Two-sided error estimates for stochastic one-step methods of higher order

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The stochastic ordinary differential equation (SODE)

We consider for $t \in [0,T]$

$$dX(t) = b^0(t, X(t))dt + \sum_{k=1}^{m} b^k(t, X(t))dW^k(t), \qquad \text{(SODE)}$$
$$X(0) = X_0,$$

where

- $b^k : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ measurable,
- W^k real, pairwise independent, standard Brownian motions, adapted to filtration $(\mathcal{F}_t)_{t\in[0,T]}$,
- X_0 initial value: $\mathcal{F}_0$-mb. with values in $\mathbb{R}^d$.

Existence and uniqueness

(A1) $\mathbb{E}(|X_0|^2) < \infty$.

(A2) There exists $K > 0$ such that

$$|b^k(t,x)| \leq K(1 + |x|)$$

and

$$|b^k(t,x) - b^k(t,y)| \leq K|x - y|$$

for all $k = 0, \dots, m$ and $x, y \in \mathbb{R}^d$, $t \in [0, T]$.

Theorem

Under assumptions (A1) + (A2) there exists a unique solution to (SODE) with

$$X(t) = X_0 + \int_0^t b^0(s, X(s))ds + \sum_{k=1}^{m} \int_0^t b^k(s, X(s))dW^k(s)$$

and $\mathbb{E}(|X(t)|^2) < \infty$ *for all* $t \in [0, T]$.

A simple numerical approximation

Let $h = \frac{T}{N}$, $N \in \mathbb{N}$, be the equidistant step size, $t_i = ih$.

The Euler-Maruyama method is given by

$$\begin{aligned} X_h(0) &= X_0, \\ X_h(t_i) &= X_h(t_{i-1}) + hb^0(t_{i-1}, X_h(t_{i-1})) \\ &\quad + \sum_{k=1}^{m} b^k(t_{i-1}, X_h(t_{i-1}))\Delta_h W^k(t_i), \quad 1 \le i \le N, \end{aligned}$$

where $\Delta_h W^k(t_i) = W^k(t_i) - W^k(t_{i-1})$.

Euler-Maruyama method converges with order $\mathcal{O}(h^{\frac{1}{2}})$.

Itô-Taylor expansions

Itô-formula for $f \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$

$$f(t, X(t)) = f(s, X(s)) + \int_s^t (L^0 f)(u, X(u))du + \sum_{k=1}^{m} \int_s^t (L^k f)(u, X(u))dW^k(u)$$

with

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{d} b^{0,i} \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} b^{k,i} b^{k,j} \frac{\partial^2}{\partial x_i \partial x_j}$$

$$L^k = \sum_{j=1}^{d} b^{k,j} \frac{\partial}{\partial x_j}, \quad k = 1, \ldots, m.$$

Itô-Taylor expansions

Iterated application of the Itô-formula gives

$$f(t, X(t)) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(s, X(s))]_{s,t} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(\cdot, X(\cdot))]_{s,t}$$

if all appearing derivatives and integrals exists.
Here

- $\alpha = (j_1, \dots, j_l)$, $j_i \in \{0, \dots, m\}$, multi-indices,
- $\mathcal{A}$ hierarchical set of multi-indices, $\mathcal{B}(\mathcal{A})$ remainder set,
- $I_\alpha[f_\alpha(\cdot, X(\cdot))]_{s,t}$ iterated (stochastic) integral, e.g. $\alpha = (0, 1)$

$$I_\alpha[f_\alpha(\cdot, X(\cdot))]_{s,t} = \int_s^t \int_s^u (L^0 L^1 f)(z, X(z)) dz \, dW^1(u).$$

Now we set $f(t, x) = x$.

Itô-Taylor schemes

Let $\gamma \in \left\{\frac{1}{2}n : n \in \mathbb{N}\right\}$ and

$$\mathcal{A}_\gamma = \left\{\alpha \in \mathcal{M} : \ell(\alpha) + n(\alpha) \leq 2\gamma \text{ or } \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\right\}$$

a set of multi-indices.
The Itô-Taylor scheme of order γ is given by

$$X_h(t_0) = X_0,$$
$$X_h(t_{k+1}) = \sum_{\alpha \in \mathcal{A}_\gamma} f_\alpha(t_k, X_h(t_k)) I_{\alpha,k}$$

with $I_{\alpha,k} = I_\alpha[1]_{t_{k-1},t_k}$.
$\gamma = \frac{1}{2} \Rightarrow$ Euler-Maruyama scheme.
$\gamma = 1 \Rightarrow$ Milstein scheme.

Assumptions

(A1) $\mathbb{E}(|X_0|^2) < \infty$

(A2) There exists $K > 0$ such that

$$|b^k(t,x)| \leq K(1 + |x|)$$

and

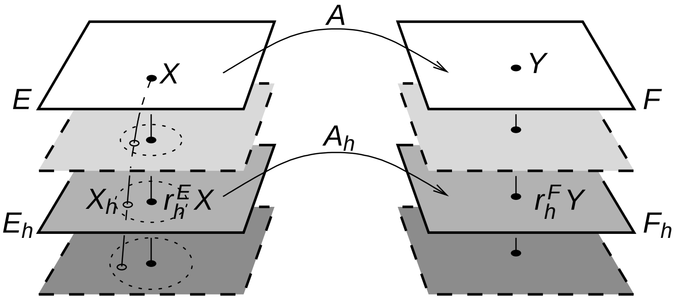
$$|b^k(t,x) - b^k(t,y)| \leq K|x - y|$$

for all $k = 0, \ldots, m$ and $x, y \in \mathbb{R}^d$, $t \in [0, T]$.

(A3) For a given order γ the Itô-Taylor expansion of $X(t)$ w.r.t. $\mathcal{A}_\gamma$ exists for all $t \in [0, T]$.

Discrete approximation theory

Schematic



To solve:

$$AX = Y$$

Numerical method:

$$A_h X_h = r_h^F Y$$

Embedding the SODE

The operator $A : E \to F$ is given by

- $E := \{X\}$,
- $F := \{Y\} = \{(X_0, 0)\}$,
- $AX := \Big(X(0), (X(t) - X(0) - \int_0^t b^0(s, X(s))ds - \sum_{k=1}^m \int_0^t b^k(s, X(s))dW^k(s))_{t\geq 0}\Big)$.

Discrete space

Let

$$\tau_h := \{t_i \mid t_i = ih,\ i = 0, \dots, N\}$$

be the time grid. Denote by

$$\mathcal{G}_h := \mathcal{G}(\tau_h, L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d))$$

set of adapted and $L^2(\Omega)$-valued grid functions.
Here $Z_h \in \mathcal{G}_h$ is adapted if $Z_h(t_i)$ is $\mathcal{F}_{t_i}$ measurable for all $t_i \in \tau_h$.

Metric space E_h

Restriction operators:

$$r_h^E : \begin{aligned} E &\to \mathcal{G}_h \\ X &\mapsto r_h^E X \end{aligned}$$

with

$$[r_h^E X](t_i) = X(t_i).$$

We define

$$E_h := \overline{B}_\rho(r_h^E X) \subset \mathcal{G}_h$$

measured in the norm

$$\|Z_h\|_0 := \max_{0 \le i \le N} \|Z_h(t_i)\|_{L^2(\Omega)}.$$

Metric space F_h

Restriction operators:

$$r_h^F : \begin{array}{l} F \to \mathcal{G}_h \\ Y \mapsto r_h^E Y \end{array}$$

with

$$[r_h^F Y](t_0) = X_0, \quad [r_h^F Y](t_i) = 0,\ 1 \leq i \leq N.$$

We define

$$F_h := \overline{B}_\sigma(r_h^F Y) \subset \mathcal{G}_h$$

measured in the stochastic Spijker-norm

$$\|Z_h\|_{-1} := \max_{0 \leq i \leq N} \|\sum_{j=0}^{i} Z_h(t_i)\|_{L^2(\Omega)}.$$

Residual mapping and the operators A_h

- Residual mapping $\mathrm{res}_h : \mathcal{G}_h \to \mathcal{G}_h$ of the Itô-Taylor scheme

$$\mathrm{res}_h(Z_h)(t_0) = Z_h(t_0) - X_0,$$
$$\mathrm{res}_h(Z_h)(t_i) = Z_h(t_i) - \sum_{\alpha \in \mathcal{A}_\gamma} f_\alpha(t_k, Z_h(t_k)) I_{\alpha,k}, \; 1 \le i \le N.$$

- Note that if X_h is the Itô-Taylor approximation then $\mathrm{res}_h(X_h) = 0$.
- Operator $A_h : \; D(A_h) \to F_h$ is defined by

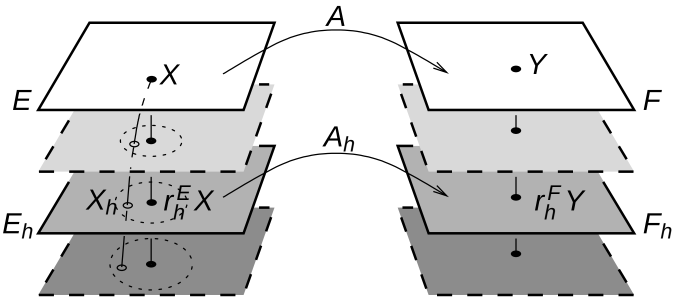
$$A_h X_h = \mathrm{res}_h(X_h) + r_h^F Y$$

with domain of definition

$$D(A_h) := \{ Z_h \in E_h \mid \|\mathrm{res}_h(Z_h)\|_{-1} \le \sigma \} \subset E_h.$$

Discrete approximation theory

Schematic



To solve:

$$AX = Y$$

Numerical method:

$$A_h X_h = r_h^F Y$$

Question: Relationship between X and the numerical solutions X_h?

Consistency

Definition

A one-step method is called consistent of order $\gamma > 0$, if there exists a constant $C > 0$ and an upper step size bound $\overline{h}$, s.t.

$$\left\| A_h r_h^E X - r_h^F A X \right\|_{-1} \leq C h^{\gamma}$$

for all $h \leq \overline{h}$.

Theorem

Under the assumptions (A1) - (A4) the stochastic Itô-Taylor method is consistent of order γ.

Assumptions

(A1) $\mathbb{E}(|X_0|^2) < \infty$

(A2) There exists $K > 0$ such that

$$|b^k(t,x)| \leq K(1 + |x|)$$

and

$$|b^k(t,x) - b^k(t,y)| \leq K|x - y|$$

for all $k = 0, \dots, m$ and $x, y \in \mathbb{R}^d$, $t \in [0, T]$.

(A3) For a given order γ the Itô-Taylor expansion of $X(t)$ w.r.t. $\mathcal{A}_\gamma$ exists for all $t \in [0, T]$.

(A4)

$$\int_0^T \mathbb{E}\left(|f_\alpha(s, X(s))|^2\right) ds < \infty$$

for all $\alpha \in \mathcal{B}(\mathcal{A}_\gamma)$.

Bistability

Definition

A one-step method is called bistable, if there exist constants $C, \tilde{C} > 0$ and an upper step size bound $\overline{h} > 0$ such that the operators $A_h : D(A_h) \to F_h$ are bijective and the inequality

$$C\|A_h Z_h - A_h \tilde{Z}_h\|_{-1} \leq \|Z_h - \tilde{Z}_h\|_0 \leq \tilde{C}\|A_h Z_h - A_h \tilde{Z}_h\|_{-1}$$

holds for all $h < \overline{h}$ and $Z_h, \tilde{Z}_h \in D(A_h)$.

Theorem

Under the assumptions (A1) - (A5) there exist parameter values $\rho, \sigma > 0$ such that the stochastic Itô-Taylor method is bistable.

Assumptions

(A1) ...

(A2) ...

(A3) For a given order γ the Itô-Taylor expansion of $X(t)$ w.r.t. $\mathcal{A}_\gamma$ exists for all $t \in [0, T]$.

(A4)

$$\int_0^T \mathbb{E}\left(|f_\alpha(s, X(s))|^2\right) ds < \infty$$

for all $\alpha \in \mathcal{B}(\mathcal{A}_\gamma)$.

(A5) For all $\alpha \in \mathcal{A}_\gamma$ there exists $L_\alpha > 0$ such that

$$|f_\alpha(t, x) - f_\alpha(t, y)| \leq L_\alpha |x - y|, \quad \forall t \in [0, T], \ x, y \in \mathbb{R}^d.$$

Convergence

Definition

A one-step method is called convergent with order $\gamma > 0$ if there exist an upper step size bound $\bar{h} > 0$ and a constant $C > 0$ such that the operators $A_h : D(A_h) \to F_h$ are bijective and

$$\|X_h - r_h^E X\|_0 \leq Ch^\gamma$$

for all $h \leq \bar{h}$. Here X_h denotes the solution to $A_h X_h = r_h^F AX$.

Two-sided error estimate

Let X_h be the solution to $A_h X_h = r_h^F AX$.

Theorem

Let the assumptions (A1) - (A5) hold and choose parameter values $\rho, \sigma > 0$ such that the stochastic Itô-Taylor method is bistable. Then the two-sided error estimate

$$C \|A_h X_h - A_h r_h^E X\|_{-1} \leq \|X_h - r_h^E X\|_0 \leq \tilde{C} \|A_h X_h - A_h r_h^E X\|_{-1}$$

holds for all $h < \bar{h}$. In particular, the stochastic Itô-Taylor method is convergent of order γ.

Maximum order of convergence

Example (Clark, Cameron 1980)

Consider the SODE

$$dX(t) = d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X_1(t) \end{pmatrix} d\begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the solution

$$X(t) = \begin{pmatrix} W^1(t) \\ \int_0^t W^1(s)dW^2(s) \end{pmatrix}$$

and the Euler-Maruyama scheme one computes

$$\|A_h r_h^E X - r_h^F AX\|_{-1} = \sqrt{\frac{1}{2}Th}.$$

Proof

One finds by the Martingale property and the Itô-isometry

$$
\begin{aligned}
&\|A_h r_h^E X - r_h^F A X\|_{-1,h}^2 \\
&= \max_{0\leq i\leq N} \mathbb{E}\left(\left|\textstyle\sum_{j=1}^i \int_{t_{j-1}}^{t_j} \left(W^1(s) - W^1(t_{j-1})\right) dW^2(s)\right|^2\right) \\
&= \max_{0\leq i\leq N} \left[\textstyle\sum_{j=1}^i \mathbb{E}\left(\left|\int_{t_{j-1}}^{t_j} \left(W^1(s) - W^1(t_{j-1})\right) dW^2(s)\right|^2\right)\right] \\
&= \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \mathbb{E}\left(\left|W^1(s) - W^1(t_{j-1})\right|^2\right) ds \\
&= \frac{1}{2} Th.
\end{aligned}
$$

Milstein scheme

Consider

$$dX(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dW^1(t) + \begin{pmatrix} 0 \\ X_1 \\ 0 \end{pmatrix} dW^2(t) + \begin{pmatrix} 0 \\ 0 \\ X_2 \end{pmatrix} dW^3(t),$$

$$X(0) = (0, 0, 0)^T,$$

and apply the Milstein scheme:

$$X_h(t_i) = X_h(t_{i-1}) + \begin{pmatrix} I_{(1),i} \\ X_h^1(t_{i-1}) I_{(2),i} + I_{(1,2),i} \\ X_h^2(t_{i-1}) I_{(3),i} + X_h^1(t_{i-1}) I_{(2,3),i}. \end{pmatrix}$$

Then

$$\|A_h r_h^E X - r_h^F A X\|_{-1} = \sqrt{\frac{1}{6} T h^2}.$$

Summary

- Discrete approximation theory provides “new” notion of consistency, stability and convergence.
- Stochastic Spijker-norm gives two-sided error estimate.

- Outlook
  - implicit schemes
  - multi-step methods
  - SPDE