Optimal Error Estimates of Galerkin Finite Element Methods for SPDEs with Multiplicative Noise

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A Semilinear SPDE with Multiplicative Noise

Consider for $t \in [0, T]$

$$\begin{aligned} \mathrm{d}X(t) + \big[AX(t) + f(X(t))\big]\,\mathrm{d}t &= g(X(t))\,\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned} \qquad \text{(SPDE)}$$

where H, U are separable Hilbert spaces,

- $X\colon [0, T] \times \Omega \to H$,
- $-A$ generator of an analytic semigroup $E(t)$ on H,
- $f\colon H \to H$ nonlinear operator,
- $(W(t))_{t\in[0,T]}$ is a U-valued Q-Wiener process, with $Q\colon U \to U$ covariance operator.
- $g\colon H \to L_2^0$, Hilbert-Schmidt op. from $U_0 = Q^{\frac{1}{2}}(U)$ to H.

We study the mild solution using the semigroup approach by [Da Prato, Zabczyk 1992].

Aim: Discretization of (SPDE)

A numerical approximation of (SPDE) needs to discretize

- H (spatial discretization),
- $[0, T]$ (temporal discretization),
- U_0 (discretization of the noise).

Today:

- Error estimates for discretization w.r.t. H by Galerkin finite element methods,
- Error estimates for discretizations w.r.t. H and $[0, T]$ by an additional linear-implicit Euler-Maruyama scheme.

Future work:

- Discretization of $U = L^2(\mathcal{D})$ if Q is given by

$$[Qu](x) = \int_{\mathcal{D}} q(x, y) u(y) \, \mathrm{d}y.$$

Assumptions on A

For simplicity, we set

- $H := L^2(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^d$ bdd. domain with smooth boundary,
- $-A := \Delta$ Laplace operator with Dirichlet b.c., generator of an analytic semigroup $E(t) = \mathrm{e}^{-At}$.

Then there exists ONB $(e_n)_{n\in\mathbb{N}} \subset H$ s.t.

$$Ae_n = \lambda_n e_n \quad \forall n \in \mathbb{N}$$

with

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty.$$

The domain $D(A)$ is given by

$$D(A) = \left\{ x \in H \ : \ \sum_{n=1}^{\infty} \lambda_n^2 (x, e_n)^2 < \infty \right\}.$$

The Spaces $\dot{H}^r$

Fractional powers of A: For $r \in \mathbb{R}$ with $r \geq 0$ define

$$A^{\frac{r}{2}} x := \sum_{n=1}^{\infty} \lambda_n^{\frac{r}{2}} x_n e_n \quad \forall x \in D(A^{\frac{r}{2}}),$$

where

$$D(A^{\frac{r}{2}}) = \left\{ x \in H \;\middle|\; (x_n)_n \subset \mathbb{R} \text{ with } \|x\|_r^2 = \sum_{n=1}^{\infty} \lambda_n^r (x, e_n)^2 < \infty \right\}.$$

Set $\dot{H}^r = (D(A^{\frac{r}{2}}), \|\cdot\|_r)$ which is a Hilbert space.
In our example

$$\dot{H}^1 = H_0^1(\mathcal{D}), \quad \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}).$$

Assumptions

Let $r \in [0,1]$ and $p \in [2,\infty)$ be given.

(A1) f maps H into H and $\exists C > 0$ s.t.

$$\|f(x) - f(y)\| \leq C\|x - y\| \quad \forall x, y \in H.$$

(A2) g maps H into L_2^0 and $\exists C > 0$ s.t.

$$\|g(x) - g(y)\|_{L_2^0} \leq C\|x - y\| \quad \forall x, y \in H.$$

Furthermore, we have that $(A^{\frac{r}{2}} \circ g(\dot{H}^r)) \subset L_2^0$ and

$$\|A^{\frac{r}{2}} g(x)\|_{L_2^0} \leq C(1 + \|x\|_r) \quad \forall x \in \dot{H}^r.$$

(A3) Initial value $X_0 \colon \Omega \to \dot{H}^{r+1}$, $\mathcal{F}_0$-mb., $\mathbf{E}[\|X_0\|_{r+1}^2] < \infty$.

Mild Solution

Theorem (Jentzen, Röckner 2010)

Under assumptions (A1) - (A3) there exists a unique mild solution X to (SPDE) *such that*

$$X(t) = E(t)X_0 - \int_0^t E(t-\sigma)f(X(\sigma))\,\mathrm{d}\sigma + \int_0^t E(t-\sigma)g(X(\sigma))\,\mathrm{d}W(\sigma) \quad \mathbf{P}\text{-}a.s.\ \forall t \in [0,T].$$

Regularity

Theorem (K., Larsson 2010)

Under assumptions (A1) - (A3) with $r \in [0,1)$ *it holds that*

$$\sup_{t\in[0,T]} \mathbf{E}\left[\|X(t)\|_s^p\right] < \infty \quad \forall s \in [0, r+1]$$

and $\exists C > 0$ *such that*

$$(\mathbf{E}[\|X(t_1) - X(t_2)\|_s^p])^{\frac{1}{p}} \leq C|t_1 - t_2|^{\min(\frac{1}{2}, \frac{r+1-s}{2})} \quad t_1, t_2 \in [0, T].$$

Galerkin Finite Element Methods

Using the methodology of [Thomée, 2006]

- Sequence of subspaces $(S_h)_{h \in (0,1]} \subset \dot{H}^1$, $N_h = \dim(S_h) < \infty$ with $\lim_{h \to 0} N_h = \infty$.
- $R_h \colon \dot{H}^1 \to S_h$ Ritz projector, which is the orthogonal projector onto S_h w.r.t.

$$a(x,y) := (A^{\frac{1}{2}} x, A^{\frac{1}{2}} y) \quad x,y \in \dot{H}^1.$$

(S1) There exists $C > 0$ such that $\|R_h x - x\| \le C h^s \|x\|_s \; \forall x \in \dot{H}^s$, $s \in \{1,2\}$, $h \in (0,1]$.

Further operators

- $A_h \colon S_h \to S_h$ discrete Laplacian, given by $a(x_h, y_h) = (A_h x_h, y_h)$ for all $x_h, y_h \in S_h$.
- $E_h(t)$ semigroup on S_h generated by $-A_h$.

Spatial Semidiscretization

- $P_h \colon H \to S_h$ is the orthogonal projector onto S_h s.t. $(P_h x, y_h) = (x, y_h)$ for all $x \in H$, $y_h \in S_h$.

Spatially semidiscrete discretization

Consider for $t \in [0, T]$

$$\begin{aligned} \mathrm{d}X_h(t) + [A_h X_h(t) + P_h f(X_h(t))]\,\mathrm{d}t &= P_h g(X_h(t))\,\mathrm{d}W(t), \\ X_h(0) &= P_h X_0. \end{aligned} \tag{1}$$

with mild solution

$$\begin{aligned} X_h(t) = E_h(t) P_h X_0 &- \int_0^t E_h(t-\sigma) P_h f(X(\sigma))\,\mathrm{d}\sigma \\ &+ \int_0^t E_h(t-\sigma) P_h g(X(\sigma))\,\mathrm{d}W(\sigma) \quad \mathbf{P}\text{-a.s. } \forall t \in [0, T]. \end{aligned}$$

Spatial Semidiscretization

Theorem (K. 2011)

Under the assumptions (A1) – (A3) with $r \in [0,1)$, $p \in [2,\infty)$, *and (S1) on* $(S_h)_{h\in(0,1]}$ *there exists* $C > 0$ *such that*

$$\|X_h(t) - X(t)\|_{L^p(\Omega;H)} \le Ch^{1+r} \quad \forall h \in (0,1], t \in (0,T].$$

- Examples for spatial discretizations: Standard finite element method, spectral Galerkin method, ...
- The order of convergence coincides with the spatial regularity $1 + r \Rightarrow$ optimal error estimate
- Former results in literature only obtain order of convergence $1 + r - \epsilon$ for every $\epsilon > 0$ or contain a logarithmic singularity of the form $\log(t/h)$.

Error Estimates for the Spatial Semidiscretization

Want to estimate

$$
\begin{aligned}
\|X_h(t) - X(t)\|_{L^p(\Omega;H)} \le\ & \|E_h(t)P_hX_0 - E(t)X_0\|_{L^p(\Omega;H)} \\
& + \Big\| \int_0^t E_h(t-\sigma)P_h f(X_h(\sigma))\,\mathrm{d}\sigma - \int_0^t E(t-\sigma) f(X(\sigma))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
& + \Big\| \int_0^t E_h(t-\sigma)P_h g(X_h(\sigma))\,\mathrm{d}W(\sigma) - \\
& \qquad - \int_0^t E(t-\sigma) g(X(\sigma))\,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)}
\end{aligned}
$$

We focus on the third summand and the case $p = 2$.

Step 1

Apply Itô-isometry

$$
\begin{aligned}
&\Big\| \int_0^t \big(E_h(t-\sigma)P_h g(X_h(\sigma)) - E(t-\sigma)g(X(\sigma))\big)\,\mathrm{d}W(\sigma)\Big\|_{L^2(\Omega;H)} \\
&= \Big(\mathbf{E}\Big[\int_0^t \|E_h(t-\sigma)P_h g(X_h(\sigma)) - E(t-\sigma)g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}} \\
&\le \Big(\mathbf{E}\Big[\int_0^t \|E_h(t-\sigma)P_h g(X_h(\sigma)) - E_h(t-\sigma)P_h g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}} \\
&+ \Big(\mathbf{E}\Big[\int_0^t \|E_h(t-\sigma)P_h g(X(\sigma)) - E(t-\sigma)g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}}
\end{aligned}
$$

For first term use $\|E_h(t)P_h\| \le C$ and (A2), then later Gronwall.
For the last term set $F_h(t) := E_h(t)P_h - E(t)$.

Step 2, First Try

Thus, it remains to estimate

$$\left(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\right)^{\frac{1}{2}}$$

First Try: Apply non-smooth error estimates

[Thomée, Th. 3.5]:

$$\|F_h(t)x\| \leq Ch^{1+r}t^{-\frac{1}{2}}\|x\|_r \quad \forall x \in \dot{H}^r, t > 0$$

and linear growth condition (A2), this gives

$$\left(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\right)^{\frac{1}{2}}$$
$$\leq Ch^{1+r}\Big(1+\sup_{t\in[0,T]}\mathbf{E}\big[\|X(t)\|_r^2\big]\Big)\underbrace{\left(\int_0^t (t-\sigma)^{-1}\,\mathrm{d}\sigma\right)^{\frac{1}{2}}}_{=\infty}$$

Step 2, Suboptimal Workaround

Thus, it remains to estimate

$$\Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(\sigma))\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}}$$

Workaround: Apply balanced non-smooth error estimates

[Thomée, Th. 3.5]:

$$\|F_h(t)x\| \le Ch^{1+r-\epsilon}t^{-\frac{1-\epsilon}{2}}\|x\|_r \quad \forall x \in \dot{H}^r, t > 0$$

and linear growth condition (A2), this gives

$$\Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(\sigma))\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}}$$

$$\le Ch^{1+r-\epsilon}\Big(1+\sup_{t\in[0,T]}\mathbf{E}\big[\|X(t)\|_r^2\big]\Big)\underbrace{\Big(\int_0^t (t-\sigma)^{-1+\epsilon}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}}}_{\le C_\epsilon}$$

Step 2, Workaround

It still remains to estimate

$$\left(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\right)^{\frac{1}{2}}$$

Idea: Apply integral version of error estimate

[Thomée, (2.28)]

$$\int_0^t \|F_h(t-\sigma)x\|^2\,\mathrm{d}\sigma \le Ch^{2(1+r)}\|x\|_r^2 \quad \forall x \in \dot{H}^r, t > 0.$$

BUT: Not directly applicable since

$$\left(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)\,\underbrace{g(X(\sigma))}_{\sigma\text{ dependent}}\,\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\right)^{\frac{1}{2}}$$

Step 2, Workaround

Therefore,

$$\begin{aligned}
&\Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}} \\
&\quad\le \Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(t))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}} \\
&\qquad + \Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)\big(g(X(\sigma))-g(X(t))\big)\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}}
\end{aligned}$$

Now, integral version of error estimate applicable to first term

$$\begin{aligned}
&\Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)g(X(t))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}} \\
&\quad\le Ch^{1+r}\Big(1+\sup_{t\in[0,T]}\mathbf{E}\big[\|X(t)\|_r^2\big]\Big)
\end{aligned}$$

Step 3, Final Step

Apply non-smooth data estimate with singularity to last term

$$
\begin{aligned}
&\Big(\mathbf{E}\Big[\int_0^t \|F_h(t-\sigma)\big(g(X(\sigma))-g(X(t))\big)\|_{L_2^0}^2\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}} \\
&\qquad \le C h^{1+r}\Big(\mathbf{E}\Big[\int_0^t (t-\sigma)^{-1-r}\|g(X(\sigma))-g(X(t))\|_{L_2^0}^2\,\mathrm{d}\sigma\Big]\Big)^{\frac{1}{2}}
\end{aligned}
$$

Finally, use Lipschitz condition and the Hölder continuity of X:

$$
\mathbf{E}\Big[\|g(X(\sigma))-g(X(t))\|_{L_2^0}^2\Big] \le C(t-\sigma)
$$

and get

$$
\begin{aligned}
&\mathbf{E}\Big[\int_0^t (t-\sigma)^{-1-r}\|g(X(\sigma))-g(X(t))\|_{L_2^0}^2\,\mathrm{d}\sigma\Big] \\
&\quad \le C\int_0^t (t-\sigma)^{-1-r}(t-\sigma)\,\mathrm{d}\sigma < \infty \quad \text{for all } r\in[0,1)
\end{aligned}
$$

A Spatio-Temporal Discretization

Combining Galerkin FEM with linear implicit Euler-Maruyama:

$$\begin{aligned} X_h^j - X_h^{j-1} + k\left(A_h X_h^j + P_h f(X_h^{j-1})\right) &= P_h g(X_h^{j-1})\Delta W^j \\ X_h^0 &= P_h X_0, \end{aligned} \tag{2}$$

where

- k constant time step size,
- $\Delta W^j = W(t_j) - W(t_{j-1})$.

(S2) There exists $C > 0$ such that
$\|P_h x\|_1 \leq C\|x\|_1 \ \forall x \in \dot{H}^1$, $h \in (0,1]$.

- Spectral Galerkin: (S2) always holds since $P_h = R_h$,
- Standard FEM: (S2) holds for quasi-uniform triangulations.

A Spatio-Temporal Discretization

Theorem (K. 2011)

Under the assumptions (A1) – (A3) with $r \in [0,1)$, $p \in [2,\infty)$, *and (S1), (S2) on* $(S_h)_{h\in(0,1]}$ *there exists* $C > 0$ *such that*

$$\|X_h^j - X(jk)\|_{L^p(\Omega;H)} \leq C(h^{1+r} + k^{\frac{1}{2}}) \quad \forall h,k \in (0,1], j = 1,2,\dots$$

- No dependencies between h and k.
- Strong order of convergence $\frac{1}{2}$ w.r.t. k is optimal for Euler-Maruyama type schemes.

Summary

- Considered semilinear SPDE with multiplicative noise under global Lipschitz and linear growth conditions
- Obtained optimal error estimates for spatial and spatio-temporal discretizations using Galerkin FEM and linear implicit Euler-Maruyama
- Idea of proof also yields optimal regularity results

Reference :

Raphael Kruse, Optimal Error Estimates of Galerkin Finite Element Methods for Stochastic Partial Differential Equations with Multiplicative Noise, arXiv: math.NA.1103.4504, March 2011.

Thank you for your attention!