varepsilon}$ $\varepsilon\mathbb{Z}^d$ periodic)

(ii) $\varepsilon$-independent regularity (Hölder) for $u^{\varepsilon}$

(iii) (i) + (ii) $\Rightarrow u^{\varepsilon} \to \bar{u}$ (subsequence)
Goal: Descibe $\bar{u}$ via eq.

- Description $\bar{u}$
  - Independent of subsequence $\{u^{\varepsilon}\}$
  - eq. for $\bar{u}$ independent of subsequence $\{u^{\varepsilon}\}$ also of $g, \Omega$

Background: Viscosity solutions of

$$F\left(D^2 u, x\right) = f(x) \quad \text{in } \Omega \qquad ((eq)_F)$$

uniformly elliptic.

(F1)

$$F(P,s)=\inf_{a\in S}\sup_{A(x)\in\mathcal{A}_a}\operatorname{tr}(A(X)P),$$

where $\mathcal{A}_a\subset\mathcal{A}=\left\{A\in\operatorname{Sym}(d\times d)\,\middle|\,\lambda\operatorname{Id}\le A(x)\le\Lambda\operatorname{Id}\ \ \forall x\in\mathbb{R}^d\right\}$ and $S$ an arbitrary index set.

(F2)

$$A(x)=\sigma^T(x)\,\sigma(x)\quad\forall A\in\mathcal{A},$$
$$|\sigma(x)-\sigma(y)|\le c\,|x-y|$$

uniformly for all $A\in\mathcal{A}$.

**4.1 Definition** (Jet-Solution)**.** $u\in USC(\Omega)$ (upper semicontinuous) is a viscosity subsolution of $\left((eq)_{per\,F}\right)$ if for all possible $x_0$ and all matrices $A$ such that

$$u(y)\le u(x_0)+(y-x_0,A(y-x_0))+o\left(|y-x_0|^2\right)$$

it holds that

$$F(A,x_0)\ge f(x_0).$$

Sign convention: $\Delta u=f$, *not* $-\Delta u=f$.
Similary $v\in LSC(\Omega)$ is a viscosity supersolution of $\left((eq)_{per\,F}\right)$ if for all possible $x_0$ and all matrices $A$ such that

$$v(y)\ge v(x_0)+(y-x_0,A(y-x_0))+o\left(|y-x_0|^2\right)$$

it holds that

$$F(A,x_0)\le f(x_0).$$

**4.2 Definition** (Comparison)**.** $u\in USC(\Omega)$ ($u\in LSC(\Omega)$) is a viscosity subsolution (supersolution) of $\left((eq)_{per\,F}\right)$ if for every time there exists $\varphi\in C^2\left(\mathbb{R}^d\right)$ such that $u-\varphi$ attains global maximum (minimum) at $x_0$, it also holds

$$F\left(D^2\varphi(x_0),x_0\right)\overset{(\le)}{\ge}f(x_0).$$

Exercise 1: Prove that these are equivalent. See: Crandall-Ishi-Lions user's guide and Crandall-Evans-Lions (1984).

**4.3 Theorem** (Comparison)**.** *Assuming (F1), (F2) and* $f \in C\left(\overline{\Omega}\right)$. *If* $u$ *is* $USC$ *subsolution of* $F\left(D^2u(x), x\right) \geq f(x)$ *in* $\Omega$ *and if* $v$ *is* $LSC$ *supersolution of* $F\left(D^2u(x), x\right) \leq g(x)$ *in* $\Omega$, *then*

$$\sup_{\overline{\Omega}} (u - v) \leq \sup_{\partial\Omega} (u - v) + \sup_{\overline{\Omega}} (f - g).$$

*Proof.* Exercise 2. Minor modification of user's guide Theorem 3.3 and exercise 3.6 and section 5.6. See also Jensen 1988 ARMA, Jensen-Lions,Songanidiz.

□

Stability: $u_n$ sequence of $USC$, $v_n$ sequence $LSC$.

**4.4 Definition** (half-relaxed limit)**.**

$$(u_n)^*(x) = \limsup_{n\to\infty}{}^{*} u_n(x) := \lim_{n\to\infty} \sup_{\substack{j\geq n,\\ |x-y|\leq \frac{1}{j}}} (u_n(y)),$$

$$(v_n)_*(x) = \liminf_{n\to\infty}{}^{*} v_n(x) := \lim_{n\to\infty} \inf_{\substack{j\geq n,\\ |x-y|\leq \frac{1}{j}}} (v_n(y)).$$

**4.5 Proposition.** *If* $u_n$ *are* $USC$ *subsolutions to* $((eq)_{per\,F})$ *then* $\bar{u} := \limsup^*_{n\to\infty}$ *also is a subsolution of* $((eq)_{per\,F})$. *(user's guide Lemma 6.1)*

**4.6 Lemma.** $\overline{u} := \limsup^*_{n\to\infty} u_n$, $\underline{u} := \liminf^*_{n\to\infty} u_n$. *If* $\overline{u} \leq \underline{u}$ *in* $\Omega$, *then there exists* $u_0 \in C(\Omega)$ *such that* $u_n \to u_0$ *locally uniformly in* $\Omega$. *(user's guide Remark 6.4)*

## 4.2 Hölder regularity

Caffarelli-Cabre, Chapter 4

**4.7 Theorem** (Interior $C^\beta$)**.** *There exists universal* $c_0 > 1$ *and* $\beta \in (0,1)$ *such that if* $u$ *is viscosity solution simultaneously of (for* $f \geq 0$, $f \in C\left(\overline{B_1}\right)$*)*

$$M^+\left(D^2u\right) \geq -f \quad \text{and} \quad M^-\left(D^2u\right) \leq f \quad \text{in } B_1$$

*then* $[u]_{C^\beta\left(B_{\frac{1}{2}}\right)} \leq c_0 \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}\right)$.

Note: Straightforward extension to $\Omega' \Subset \Omega$, $c_0$ depends on $\operatorname{dist}(\Omega', \Omega)$.

**4.8 Proposition** (Global Hölder)**.** *Assume $\Omega$ satisfies uniform (in radius) exterior ball condition (say $B_r(x)$, $r \geq r_0$). If $u \restriction_{\partial\Omega} = \varphi \in C^3(\partial\Omega)$ and $u$ simultaneously solves in $\Omega$ same inequalities of Theorem 4.7 ($f \geq 0$, $f \in C(\overline{\Omega})$), then*

$$\|u\|_{C^\gamma(\overline{\Omega})} \leq Kc_0$$

*for universal $\gamma \in (0,1)$ whenever*

$$[\varphi]_{C^{B'}(\partial\Omega)}, \ \|f\|_{L^d(\Omega)} \leq K.$$

*For notation see CC. $u \in S^*(\lambda, \Lambda, |f|)$.*

Existence: Basically corollary of comparison (Theorem 4.3) and $\partial\Omega$ regularity (barriers) Proposition 4.8.

**4.9 Definition.** $\Omega$ satisfies exterior ball condition if $\forall x_0 \in \partial\Omega$ there exists $\hat{x}_0$, $B_r(\hat{x}_0) \subset \Omega^{\mathsf{C}}$ and $\overline{B_r(\hat{x}_0)} \ni x_0$. „$B_r(\hat{x}_0)$ tangent to $\partial\Omega$ at $x_0$".

**4.10 Theorem.** *Assume $\Omega$ has uniformly exterior ball condition, (F1), (F2), $f \in C(\overline{\Omega})$, $g \in C^{\beta'}(\partial\Omega)$. Then there exists a solution to ($(eq)_F$) which is $C(\overline{\Omega})$ and agrees with $g$ on $\partial\Omega$.*

*Idea of proof.* The key point is: Exercise 3: $w(x) := \sup(u(x)) \to u$ subsolution of ($(eq)_F$) and $w(x) \to u \leq g$ on $\partial\Omega$ is both a sub- and supersolution of ($(eq)_F$) in $\Omega$. For $w = g$ on $\partial\Omega$ use barriers (CC Prop. 4.12).

□

## 4.3 Careful treatment of periodic, unif. ellipt. non-divergence homogenization

$$\begin{aligned} a_{ij}\left(\frac{x}{\varepsilon}\right) u^{\varepsilon}_{x_i x_j}(x) &= f\left(\frac{x}{\varepsilon}\right) && \text{in } \Omega, \\ u^{\varepsilon}(x) &= g && \text{on } \partial\Omega \end{aligned} \qquad ((eq)_\varepsilon)$$

Note unique solution.

$$\begin{aligned} \bar{a}_{ij}\bar{u}_{x_i x_j} &= \bar{f} && \text{in } \Omega, \\ \bar{u} &= g && \text{on } \partial\Omega \end{aligned} \qquad ((\overline{eq}))$$

Exercise 4: Find some $f\colon \mathbb{R} \to \mathbb{R}$ such that $f$ is $C^1$ but fails the exterior ball condition at $x = 0$. $\Omega = \{(x,y) \mid y < f(x)\}$.

**Assumptions.**

(a0) $\Omega \subset \mathbb{R}^d$ open, bounded, uniformly exterior ball condition

(a1) $a\colon \mathbb{R}^d \to \operatorname{Sym}(d \times d)$ such that $a(x) = \sigma^T(x)\,\sigma(x)$

(a2) $\exists\, 0 < \lambda < \Lambda$ s. th.

$$\lambda\,\mathrm{Id} \leq a(x) \leq \Lambda\,\mathrm{Id} \quad \forall\, x \in \mathbb{R}^d$$

(a3) $g \in C^{\beta'}(\partial\Omega)$

(a4) $a$, $f$ both $\mathbb{Z}^d$ periodic

(a5) $f \in C\left(\overline{\Omega}\right)$

Note $a = (a_{ij})$.

**Remark.** *(a0)-(a3)+(a5) basically uniqueness (Theorem 4.3) only!*
*(a4) for homogenization.*

**4.11 Theorem** (Bensoussan-Lions-Papanicolaou)**.** *Assume (a0)-(a5). Then there exists a unique $\bar{a} \in \operatorname{Sym}(d \times d)$ with $\lambda\,\mathrm{Id} \leq \bar{a} \leq \Lambda\,\mathrm{Id}$ which is independent of $g$ and $\Omega$ such that $\underline{u^\varepsilon \to u}$ locally uniformly in $\Omega$, where $\bar{u}$ is a unique solution to $((\overline{eq}))$.*

**Remark.** *Uniform Hölder estimates always subsequence*
*$u^\varepsilon \to \bar{u}$ and $\bar{u}$ will have an eq. $\bar{a}$*
*$\to$ do this independent of subsequence!*
*$\to$ convergence of whole sequence!*

Incorrect, but useful:
Expect (hope)

$$u^\varepsilon(x) = \bar{u}(x) + \varepsilon w_1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(\frac{x}{\varepsilon}\right) + \text{H.O.T (in } \varepsilon)$$

into $((eq)_\varepsilon)$

$$a_{ij}\left(\frac{x}{\varepsilon}\right)\bar{u}_{x_i x_j}(x) + \frac{1}{\varepsilon}a_{ij}\left(\frac{x}{\varepsilon}\right)w_{1_{x_i x_j}}\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)w_{2_{x_i x_j}}\left(\frac{x}{\varepsilon}\right) = f\left(\frac{x}{\varepsilon}\right)$$

make sense of equation above for all $\varepsilon \to 0$
$\to w_1$ should be affine $c_0 + px$

$$\varepsilon w_1\left(\frac{x}{\varepsilon}\right) = \varepsilon c_0 + px$$

(remove (up to $\varepsilon c_0$) by change of unknown)

$$u^\varepsilon (x) = \bar{u} (x) + \varepsilon^2 v \left(\frac{x}{\varepsilon}\right) + \text{H.O.T}$$

ideally, make $a_{ij} \left(\frac{x}{\varepsilon}\right) \bar{u}_{x_i x_j} (x) + a_{ij} \left(\frac{x}{\varepsilon}\right) v_{x_i x_j} \left(\frac{x}{\varepsilon}\right) = f \left(\frac{x}{\varepsilon}\right) \forall x, \varepsilon$

$$Lu^2 = \text{const.} = f \left(\frac{x}{\varepsilon}\right)$$

goal??

$$a_{ij} (y) \underbrace{\bar{u}_{x_i x_j} (x)}_{=P_{ij}} + a_{ij} (y) v_{y_i y_j} (y) = f (y)$$

Give $P \in \text{Sym} (d \times d)$ can you find a periodic $v$ solving

$$a_{ij} (y) P_{ij} + a_{ij} (y) v_{y_i y_j} (y) = F$$

Q1: $P$ fixed. Given $F$ periodic there are very few times when such $v$ exists!
To come $m$ invariant measure for

$$\begin{aligned} & a_{ij} (y) \quad \text{then } v \text{ exists} \\ \Leftrightarrow & \int_Q (F (y) - a_{ij} (y) P_{ij}) \ m (y) \ \mathrm{d}y = 0 \end{aligned}$$

Fredholm's Alternative (Evans PDE Appendix D.5)
$H$ is Hilbert space, $K$ linear and compact $H \to H$

i) $\text{Null} (I - K)$ finite dim.

ii) $\text{Ran} (I - K)$ clsoed

iii) $\text{Ran} (I - K) = \text{Null} (I - K^*)^\perp$

iv) $\text{Null} (I - K) = \{0\} \ \Leftrightarrow \ \text{Ran} (I - K) = H$

v) $\dim \text{Null} (I - K) = \dim \text{Null} (I - K^*)$

$\to$ severe restrictions on possibility of given $F$, $\exists\, v$ solving $((eq)_{per\,F})$, where

$$a_{ij} (y) v_{y_i y_j} (y) = F (y) \quad \text{in } \mathbb{R}^d, \ a_{ij}, F \text{ periodic} \qquad ((eq)_{per\,F})$$

Use Fred. what should $K$ be? For $\sigma > 0$ large enough, there exists a unique $v_\sigma \, (= v)$

$$-\sigma v_\sigma + L v_\sigma = F$$

with $L = a_{ij}(y)\, v_{y_i y_j}(y)$ and $W^{2,p}$ estiamtes $\forall\, p > 1$ specifically $p = 2$ (GT 9.14)
G.T. 9.14, $p = 2 \Rightarrow \exists$ universal constant there exists unique $v_\sigma$ solving

$$-\sigma v_\sigma + L v_\sigma = F$$

and

$$\|v_\sigma\|_{W^{2,2}(Q)} \leq c \|L v_\sigma - \sigma v_\sigma\|_{L^2(Q)}$$

$H = L^2$. Assume $v_\sigma \in L^2$

$$\sigma v \underbrace{-\sigma v + L v}_{= L_\sigma} = F$$
$$\Leftrightarrow L_\sigma v = F - \sigma v$$

$$v = L_\sigma^{-1}(F - \sigma v) = L_\sigma^{-1} F - L_\sigma^{-1} \sigma v = L_\sigma^{-1} F - \sigma L_\sigma^{-1} v$$
$$v + \underbrace{\sigma L_\sigma^{-1} v}_{K} = L_\sigma^{-1} F = h \in L^2$$

$$(I + K)_v = L_\sigma^{-1} F = h$$
$$\Leftrightarrow L v = F$$

plus extra computations global comparison + periodicity
G.T. 9.4 $\Rightarrow K$ compact $L^2 \to L^2$
What does Fred. say about this $K$?

**4.12 Lemma.** *Assume (a1), (a2), (a4). There exists an unique* $m \in L^2_{per}$*, where* $L^2_{per}(Q) = \left\{ u \in L^2(Q) \,\middle|\, u \text{ is periodic} \right\}$*, with* $m > 0$*,* $\int_Q m \, \mathrm{d}y = 1$*,*

$$\left(a_{ij}(y)\, m(y)\right)_{y_i y_j} = 0.$$

Either exercise or give proof later.

**4.13 Lemma.** *Assume (a1), (a2), (a4). Given* $F \in L^2_{per}$ *there exists a solution* $v \in L^2_{per}$ *of* $((eq)_{per\,F}) \Leftrightarrow \int_Q mF \, \mathrm{d}y = 0$

*Proof of both Lemmas.* Exercise 6: Show that formal $L^2$ adjoint of $K$ is $K^*$, $K^*G = u$ whenever

$$-\sigma u + \left(a_{ij}\left(y\right)u\left(y\right)\right)_{y_iy_j} = G$$

□

**4.14 Lemma.** *$v \in L^2_{per}$, $Lv = 0 \Leftrightarrow v =$ const.*

*Proof.* Exercise 7 or later. □

Back to Fred. $K$ is linear.

$$\begin{aligned} \frac{1}{\sigma}K\left(0\right) = 0 \quad &\Leftrightarrow \quad v \text{ const.} \\ \Rightarrow \dim \operatorname{Null}\left(I - \left(-K\right)\right) &= 1 \\ \dim \operatorname{Null}\left(I - \left(-K^*\right)\right) &= 1 \end{aligned}$$

note $\left(I - \left(-K^*\right)\right)m = 0$ $\left(m \in \operatorname{Null}\left(I - \left(-K^*\right)\right)\right) \Leftrightarrow \left(a_{ij}\left(y\right)m\left(y\right)\right)_{y_iy_j} = 0$, $m \in L^2_{per}$
$\Rightarrow$ as long as $m_0$ such that

$$\left(a_{ij}\left(y\right)m_0\left(y\right)\right)_{y_iy_j} = 0 \quad \text{and} \quad \int_Q m_0 \neq 0$$

then $\operatorname{Null}\left(I - \left(-K^*\right)\right) = \operatorname{span}\left(m_0\right)$. Proof of Lemma 4.12 done up to fact $\exists\, m > 0$ in $Q$.
Known: Invariant measures, stochastic processes.
By Fred. (iii) $\exists\, v$ such that

$$\begin{aligned} & v + Kv = \frac{1}{\sigma}KF \quad (= h) \\ & \Leftrightarrow \frac{1}{\sigma}KF \;\perp\; \operatorname{Null}\left(I + K^*\right) = \operatorname{span}\left(m\right) \\ & \Leftrightarrow \int \frac{1}{\sigma}K\left(F\right)m \,\mathrm{d}y = 0 \\ & \Leftrightarrow \int K\left(F\right)m \,\mathrm{d}y = 0 \\ & \Leftrightarrow \int F\left(K^*m\right)\,\mathrm{d}y = 0 \\ & \Leftrightarrow -\sigma \int F\left(m\,\mathrm{d}y\right) + \int F\left(a_{ij}\left(y\right)m\left(y\right)\right)_{y_iy_j}\,\mathrm{d}y = 0 \\ & \Leftrightarrow \int Fm \,\mathrm{d}y = 0 \\ & \Rightarrow \text{Lemma 4.13 complete} \end{aligned}$$

What did we learn about original expansion?

$$\text{"}u^{\varepsilon}(x) = \bar{u}(x) + \varepsilon^2 w\left(\frac{x}{\varepsilon}\right) + \text{H.O.T."}$$

(incorrect) wanted $w$ to solve

$$a_{ij}(x)\, w_{y_i y_j}(y) = f(y) - a_{ij}(y)\, P$$

$$P = D^2\bar{u}(x), \quad x \text{ fixed}$$

Lemma 4.13$\Rightarrow$ not going to happen!
$\rightarrow$ need help!

$$\Theta(P) = -\int f m \,\mathrm{d}y + \int a_{ij}(y)\, P_{ij} m \,\mathrm{d}y$$

$\Rightarrow$ good by Lemma 4.13.
Given any $x$ fixed, $D^2\bar{u}(x)$ fixed.
locally $u^2$ does look like $(|x-y| \ll 1)$ , $P = D^2\bar{u}(x)$,

$$u^2(y) = \bar{u}(y) + \varepsilon^2 w_p\left(\frac{x}{\varepsilon}\right)$$

$w_p$ solves

$$a_{ij}(y)\, w_{p_{y_i y_j}}(y) = f(y) - a_{ij}(y)\, P_{ij} + \Theta(P) \qquad ((eq)_{cor\,P})$$

**4.15 Proposition** ("True" corrector eq.)**.** *Given $P \in \mathrm{Sym}(d \times d)$ fixed, there exists a unique scalar $\Theta(P)$ such that $((eq)_{cor\,P})$ admits a periodic solution $w_p$.*

**Remark.** *Schauder $\rightarrow W^{2,p}$, $W_p$ actually $C^2$*

Proof of 4.15 contained in all of above work.

(1) Fix / modify $H$ last time

(2) Using Prop. 4.15 (correct eq)
Give convergence $u^{\varepsilon} \rightarrow \bar{u}$ (finishes Theorem 4.11 linear case)

(3) Redo Theorem 4.11 for nonlinear $F$

Why linear case?
See precisely solvability of $((eq)_{per\,F})$ with compatibility condition

$$\int (f(y) - a_{ij}(y)\, P_{ij})\, m(y)\,\mathrm{d}y - \Theta(P) = 0$$

Fredholm Alternative
Finding good $v$ (depending on $P$)

$$u^{\varepsilon}(x) \text{"} = \text{"} \bar{u}(x) + \varepsilon^2 u\left(\frac{x}{\varepsilon}\right)$$

useful in neighbourhood of $x \to$ compatibility condition using $m$

$$(a_{ij}(y)\, m(y))_{y_i y_j} = 0 \quad \text{(all info for } a_{ij})$$

<u>Main goal last time</u>

**4.16 Proposition** (True corrector)**.** *Give $P \in \mathrm{Sym}(d \times d)$ fixed $\left(P = D^2 u(x)\right)$, there exists a unique scalar $\Theta(P)$ such that ($(eq)_{cor\,P}$) admits a periodic $W^{2,2}_{per}$ solution $v$, actually assumptions on a*

$\Rightarrow v \in C^{2,\alpha}$ By Lemma 4.13, $\Theta(P) := -\underbrace{\int f\, m\, \mathrm{d}y}_{:=\Theta_1} + \underbrace{\int a_{ij}(y)\, P_{ij}\, m(y)\, \mathrm{d}y}_{:=\Theta_2} \Rightarrow$ unique

$\Theta$
Via Fredholm Alternative for (correction) $H = W^{2,2}_{per}$ (instead of $L^2_{per}$)

$$KF = V \quad \Leftrightarrow \quad \overbrace{\underset{L = a_{ij}(y) v_{y_i y_j}}{-\sigma v + Lv}}^{= L\sigma} = F$$

and

$$KF = L_{\sigma}^{-1} \colon W^{2,2}_{per} \to W^{2,2}_{per}$$

Assume on $a$ $C^{1,1}\left(\overline{Q}\right)$, where $\overline{Q}$ unit cell
Theorem 9.19 (GT) $\Rightarrow \|v\|_{W^{4,2}} = C\|F\|_{W^{2,2}} \Rightarrow K$ compact
Replace $L^2_{per}$ by $W^{2,2}_{per}$ in ($(eq)_{per\,F}$)
„solution" strong unique $W^{2,2}$ solution Chapter 9 GT
$a_{ij}(y)\, v_{y_i y_j}(y) = F(y)$ in $\mathbb{R}^d$ periodic
Fredholm Alternative given $F \in W^{2,2}_{per}$

$$\exists\, v \text{ solving } ((eq)_{per\,F}) \quad \Leftrightarrow \quad \int F(y)\, m(y)\, \mathrm{d}y = 0$$

$m$ is unique inverse measure for $a$ appearing Lemma 4.12. $m > 0$, $\int m = 1$, $(a_{ij}(y)\, m(y))_{y_i y_j} = 0$

make heuristic expansion

$$\underbrace{u^{\varepsilon}(x) = \bar{u}(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}\right)}_{\text{not true}}$$

Finish Theorem 4.11

Observation $P \mapsto \Theta_2(P)$ is linear, $\mathrm{Sym}(d \times d) \to \mathbb{R} \overset{\text{Riesz}}{\Rightarrow} \exists\, \bar{a} \in \mathrm{Sym}(d \times d)$, $P \mapsto \Theta_2(P)$ represented, $P \mapsto \mathrm{tr}(\bar{a}P)$.

**4.17 Proposition.** *If $\bar{a}$ is unique matrix representing $P \mapsto \Theta_2(P)$, e.g. $\Theta_2(P) = \mathrm{tr}(\bar{a}P)$ then $\bar{a} \in \mathrm{Sym}(d \times d)$ and $\lambda\,\mathrm{Id} \le \bar{a} \le \Lambda\,\mathrm{Id}$ (wait on Prop 4.17)*

**4.18 Proposition.** *If $\bar{u}$ solves (($\overline{eq}$)) with $\bar{a}$ from Propositions 4.15, 4.17, then $u^{\varepsilon} \to \bar{u}$ locally uniformly in $\Omega$.*

**Remark.** *Result appears in BLP (book) but not same proof. This proof Lions-Papanicoloau-Varandhan first order Hamilton Jacobi, appears in Evans '92.*

Extra fact of viscosity solutions:

**4.19 Definition** (Strict comparison)**.** Replace phrases in Def (comparison) by

$$\text{"}u - \varphi \text{has (strict) global max"},$$
$$\text{"}v - \varphi \text{has (strict) global min"}$$

**4.20 Lemma.** *All 3 definitions of viscosity solutions are equivalent.*

Exercise: Crandall-Evans-Lions + user's guide

**Remark.** *If $u$ is viscosity solution and $C^2$, then it is classical (consistency of weak solution)*

*Proof of 4.18.* First show $(u^{\varepsilon})^*$ is subsolution (viscosity) of (($\overline{eq}$)), by contradiction. Suppose $\exists\, \varphi$ smooth and $x_0 \in \Omega$ such that $u - \varphi$ has strict global max at $x_0$ but equation fails, e.g.

$$\mathrm{tr}\left(\bar{a}D^2\varphi(x)\right) \le -\delta + \bar{f}$$

$$F = \int f(y)\, m(y)\, \mathrm{d}y \quad (= -\Theta_1(P))$$

(($\overline{eq}$)) is $\bar{a}_{ij}\bar{u}_{x_i x_j}(x) = \bar{f}$. Contradiction must use $u^{\varepsilon}$, $((eq)_{\varepsilon})$ if $v_p$ is a periodic solution to $((eq)_{cor\,P})$, then we can make

$$w^{\varepsilon}(x) = \varphi(x) + \varepsilon^2 v_p\left(\frac{x}{\varepsilon}\right)$$

a supersolution of $((eq)_\varepsilon)$ on some small $B_{r_0}(x_0)$, $r_0$ fixed depending on $\varphi$. Convenient that $v_p$ is $C^2$ by Schauder. but not neccessary (see argument in linear case).

$$L\left(\varphi+\varepsilon^2 v_p\left(\frac{x}{\varepsilon}\right)\right)=a_{ij}\left(\frac{x}{\varepsilon}\right)\quad(x)+\underbrace{a_{ij}\left(\frac{x}{\varepsilon}\right)v_{p_{x_ix_j}}\left(\frac{x}{\varepsilon}\right)}_{\text{classically}}$$

$$=f\left(\frac{x}{\varepsilon}\right)-a_{ij}\left(\frac{x}{\varepsilon}\right)P_{ij}-\underbrace{\bar f}_{=-\Theta_!(P)}+\underbrace{\Theta_2(P)}_{=\mathrm{tr}(\bar aP)\le-\delta+\bar f}$$

$$a_{ij}\left(\frac{x}{\varepsilon}\right)\varphi_{x_ix_j}(x)-a_{ij}\left(\frac{x}{\varepsilon}\right)P_{ij}\le\|a\|_{L^\infty}\qquad\underbrace{\left|D^2\varphi(x)-P_{ij}\right|}_{\text{choose }c_0\text{depending on }\varphi}\qquad<\frac{\delta}{2}$$

$B_{r_0}(x_0)\subset\Omega$. Collect terms

$$\begin{aligned}Lw^\varepsilon(x)&=a_{ij}\left(\frac{x}{\varepsilon}\right)\left(\varphi_{x_ix_j}(x)-P_{ij}\right)+f\left(\frac{x}{\varepsilon}\right)-\bar f+\mathrm{tr}(\bar aP)\\&\le\frac{\delta}{2}+f\left(\frac{x}{\varepsilon}\right)-\bar f+\bar f-\delta\quad\text{in }B_{r_0}(x_0)\\&=f\left(\frac{x}{\varepsilon}\right)-\frac{\delta}{2}\quad\text{in }B_{r_0}(x_0)\end{aligned}$$

Comparison with $u^\varepsilon$ (solution)

$$\underline{\sup}_{B_{r_0}(x_0)}u^\varepsilon-w^\varepsilon\le\sup_{\partial B_{r_0}(x_0)}u^\varepsilon-w^\varepsilon$$

$$\sup\left(u^\varepsilon-\varphi-\varepsilon^2 v\left(\frac{\cdot}{\varepsilon}\right)\right)\le\sup_{\partial B_{r_0}(x_0)}u^\varepsilon-\varphi-\varepsilon^2 v\left(\frac{\cdot}{\varepsilon}\right)$$

Will contradict $u-\varphi$ strict max

$$u^\varepsilon(x_0)-\varphi(x_0)-\varepsilon^2 u\left(\frac{x_0}{\varepsilon}\right)\le\sup_{\partial B_{r_0}(x_0)}u^\varepsilon-\varphi-\varepsilon^2 v\left(\frac{\cdot}{\varepsilon}\right)$$

note $\varepsilon^2 v\left(\frac{\cdot}{\varepsilon}\right)\to 0$ uniformly by periodicity (true if $v$ subquadratic growth at $\infty$)
$(\ )^*$ limit pass inside sup
(apply $(\ )^*$ both sides)

$$\left(u^\varepsilon\right)^*(x_0)-\varphi(x_0)\le\sup_{\partial B_{r_0}(x_0)}\left(\left(u^\varepsilon\right)^*-\varphi\right)$$

contradiction to strict max

$$\sup_{\partial B_{r_0}(x_0)} \left((u^\varepsilon)^* - \varphi\right) < (u^\varepsilon)^*(x_0) - \varphi(x_0)$$

$\Rightarrow (u^\varepsilon)^*$ is subsolution of $((\overline{eq}))$ in $\Omega$ (have not commented on $\partial\Omega$ yet!!!)
Revised all arguments.... $(u^\varepsilon)_*$ supersolution of $((\overline{eq}))$ in $\Omega$. Comparison of viscosity solutions ($\bar{a} \geq 0$)

$$\begin{aligned}
&\Rightarrow \sup_{\Omega} (u^\varepsilon)^* - (u^\varepsilon)_* \leq \sup_{\partial\Omega} \left((u^\varepsilon)^* - (u^\varepsilon)_*\right) = 0 \\
&\Rightarrow (u^\varepsilon)^* \leq (u^\varepsilon)_* \leq (u^\varepsilon)^* \quad \text{(first by equations, secondly by def.)} \\
&\Rightarrow (u^\varepsilon)^* = (u^\varepsilon)_* = \bar{u} \quad \leftarrow \ \bar{u} \text{is a solution}
\end{aligned}$$

Done!

□

*Proof of 4.17.* Only piece left is $\lambda\,\mathrm{Id} \leq \bar{a} \leq \Lambda\,\mathrm{Id}$. This is true $\Leftrightarrow$

$$\underbrace{\lambda\,\mathrm{tr}(S) \leq \mathrm{tr}(\bar{a}S)}_{\text{we will show this}} \leq \Lambda\,\mathrm{tr}(S) \quad \forall\, S \text{ sym. } \geq 0$$

By contradiction, suppose it fails, so $\exists\, S \geq 0$ such that

$$\mathrm{tr}(\bar{a}S) < \lambda\,\mathrm{tr}(S)$$

Only have $\left((eq)_{cor,?}\right)$ at our disposal! Look at $v_0$ & $v_s$ solving $\left((eq)_{cor,0}\right)$, $\left((eq)_{cor,S}\right)$ eq. for $v_0 - v_s$?

$$\begin{aligned}
v_0 &\sim a_{ij}(y)\, v_{0_{y_i y_j}}(y) = f(y) + \Theta_1(0) + \Theta_2(0) - a_{ij}(y)\, 0 \\
v_s &\sim a_{ij}(y)\, v_{s_{y_i y_j}}(y) = f(y) + \Theta_1(0) + \Theta_2(S) - a_{ij}(y)\, S_{ij} \\
&\sim a_{ij}(y)\, v_{s_{y_i y_j}}(y) < f(y) + \Theta_1(S) + \underbrace{\Theta_2(S)}_{=\mathrm{tr}(\bar{a}S)} - \mathrm{tr}(\bar{a}S)
\end{aligned}$$

Subtract:

$$a_{ij}(y)\,(v_0 - v_s)_{y_i y_j}(y) > 0$$

($v_0 - v_s$ $C^2$)
$\Rightarrow v_0 - v_s$ can't attain a loc max
$\Rightarrow$ contradiction because $v_0 - v_s$ periodic

□

Lecture 5

- Comments & clarifications for invariant measures from Lemma 4.12
- Fully nonlinear equation (periodic) (completely includes Theorem 4.11)

Recall set-up Fredholm Alternative
$\sigma > 0$ large enough such that

**4.21 Lemma.** *$\sigma \geq \sigma_0$ (universal), givem $F \in L^2_{per}$, $\exists$ unique $v \in W^{2,2}_{per}$ such that $-\sigma v + Lv = F$.*

Recall $Lv = a_{ij}(y) v_{y_i y_j}(y)$
G.T. 9.14 (adapt to periodic case)
Universal $c$: $\|v\|_{W^{2,2}_{per}} \leq c\| - \sigma v + Lv\|_{L^2_{per}}$ $(\sigma \geq \sigma_0)$
We're looking for possibility to solve

$$Lv = F$$

Fredholm Alternative $H = W^{2,2}_{per}$

$$\begin{aligned} L_\sigma v &:= -\sigma v + Lv \\ L_\sigma v &= F \quad \text{uniquely } F \in L^2_{per} \ \left(F \in W^{2,2}_{per}\right) \end{aligned}$$

$K := L_\sigma^{-1}$ compact by $W^{2,2}$ estimates (GT 9.14 modified)
$K^*$ corresponding operator for $L^*_\sigma$ $(-\sigma v + L^* v)$

$$\text{Fred.} \Rightarrow \begin{cases} \operatorname{Ran}(I - (-K)) = \operatorname{Null}(I - (-K^*))^\perp \\ \dim \operatorname{Null}(I - (-K)) = \dim (I - (-K^*)) \end{cases}$$

$$\begin{aligned} L^* m = F \Leftrightarrow (I + K^*)\, m &= \frac{1}{\sigma} K^* F \\ Lv = F \Leftrightarrow \quad (I + K)\, v &= \frac{1}{\sigma} K F \end{aligned}$$

Two key components were Lemma 4.12, 4.14. Conclusion

$$\begin{pmatrix} Lv = F \\ \text{there exists } v \end{pmatrix} \Leftrightarrow \int F\, m \, \mathrm{d}y = 0$$

**4.22 Lemma.** *$\sigma \geq \sigma_0$, $v \in W^{2,2}_{per}$, $-\sigma v + Lv = 0 \;\Leftrightarrow\; v \equiv 0$ (immediate consequence Lemma 4.21, $-\sigma v - Lv = F = 0$)*

*Proof of 4.12.* Fredh. + Lemma 4.14 implies sol. space of $L^* m = 0$ is 1-dim

$$(\mathrm{Null}\,(I - (-K)) = \mathrm{span}\,(1))$$

Claim, $\exists\, m$ solving $L^* m = 0$ , $m > 0$ in $Q$ $\left(m \in W^{2,2}_{per}\right)$

$$\begin{aligned}
&-\sigma v + Lv = 0 \quad \Leftrightarrow \quad v \equiv 0 \\
&\Leftrightarrow Lv = \sigma v \\
&\overset{\text{Fred}}{\Leftrightarrow} \sigma v \perp m \quad (m \in \mathrm{Null}\,(I - (-K^*)) = \mathrm{span}\,(m)) \\
&\Leftrightarrow \int \sigma v m = 0
\end{aligned}$$

Observatio: $\forall\, f \in W^{2,2}_{per}$, $\int vm = 0 \;\Leftrightarrow\; v \equiv 0$ ? verify next lecture

□

**Fully nonlinear eq.**

Was:

$$F\,(P, x) = \inf_{\alpha \in S} \sup_{A \in \mathcal{A}_\alpha} (\mathrm{tr}\,(A \otimes P))$$

$$f^q_A\,(x) \in C\,(\overline{\Omega})$$

$$\inf \sup \left(\mathrm{tr}\,(A\,(x))^2\, u\,(x)\right) = f\,(x)$$

less general

$$\inf \sup \left(-f^q_A\,(x)\,\mathrm{tr}\,(A\,(x))^2\, u\,(x)\right) = 0$$

$$\mathrm{tr}\left(A\left(\frac{x}{\varepsilon}\right) D^2 u\,(x)\right) + f\left(\frac{x}{\varepsilon}\right)$$

$$\bar{F}\,(P) = \Theta_2\,(P) = -\Theta_1\,(P)$$

$$\Theta\,(P) = \int a_{ij}\,(y)\, P_{ij}\, m\,(y)\; \mathrm{d}y - \int f\, m\, \mathrm{d}y$$

Uniform ellipticity

$$\begin{aligned}\mathcal{M}^+(P) &= \sup_{\lambda Id \le A \le \Lambda Id} (\operatorname{tr}(AP)) \\ &= \Lambda \sum_{e_i \ge 0} e_i(P) + \lambda \sum_{e_i < 0} e_i(P)\end{aligned}$$

$$\mathcal{M}^-(P-Q) \le F(P,x) - F(Q,x) \le \mathcal{M}^+(P-Q) \quad \forall\, P,Q \in \operatorname{Sym}(d \times d)$$

$$\lambda \operatorname{tr}(S) \le F(P+S) - F(P) \le \Lambda \operatorname{tr}(S) \quad \forall\, S \ge 0,\ S \in \operatorname{Sym}(d \times d)\ \forall\, P \in \operatorname{Sym}(d \times d)$$

**4.23 Theorem** (Evans 92)**.** *There exists unique $\bar{F}\colon \operatorname{Sym}(d \times d) \to \mathbb{R}$ which is uniformly elliptic with same $\lambda, \Lambda$ such that $\forall\, g \in C^\beta(\partial\Omega)$, $u^\varepsilon \to \bar{u}$ loc. uniform. in $\Omega$ ($u^\varepsilon$, $\bar{u}$ solve $((eq)_\varepsilon)$, $((\overline{eq}))$).*

Basically Lions-Papanicolav-Varandhan unpublished applied + 2nd order eq.
LPV did $u_t + H(Du, x) = 0$
Some ideas as linear case... expansion

$$u^\varepsilon(x) = \bar{u}(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}\right) + H.O.T. \text{ (in } \varepsilon)$$

plug into $((eq)_\varepsilon)$

$$F\left(D^2\bar{u}(x) + D^2 v\left(\frac{x}{\varepsilon}\right), \frac{x}{\varepsilon}\right) = 0$$

For $v$ to be helpful, we need this „indep. of $\varepsilon$" e.g. = constant
Seperate $x, \frac{x}{\varepsilon} = y$
We can solve (given $P$)

$$F\left(P + D^2 v(y), y\right) = 0$$

in $\mathbb{R}^d$ for periodic $v$ in viscosity sense ($\to$ typically no!)
help by $\Theta(P)$

**4.24 Proposition.** *There exists a unique scalar $\Theta(P)$ such that*

$$F\left(P + D^2 v(y), y\right) = \Theta(P) \quad \text{in } \mathbb{R}^d \qquad ((eq)_{cor,F,P})$$

*admits as least one periodic viscosity solution $v_0$*

Previously did special case when

$$F\left(D^2 u, x\right) = \operatorname{tr}\left(A(x) D^2 u(x)\right) - f(x)$$

Comments for building proof of 4.24: Recall in linear case used Fredh. needed auxillary operator

$$-\sigma v + Lv$$

get good:
→ uniqueness eq in $\mathbb{R}^d$
→ $W^{2,2}$ estimates ⇒ compactness
→ able get $\Theta(P)$ as unique compatibility condition
note: $P$ fixed

$$F\left(P + D^2 v, y\right) = 0 \quad \text{in } \mathbb{R}^d$$

does not have unique solutions

$$-v^2 + F\left(P + D^2 v^\varepsilon, \frac{y}{\varepsilon}\right) = 0 \quad \text{in } \mathbb{R}^d$$

does! (bounded, „subquadratic")
Sake of motivation $P = 0$
→ $w^\varepsilon(y) = \frac{1}{\varepsilon^2} w(\varepsilon y)$

$$-\varepsilon^2 w + F\left(D^2 w^\varepsilon, \frac{\varepsilon x}{\varepsilon}\right) = 0$$

or $\alpha = \varepsilon^2$

$$-\alpha w^\alpha + F\left(D^2 w^\alpha, x\right) = 0 \quad \text{in } \mathbb{R}^d \qquad ((eq)_\alpha)$$

$$F(0, y) = 0$$
$$\bar{F}(0) = 0$$

$w^\alpha$ periodic + $C^\beta$ estimates

$$\mathcal{M}^+ D^2 w^\varepsilon \geq -c$$
$$\mathcal{M}^- D^2 w^\varepsilon \leq +c$$

+ periodic
⇒ $w^\varepsilon$ →constant
at least on subsequence
can we say about all subseq.

*Proof of 4.24.* Let $\alpha > 0$ be fixed & $w^\alpha$ solve $((eq)_\alpha)$

By uniqueness $w^\alpha$ is periodic
($\tilde{w}^\alpha = w^\alpha (\cdot + z)$ $z \in \mathbb{Z}^d$ solves exactly same $((eq)_\alpha)$)
exercise: Write all details for this claim carefully using def of viscosity solution
goal: $\alpha w^\alpha (0) \to$ const. $(:= \Theta (P))$
note: Want to take limits on $w^\alpha$, but $w^\alpha$ typically unbounded in $\alpha$!
step 1: $v^\alpha = w^\alpha - w^\alpha (0)$ is bounded unif. in $\alpha$
step 2: Hölder estimates + stability of solutions $v^\alpha \to v$, $v$ must solve

$$F\left(P + D^2 v(y), y\right) = \lim_{\alpha \to 0} \alpha w^\alpha (0)$$

step 1: By contradiction

$$C_\alpha = \| w^\alpha - w^\alpha (0) \|_{L^\infty} = \| v^\alpha \|_{L^\infty} \xrightarrow{\alpha \to 0} \infty$$

$$-\alpha v^\alpha + F\left(P + D^2 v^\alpha (y), y\right) = \alpha w^\alpha (0)$$

Let $V^\alpha = \frac{1}{C_\alpha} v^\alpha$

$$-\alpha V^\alpha + \tilde{F}\left(\frac{1}{C_\alpha} P + D^2 v^\alpha (y), y\right) = \frac{\alpha w^\alpha (0)}{C_\alpha}$$

$$\tilde{F}(Q, y) = \inf \sup \left(\frac{f_A^a}{C_\alpha} + \operatorname{tr}(A(x) Q)\right)$$
$$F(Q, y) = \inf \sup \left(f_A^a (x) + \operatorname{tr}(A(x) Q)\right)$$

Same arguments as above

$$\mathcal{M}^+ \left(D^2 V^\alpha\right) \le C$$

$\Rightarrow$ Hölder for $V^\alpha$
W.l.o.g. let $\alpha$ be subseq.
$V^\alpha \to V$ loc. unif.
viscos. sol. stable w.r.t. loc. unif. convergence

$$F\left(D^2 V(y), y\right) = 0 \quad \text{in } \mathbb{R}^d$$

Liouville thm $\Rightarrow V =$ const.

$$\max(V) = 1 \quad \left(V^\alpha\left(x_\alpha^+\right) = 1 \quad \forall\,\alpha\right)$$
$$\min(V) = 0 \quad \left(V^\alpha(0) = 0 \quad \forall\,\alpha\right)$$

Contradiction! $\Rightarrow C_\alpha$ does not $\nearrow \infty$ (step 1 done) □

Lecture 6

- Prove Lemma 4.12
- finish Nonlinear Periodic theory

Exercise 9: Use user's guide plus (argument) to check that $-v + F\left(D^2 v(y), y\right)$ admits unique viscosity solutions.
Cancel the claimed proof of Lemma 4.12 last time.

**Intuition for Lemma 4.12** When does $Lv = F$ have periodic solutions in $\mathbb{R}^d$? What if $F$ smooth, $v \in C^2_{per}$? Compatible with expected result $m > 0$. $\exists\, v \;\Leftrightarrow\; \int F\, m\, \mathrm{d}y = 0$.

**4.25 Lemma.** *If $v \in C^2_{per}$, $F \not\equiv 0$ $C_{per}$ and $Lv = F$ in $\mathbb{R}^d$, then there exists $x_1 \neq x_2$ such that one of the following holds*

*(i) $F(x_1) > 0$ and $F(x_2) \leq 0$*

*(ii) $F(x_1) < 0$ a nd $F(x_2) \geq 0$*

*Proof.* Let $x_{min}$, $x_{max}$ achieve respectively $\min(v)$, $\max(v)$.

$$Lv = F \Rightarrow \left.\begin{array}{l} F(x_{min}) = Lv(x_{min}) \geq 0 \\ F(x_{max}) = Lv(x_{max}) \leq 0 \end{array}\right\} \overset{F \not\equiv 0}{\Rightarrow} \text{either (i) or (ii) holds}$$

□

*Proof of Lemma 4.12.* Know at least one $m$ exists $L^* u = 0$, all solutions are span$\{m\}$. Need check that $m \geq 0$ can the attained. By contradiction suppose $E = \{m \leq 0\}$. $|E| > 0$. Let $F_1 \geq 0$ be smooth approximation to $\mathbb{I}_E$ such that $\int m F_1 < 0$. Then there exists $c_1 > 0$, $c_2 > 0$ such that $F := c_2 + c_1 F_1$

$$\int F\, m\, \mathrm{d}y = 0$$

We already know (Fred) this implies there exists periodic $v$ solving

$$Lv = F \quad \text{in } \mathbb{R}^d$$

$F$ smooth, Schauder $\Rightarrow v \in C^{2,\alpha}_{per}$ this contradicts Lemma 4.25. ($F \geq c_2 > 0$ in $\mathbb{R}^d$) $\Rightarrow$ $m > 0$ is correct. $(\mathrm{Ran}\,(I - (-K)) = \mathrm{Null}\,(I - (-K))^{\perp})$

□

Back to nonlinear setting

$$F\left(Q, \frac{x}{\varepsilon}\right) = \inf_{a \in S} \sup_{A \in \mathcal{A}_a} \left( f^a_A \left(\frac{x}{\varepsilon}\right) + \mathrm{tr}\left( A \left(\frac{x}{\varepsilon}\right) Q\right)\right)$$

study $((eq)_\varepsilon)$

$$\begin{cases} F\left(D^2 u^\varepsilon (x), \frac{x}{\varepsilon}\right) = 0 & \text{in } \Omega \\ u^\varepsilon = g & \text{on } \partial\Omega \end{cases}$$

Find $\bar{F}$ translation invariant (function of $D^2\bar{u}$ only, no $x$ dependence)

**4.26 Proposition.** *Given $P \in \mathrm{Sym}\,(d \times d)$ fixed. There exists a unique scalar $\Theta(P)$ such that*

$$F\left(P + D^2 v(y), y\right) = \Theta(P) \qquad ((eq)_{cor,F,P})$$

*admits a global periodic periodic viscosity solution.*

*Proof.* Using unique sol $W^\alpha$

$$-\alpha w^\alpha + F\left(P + D^2 w^\alpha, y\right) = 0 \quad \text{in } \mathbb{R}^d$$

goal: $\alpha w^\alpha(0) \to \mathrm{const.}\,(= \Theta(P))$
problem: expect $w^\alpha$ unbounded in $\alpha \to 0$.

$$v^\alpha := w^\alpha - \alpha w^\alpha(0)$$

(0 irrelevant, any fixed $x_0$ works)
step 1: $v^\alpha$ bounded.... done
step 2: extract limit (subseq.) of $v^\alpha$

$$\underbrace{\alpha v^\alpha + F\left(P + D^2 v^\alpha, y\right)}_{\text{const. sub\&super solutions}} = \underbrace{\alpha w^\alpha(0)}_{\text{bounded}}$$

(note $v^\alpha$ periodic .... same as $w^\alpha$)

$$\mathcal{M}^+ \left(D^2 v^\alpha\right) \le C, \quad \mathcal{M}^- \left(D^2 v^\alpha\right) \ge -C$$

$$[v^\alpha]_{C^\beta(Q)} \le C^2 \quad \text{indep. of } \alpha$$

Arzela-Ascoli $\Rightarrow \exists\, v^{\alpha'}$, $v$ s.th. $v^{\alpha'} \to v$ loc. unif. $\mathbb{R}^d$ (uniformly by periodicity)
Stability of visc. sol. loc. unif. conv.
$v$ solves

$$F\left(P + D^2 v, y\right) = \lim_{\alpha \to 0} \alpha w^\alpha (0) \quad (= \Theta(P))$$

$\Theta$ exists. Show $\Theta(P)$ is unique
Go back to $v^\alpha$ eq.
Suppose $\Theta(P) < \hat{\Theta}(P)$ two possible scalars. Let $v$, $\hat{v}$ solve

$$F\left(P + D^2 \hat{v}, y\right) = \hat{\Theta}(P) \quad \text{in } \mathbb{R}^d$$

($v$ without hats) Look for contradiction. ($(eq)_{cor,F,P}$) is invariant by add. contants to $v$
W.l.o.g. we take $v < \hat{v}$
eq for $v$ relate to eq for $\hat{v}$?

$$F\left(P + D^2 v, y\right) = \Theta(P) < \hat{\Theta}(P) = F\left(P + D^2 \hat{v}, y\right)$$

by convergence of $v^{\alpha'} \to v$, $\hat{v}^{\alpha'} \to \hat{v}$ ($v^\alpha$, $\hat{v}^\alpha$ solving $v^\alpha$ eq.) for $\alpha$ small enough

$$\alpha v^\alpha + F\left(P + D^2 v, y\right) < \alpha \hat{v}^\alpha + F\left(P + D^2 \hat{v}^\alpha, < y\right)$$

Contradiction because global comparison for some sub, super sols. of ($v^\alpha$ eq) says

$$\sup_{\mathbb{R}^d} \hat{v}^\alpha - v^\alpha \le 0$$

$$\hat{v}^\alpha \le v^\alpha \quad \forall\, \alpha \text{ small enough}$$

$\lim_{\alpha \to 0} c$, $\hat{\Theta} = \Theta$

$$\hat{v} \le v$$

contradiction.

□

**4.27 Proposition.** *$P \mapsto \Theta(P)$, ($P \in \mathrm{Sym}(d \times d)$,) is uniformly elliptic in sense*

$$\mathcal{M}^-(P-Q) \le \Theta(P) - \Theta(Q) \le \mathcal{M}^+(P-Q)$$

$\forall\ P, Q \in \mathrm{Sym}(d \times d)$. *(Ellipticity (choice of $\lambda, \Lambda$ for $F$ is preserved via corrector eq)*

Recall $\mathcal{M}^-(P-Q) \le F(P,y) - F(Q,y) \le \mathcal{M}^+(P-Q)$

*Proof.* Very similar to linear case. Note ellipt. equivalent.
$\forall\, S \ge 0$ sym.

$$\lambda \operatorname{tr}(S) \le \Theta(P+S) - \Theta(P) \le \Lambda \operatorname{tr}(S)$$

we will establish „$\le$", „$\ge$" similiar.
by contradiction, suppose $\exists\, S$ for which „$\le$" fails, $\lambda \operatorname{tr}(S) > \Theta(P+S) - \Theta(P)$. Work at level of $v_{P+S}$, $v_P$ solving respectively $(eq)_{cor,F}$ with $P+S$ and $P$.

$$F\left(P+S+D^2 v_{P+S}, y\right) = \Theta(P+S)$$
$$F\left(P+D^2 v_P, y\right) = \Theta(P)$$

in $\mathbb{R}^d$. $v_{P+S}$ periodic. $v_p$ periodic (?). Recall eq are inv, by add constants, so wlog assume

$$v_{P+S} < v_S$$

eq for $v_{P+S}$ relative to eq for $v_P$. In all that follows must justify at level of viscosity!
First do it without, then viscosity solutions later.
Will show $v_{P+S}$ strict supersolution of eq for $v_P$

$$\begin{aligned} F\left(P+D^2 v_{P+S}, y\right) &= F\left(P+S+D^2 v_{P+S}\right) \quad + F\left(P+D^2 v_{P+S}, y\right) \\ &= \qquad\qquad\qquad\qquad\quad - F\left(P+S+D^2\left(v_{P+S}, y\right)\right) \\ &\le F\left(P+S+D^2 v_{P+S}, y\right) - \lambda \operatorname{tr}(S) \\ &= \Theta(P+S) - \lambda \operatorname{tr}(S) \\ &< \Theta(P) \qquad\qquad\qquad\quad = F\left(P+D^2 v_P, y\right) \end{aligned}$$

Use this to go to $v^\alpha eq$
$v^\alpha_{P+S}$ solve $v^\alpha eq$ with $P+S$ inside
$v^\alpha_P$ solve $v^\alpha eq$ with $P$ inside
$v^\alpha_{P+S} \to v_{P+S}$, $v^\alpha_P \to v_P$ loc unif as $\alpha \to 0$ (recall $\Theta(P) = \lim \alpha w^\alpha(0)$)
$\Rightarrow$

$$\alpha v^\alpha_{P+S} + F\left(P+D^2 v_{P+S}, y\right) < \alpha v^\alpha_P + F\left(D^2 v_P, y\right)$$

global comparison $\Rightarrow v_P^\alpha \leq v_{P+S}^\alpha$ $\alpha$ small enough
$\Rightarrow v_P \leq v_{P+S}$ contradicition. modulo visc. sol. argument.

□

Claim: $v_{P+S}$ is a strict visc supersol of $F\left(P+D^2v,y\right) \underset{<}{=} \Theta(P)$

*Proof.* Let $v_{P+S}-\varphi$ have a global min at $x_0$
$\Rightarrow F\left(P+S+D^2\varphi(x_0),x_0\right) \leq \Theta(P+S)$
$\Rightarrow F\left(P+D^2\varphi(x_0),x_0\right) \leq \Theta(P+S)+F\left(P+S+D^2\varphi(x_0),x_0\right)-F\left(P+D^2\varphi(x_0),x_0\right)$

$$F\left(P+D^2\varphi(x_0),x_0\right) \leq \Theta(P+S)-\lambda\operatorname{tr}(S) < \Theta(P)$$

hence $v_{P+S}$ solves $F\left(P+D^2v_{P+S},y\right) < \Theta(P)$ in visc sense. □

All tools to prove Theorem 4.23
show that or $\bar{F}(P) := \Theta(P)$ $u^\varepsilon \to \bar{u}$ , $u^\varepsilon$, $\bar{u}$ solve $((eq)_\varepsilon)$, $((\overline{eq}))$ resp.
($\bar{F}$ elliptic function of Hessian only $\Rightarrow \exists$ unique sol. of $((\overline{eq}))$)

*Proof.* $(u^\varepsilon)^*$, $(u^\varepsilon)_*$ respectively sub & super of $((\overline{eq}))$. We will show argument for $(u^\varepsilon)^*$
Let $(u^\varepsilon)^*-\varphi$ have strict global max at $x_0$ (need $\bar{F}\left(D^2\varphi(x_0) \geq 0\right)$. Proceed by contradiction, suppose

$$\bar{F}\left(D^2\varphi(x_0)\right) \leq -\delta < 0 \quad (\delta > 0)$$

go back to $((eq)_\varepsilon)$ for any good information
can't just take $\varphi$ to$((eq)_\varepsilon)$
Let $P = D^2\varphi(x_0)$. Let $v$ be any viscosity sol to $((eq)_{cor,F,P})$ periodic global sol of $F\left(P+D^2v,y\right) = \Theta(P)$ in $\mathbb{R}^d$

$$w^\varepsilon(x) := \varphi(x)+\varepsilon^2 v\left(\frac{x}{\varepsilon}\right)$$

Claim $w^\varepsilon$ solves $F\left(D^2w^\varepsilon,\frac{x}{\varepsilon}\right) < 0$ in $B_{r_0}(x_0)$ in viscosity sense for some $r_0 > 0$ chosen by $C^2$ norm of $\varphi$. Use unif. ellipticity of all $A \in \mathcal{A}$,

$$\left|\operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right)D^2\varphi(x)\right)-\operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right)D^2\varphi(x_0)\right)\right| \leq C\left|D^2\varphi(x)-D^2\varphi(x_0)\right|$$

unif in $A \in \mathcal{A}$, $\varepsilon > 0$. For $r_0$ small enough that $\left|D^2\varphi(x)-D^2\varphi(x_0)\right| < \frac{\delta}{2}$ $\forall x \in B_{r_0}(x_0)$

$$\left|F\left(D^2\varphi(x)+Q,\frac{x}{\varepsilon}\right)-F\left(D^2\varphi(x_0)+Q,\frac{x}{\varepsilon}\right)\right| < \frac{\delta}{2}$$

for all $Q \in \mathrm{Sym}(d \times d)$

$$\begin{aligned} F\left(D^2 w^\varepsilon, \frac{x}{\varepsilon}\right) &= F\left(D^2\varphi(x) + D^2 v\left(\frac{x}{\varepsilon}\right), \frac{x}{\varepsilon}\right) \\ &\le F\left(D^2\varphi(x_0) + D^2 v\left(\frac{x}{\varepsilon}\right), \frac{x}{\varepsilon}\right) + \frac{\delta}{2} \\ &= \underbrace{\Theta(P)}_{=F(P)} + \frac{\delta}{2} \\ &< -\delta + \frac{\delta}{2} < 0 \end{aligned}$$

$\Rightarrow w^\varepsilon$ supersol of same eq as $u^\varepsilon$ (in $B_{r_0}(x_0)$) comparison in $B_{r_0}(x_0)$ in $B_{r_0}(x_0)$

$$u^\varepsilon(x) - w^\varepsilon(x) \le \sup_{\overline{B_{r_0}(x_0)}} u^\varepsilon - w^\varepsilon \le \sup_{\partial B_{r_0}(x_0)} u^\varepsilon - w^\varepsilon$$

$$(u^\varepsilon)^*(x_0) - \varphi(x_0) \le \sup_{\partial B_{r_0}(x_0)} u^\varepsilon - \varphi$$

contradiction ti strict max $(u^\varepsilon)^* - \varphi$ at $x_0$
similarily $(u^\varepsilon)_*$ super of $((\overline{eq}))$. $\bar{F}$ unif elliptic $\Rightarrow$ barriers on $\partial\Omega \Rightarrow (u^\varepsilon)^* = (u^\varepsilon)_* = g$ on $\partial\Omega$. Comparison of sub, super sol

$$(u^\varepsilon)^* \le \bar{u} \le (u^\varepsilon)_* \le (u^\varepsilon)^*$$

$$\Rightarrow (u^\varepsilon)^* = (u^\varepsilon)_* = \bar{u} \quad \text{loc unif limit}$$

□

Lecture 7:

- Change of plans! Skip Papanicolaou-Varadhan (random linear homogenization)
- Recap done so far
- Moving from $F$ periodic to $F$ stationary ergodic (defined later)
- Begin nonlinear random case

Heuristically

$$u^\varepsilon(x) = \bar{u}(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}\right) + o\left(\varepsilon^2\right)$$

not true exactly! But good enough in small neighborhood where

$$D^2\bar{u}(x) = \text{const.} = P$$

periodicity puts compatibility condition on solving for an appropiate $v$, given $P$
Unique $\Theta(P)$ such that

$$F\left(P+D^2v,y\right)=\Theta(P) \quad \text{in } \mathbb{R}^d$$

admits a periodic viscosity solution
Linear case, we had Fred. Alt.

$$\Theta(P)=-\int_Q \underbrace{f(y)}_{\text{RHS}} \quad \underbrace{m(y)}_{\text{invariant measure}} \quad dy+\int a_{ij}(y)\,P_{ij}\,m(y)\,dy$$

Restatement of why corrector eq. is good enough

**4.28 Lemma.** *If $\bar{F}$ is defined as $\Theta(P)$ from Prop. 4.26 and $\varphi\in C^2$, with $\bar{F}\left(D^2\varphi(x_0)\right)<-\delta<0$, then there exists a $v^\varepsilon$ such that $\varphi+v^\varepsilon$ is visc. sub sol. to $((eq)_\varepsilon)$ in some $B_{r_0}(x_0)$, $r_0$ small depending on $\varphi$.*

($v^\varepsilon(x)=\varepsilon^2 v\left(\frac{x}{\varepsilon}\right)$ from true corrector for $P=D^2\varphi(x_0)$)
Justifying Lemma 4.28 was given as an exercise (do calc in viscosity sense) should do it (inside Theorem 4.23). Cheat, look in either Evans '92 or Schwab SIAM (periodic nonlinear case)
Why Lemma 4.28 good? Given $\bar{u}$, what does it mean to determine $\bar{F}\left(D^2\bar{u}\right)=0$ in viscosity sense? $\bar{u}$ puts a restriction on which matrices, $Q$ are allowed to given

$$u(y)\le u(x)+\langle (y-x),Q(y-x)\rangle+o\left(|y-x|^2\right) \quad (\text{or } \ge)$$

Only possible for those $Q$ which also satisfy $\bar{F}(Q)\ge 0$ or $\le 0$. Only have information at $\varepsilon$-level inside $F\left(\cdot,\frac{x}{\varepsilon}\right)$ & $u^\varepsilon$!
Lemma 4.28 says any $Q$ such that $\bar{F}(Q)<0$ is impossible for $\bar{u}-Q$ to have local max.
Change local max to strict global max for some $\varphi$, $D^2(\varphi(x_0))=Q$. Then Lemma 4.28 $\Rightarrow$ (comparison with $u^\varepsilon$)

$$\sup_{B_{r_0}(x_0)} u^\varepsilon-(\varphi+v^\varepsilon)\le\sup_{\partial B_{r_0}(x_0)} u^\varepsilon-(\varphi+v^\varepsilon)$$

$u^\varepsilon$ sol (sub), $\varphi+v^\varepsilon$ super
gives contradiction in passing $\varepsilon\to 0$ (all appeared in proof of Theorem 4.23)

Did we really need $v^{\varepsilon} = \varepsilon^2 v\left(\frac{x}{\varepsilon}\right)$ where $v$ periodic & solved

$$(\Theta(P)) = \bar{F}(P) = F\left(P + D^2 v, y\right) \quad \text{in } \mathbb{R}^d$$

quite lucky, just 1 $v$ works for all $\varepsilon$ simultaneously.
Key facts from proof Theorem 4.23 $\to \varphi + v^{\varepsilon}$ made to solve $F\left(D^2\left(\varphi + v^{\varepsilon}\right), \frac{x}{\varepsilon}\right) \leq 0$ in some $B_{r_0}(x_0)$ whenever $\bar{F}\left(D^2\varphi(x_0)\right) < 0 \to \varphi + v^{\varepsilon} \xrightarrow{\varepsilon\to 0} \varphi$ unif. in $B_{r_0}(x_0)$. Why? $\varphi + v^{\varepsilon}$ is super of same equation for which $u^{\varepsilon}$ is a sol (sub)

$$\sup_{B_{r_o}(x_0)} u^{\varepsilon} - \left(\varphi + v^{\varepsilon}\right) \leq \sup_{\partial B_{r_0}(x_0)} u^{\varepsilon} - \varphi + v^{\varepsilon}$$

$\Rightarrow (u^{\varepsilon})^* - \varphi$ can't have a strict max at $x_0$ (otherwise there is contradiciton)
Know by continuity of $F$, can localize $F\left(D^2\varphi(x) + Q, \frac{x}{\varepsilon}\right)$ ($Q$ fixed) to $F\left(D^2\varphi(x_0) + Q, \frac{x}{\varepsilon}\right) + \frac{\delta}{2}$ error uniformly in any $Q \in \mathrm{Sym}(d \times d)$

$$\left|\mathrm{tr}\left(A\left(\frac{x}{\varepsilon}\right) D^2\varphi(x)\right) - \mathrm{tr}\left(A\left(\frac{x}{\varepsilon}\right) D^2\varphi(x_0)\right)\right| \leq \|A\|_{L^\infty}\left|D^2\varphi(x) - D^2\varphi(x_0)\right|$$

$\|v^{\varepsilon}\|_{L^\infty\left(B_{r_0}(x_0)\right)} \to 0$ for $\varepsilon \to 0$ is compatibility condition

**4.29 Lemma.** *$\bar{F}(P)$ (from Prop. 4.26) is the unique scalar such that the familiy $w^{\varepsilon}$ of unique solutions to*

$$\begin{aligned} F\left(P + D^2 w^{\varepsilon}(x), \frac{x}{\varepsilon}\right) &= \bar{F}(P) \quad \text{in } B_{r_o}(x_0) \\ w^{\varepsilon} &= 0 \quad\quad\;\; \text{on } \partial B_{r_0} \end{aligned}$$

*also satisfies*

$$\|w^{\varepsilon}\|_{L^\infty\left(B_{r_0}(x_0)\right)} \xrightarrow{\varepsilon\to 0} 0$$

In reality Lemma 4.29 is all that proof of Theorem 4.23 required! (discussion before Lemma 4.29)
Why good? If periodicity of $F$ removed you do not expect existece of periodic $V$ solving $((eq)_{cor,F,P})$! Prop 4.26 is no longer true! Lemma 4.29 still holds in stationary ergodic setting (good)

## 4.4 Stationary ergodic

Must introducte a probability space to model a random family of of equations

$$F\left(P, \frac{x}{\varepsilon}, \omega\right) \in (\Omega, \mathcal{F}, \mathbb{P})$$

Fix probability Space $(\Omega, \mathcal{F}, \mathbb{P})$ think of coefficients of PDE

$$(a_{ij}) : \mathbb{R}^d \times \Omega \to \mathrm{Sym}\,(d \times d)$$

measurable. Need notion of transformation on $\Omega$ to track group on $\mathbb{R}^d$ (additive translations). Assume existence of $\tau_x \colon \Omega \to \Omega\, \forall x \in \mathbb{R}^d$, preserves measures on $\Omega$. ($\mathbb{P}\left(\tau_x^{-1}(E)\right) = \mathbb{P}(E)\ \forall\ E \in \mathcal{F}$) And $\tau$ is group $\tau_{x+y} = \tau_x \tau_y$. Stationarity means that $\mathrm{Law}\,(a_{ij}(x, \cdot))$ same as $\mathrm{Law}\,(a_{ij}(x+z, \cdot))\ \forall\, x, z$.

**4.30 Definition.** $f \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is stationary w.r.t. $\tau$ if $\forall$ $k$-tuple $x_1, \ldots, x_k$ the law of $(f(x_1, \cdot), \ldots, f(x_k, \cdot))$ have same distribution (law) as $(f(x_1+y, \cdot), \ldots, f(x_k+y, \cdot))$ $\forall\, y \in \mathbb{R}^d$. (stationary w.r.t. $\tau$ means $f(x+y, \omega) = f(x, \tau_y \omega)\ \forall\ x, y$)

(bonus way to check if such $\tau$ exists)
Ergodic? Assume such a $\tau$ exists, then

**4.31 Definition.** $\mathbb{P}$ is ergodic w.r.t. $\tau$ if $\forall\ A \in \mathcal{F}$ s.th.

$$\tau_x^{-1} A = A \quad \forall\ x \in \mathbb{R}^d$$

then must be that

$$P(A) = 0 \quad \text{or} \quad P(A) = 1$$

(The only $\tau$-invariant subsets are trivial)

Where do these ideas come from, what do they mean? $\to$ Breiman (Leo) Probability Chp. 6 stationarity & ergodicity. Simple example: $\infty$-coin tossing. $X_i \in \{0,1\}$ for $i \in \mathbb{Z}$ are outcomes of independent coin tosses. Think of $\Omega = \{0,1\}^{\mathbb{Z}^d}$, product measure induced by $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. $\omega \in \Omega$ outcome $(\ldots, \omega_{i-1}, \omega_i, \omega_{i+1}, \ldots)$. Natural transformation on $\Omega$ shift

$$(T\omega)_i = \omega_{i+1}$$

maps $\Omega \to \Omega$. Think of stationary function $f \colon \mathbb{Z} \times \Omega \to \mathbb{R}$,

$$f(z+k, \omega) = f\left(z, T^k \omega\right)$$

with $T^k = \underbrace{T \circ \cdots \circ T}_{k-\text{times}}$. Typically realized as

$$f(k,\omega) = \hat{f}\left(T^k\omega\right), \quad \hat{f}\colon \Omega \to \mathbb{R}$$

(More or less all stationary $f$ are of this form)
E.g. $\hat{f}(\omega) = \omega_0$ or $\hat{g}(\omega) = \sum_{i=1}^{10} \omega_i$ etc.
Ergodicity?
Idea 1: For coin tossing (or any i.i.d. infinite experiment) there is famous $0-1$-law to events by Kolmogorov.

$$\mathcal{F}(n) = \sigma\left(\{X_i\}_{|i|\geq n}\right)$$

or tail-$\sigma$-alg. $= \bigcap_{n=0}^{\infty} \mathcal{F}(n)$ e.g.

$$E = \left\{\omega \in \Omega \,\middle|\, \frac{1}{N}\sum_{i=0}^{N} \omega_i \text{ is convergent}\right\}$$

assume $X_i$ i.i.d. as above. $E$ is tail event.
Kolmogorov: If $E$ is tail event, then $\mathbb{P}(E) = 0$ or $\mathbb{P}(E) = 1$.
Idea 2: $\Omega$ =physical state space of a dynamical system (e.g. pos & momentum), $T\colon \Omega \to \Omega$ is time 1 evolution of dynamics. $\omega$ =initial condition, then $T^k\omega$ =state at time $k$.
Gibbs: The systems we care about have equilibrium distributions of states, some measure $\mathbb{P}$ on $\Omega$. Also our systems should be such that orbits by $T$ visit all of phase space! Given $\omega_0$ & neighborhood $T^k\omega$ should be in nbhd. of $\omega_0$ for $\infty$-many $k$. Should be $f$ „observable" $f \in C(\Omega)$
information produced by $f\left(T^k\omega\right)$ approximates information in $\mathbb{P}$

$$\frac{1}{N}\sum_{k=0}^{N} f\left(T^k\omega\right) \to \int_{\Omega} f \,\mathrm{d}\mathbb{P}$$

all initial $\omega$, as $N \to \infty$.
What if $\mathbb{P}(A) > 0$, yet $T^{-1}A = A$ ($TA \subset A$) A invariant by $T$, then $\omega \in A$, $T^k\omega$ never leaves $A$.
Birkhoff: If $T$ is ergodic w.r.t. $\mathbb{P}$ (as above) there exists a.e. $\omega$ w.r.t. $\mathbb{P}$

$$\frac{1}{N}\sum_{k=0}^{N} f\left(T^k\omega\right) \xrightarrow{N\to\infty} \int f \,\mathrm{d}\mathbb{P}$$

The Birkhoff ergodic thm.

21.6.13: to be completed...

24.6.13: to be completed...

# Literaturverzeichnis

[0] Foo Bar, QuuX (and B3y0nd), 2013