A stability concept for stochastic onestep and multistep methods

Raphael Kruse

CRC 701: Spectral Structures and Topological Methods in Mathematics
Faculty of Mathematics
Bielefeld University

RDS’09
Nov 18th - 20th, 2009, Bielefeld

Outline

Overview
- SODE
- Numerical schemes
- Discrete approximation theory
- Main results
  - Characterization of bistability
  - Two-sided error estimates

Stability
- Bistability and Spijker’s norm
- Sketch of proof
  - Dahlquist’s strong root condition

Consistency
- Tools for the proof
- Optimal order of convergence

The stochastic ordinary differential equation (SODE)

Consider for $t \in [0, T]$

$$
\begin{aligned}
dX(t) &= b^0(t, X(t))dt + \sum_{r=1}^{m} b^r(t, X(t))dW^r(t), \\
X(0) &= X_0,
\end{aligned} \tag{SODE}
$$

where

- $b^r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ measurable,
- W^r real, independent, standard Brownian motions, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$,
- X_0 initial value: $\mathcal{F}_0$-mb. with values in $\mathbb{R}^d$.

Existence and uniqueness

(A1) $\mathbb{E}(|X_0|^2) < \infty$.

(A2) There exists $K > 0$ such that

$$|b^r(t,x)| \leq K(1 + |x|)$$

and

$$|b^r(t,x) - b^r(t,y)| \leq K|x - y|$$

for all $r = 0, \ldots, m$ and $x, y \in \mathbb{R}^d$, $t \in [0,T]$.

Theorem

Under assumptions (A1) + (A2) there exists a unique solution X to (SODE) with

$$X(t) = X_0 + \int_0^t b^0(s, X(s))ds + \sum_{r=1}^{m} \int_0^t b^r(s, X(s))dW^r(s)$$

and $\mathbb{E}(|X(t)|^2) < \infty$ *for all* $t \in [0,T]$.

Numerical schemes – general form

- equidistant time grid, $h = \frac{T}{N}$,

$$\tau_h = \{t_i = ih \,|\, i = 0, \ldots, N\}$$

- general k-step method

$$Y_i = \tilde{X}_i, \qquad \text{for } i = 0, \ldots, k-1,$$

$$\sum_{j=0}^{k} a_j Y_{i+j-k} = \Psi_h(t_i, Y_{i-k}, \ldots, Y_i, (I_\alpha^{t_{i+j-k}})_{\alpha \in \mathcal{A}, j=1,\ldots,k}),$$

for $i = k, \ldots, N$, where $\mathcal{A}$ is a finite set of multi-indices $\alpha = (j_1, \ldots, j_\ell)$, $j_n \in \{0, \ldots, m\}$, and

$$I_\alpha^{t_i} = \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{\ell-1}} dW^{j_1}(s_\ell) \ldots dW^{j_{\ell-1}}(s_2) dW^{j_\ell}(s_1).$$

Numerical schemes – Stochastic theta method

Let $\theta \in [0, 1]$,

$$\begin{aligned} Y_0 &= \tilde{X}_0, \\ Y_i - Y_{i-1} &= h\left((1-\theta)b^0(t_{i-1}, Y_{i-1}) + \theta b^0(t_i, Y_i)\right) \\ &\quad + \sum_{r=1}^{m} b^r(t_{i-1}, Y_{i-1})\Delta W^r(t_i) \end{aligned}$$

for $i = 1, \ldots, N$, where

$$\Delta W^r(t_i) = I_{(r)}^{t_i} = W^r(t_i) - W^r(t_{i-1}).$$

Note

$$\theta = 0 \quad \Rightarrow \text{ Euler-Maruyama}$$

Numerical schemes – Itô-Taylor schemes

Let $\gamma \in \left\{\frac{1}{2}n : n \in \mathbb{N}\right\}$ and

$$\mathcal{A}_\gamma = \left\{\alpha \in \mathcal{M} : \ell(\alpha) + n(\alpha) \leq 2\gamma \text{ or } \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\right\}$$

a set of multi-indices.

The Itô-Taylor scheme of order γ is given by

$$Y_0 = \tilde{X}_0,$$
$$Y_i - Y_{i-1} = \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{\nu\}} f_\alpha(t_{i-1}, Y_{i-1}) I_\alpha^{t_i}.$$

$\gamma = \frac{1}{2} \Rightarrow$ Euler-Maruyama scheme.

$\gamma = 1 \Rightarrow$ Milstein scheme.

(for details: e.g. P. E. Kloeden, E. Platen 1992)

Numerical schemes – BDF2-Maruyama scheme

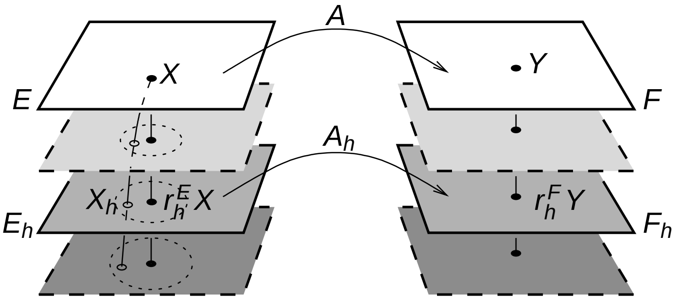
BDF2-Maruyama scheme as prototype for stochastic drift-linear multistep methods (E. Buckwar, R. Winkler 2006)

$$\begin{aligned} Y_0 &= \tilde{X}_0,\ Y_1 = \tilde{X}_1, \\ Y_i - \frac{4}{3}Y_{i-1} + \frac{1}{3}Y_{i-2} &= \frac{2}{3}hb^0(t_i, Y_i) \\ &\quad + \sum_{r=1}^{m} b^r(t_{i-1}, Y_{i-1})\Delta W^r(t_i) \\ &\quad - \frac{1}{3}\sum_{r=1}^{m} b^r(t_{i-2}, Y_{i-2})\Delta W^r(t_{i-1}) \end{aligned}$$

for $i = 1, \ldots, N$.

Discrete approximation theory

Schematic



(cf. F. Stummel, 1973)

To solve:

$$AX = Y$$

Numerical method:

$$A_h X_h = r_h^F Y$$

Embedding the SODE

The operator $A : E \to F$ is given by

- $E := \{X\}$,
- $F := \{Y\} = \{(0, \underline{0})\}$,
- $AX := \big(X(0) - X_0, (X(t) - X(0) - \int_0^t b^0(s, X(s))ds - \sum_{r=1}^m \int_0^t b^r(s, X(s))dW^r(s))_{t\geq 0}\big)$.

Discrete spaces

For $h = \frac{T}{N}$ denote by

$$\mathcal{G}_h := \mathcal{G}(\tau_h, L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d))$$

space of adapted and $L^2(\Omega)$-valued grid functions.

- $E_h := (\mathcal{G}_h, \|\cdot\|_0)$ with the norm

$$\|V_h\|_0 = \left(\mathbb{E}\left(\max_{0\le i\le N} |V_h(t_i)|^2\right)\right)^{\frac{1}{2}}.$$

- $F_h := (\mathcal{G}_h, \|\cdot\|_{-1})$ with the stochastic Spijker-norm

$$\|V_h\|_{-1} = \sum_{j=0}^{k-1} \|V_h(t_j)\|_{L^2(\Omega)} + \left(\mathbb{E}\left(\max_{k\le i\le N} |\textstyle\sum_{j=k}^{i} V_h(t_j)|^2\right)\right)^{\frac{1}{2}}$$

Discrete operators

The operator $A_h : E_h \to F_h$ is given by

$$[A_h V_h](t_i) = V_h(t_i) - \tilde{X}_i,$$

for $0 \leq i \leq k-1$ and

$$[A_h V_h](t_i) = \sum_{j=0}^{k} a_j V_h(t_{i+j-k})$$
$$- \Psi_h(t_i, V_h(t_{i-k}), \ldots, V_h(t_i), (I_\alpha^{t_{i+j-k}})_{\alpha \in \mathcal{A}, j=1,\ldots,k}),$$

for $i = k, \ldots, N$.
The operators $r_h^E : E \to E_h$ and $r_h^F : F \to F_h$ are the restriction operators to the time grid τ_h, i.e. $[r_h^E X](t_i) = X(t_i)$.

Consistency

Definition

A multistep method $(A_h)_{h>0}$ is called consistent of order $\gamma > 0$, if there exist constants $C, \overline{h} > 0$, s.t.

$$\left\| A_h r_h^E X - r_h^F A X \right\|_{-1} \leq C h^{\gamma}$$

for all $h \leq \overline{h}$.

Bistability

Definition

A multistep method $(A_h)_{h>0}$ is called bistable, if there exist constants $C, \tilde{C}, \overline{h} > 0$ such that the operators A_h are bijective and the inequality

$$C\|A_h Z_h - A_h \tilde{Z}_h\|_{-1} \le \|Z_h - \tilde{Z}_h\|_0 \le \tilde{C}\|A_h Z_h - A_h \tilde{Z}_h\|_{-1}$$

holds for all $h < \overline{h}$ and $Z_h, \tilde{Z}_h \in E_h$.

Characterization of bistability

Definition

The polynomial

$$\rho(z) = \sum_{j=0}^{k} a_j z^j$$

is the characteristic polynomial of the multistep method $(A_h)_{h>0}$.

Dahlquist's strong root condition

$\rho(\zeta) = 0,\ \zeta \in \mathbb{C} \quad \Rightarrow \quad |\zeta| < 1$ or $\zeta = 1$ simple.

Characterization of bistability

Theorem (Bistability)

Assume Ψ_h *satisfies the Lipschitz condition (S1) (see below) and* $\rho(1) = 0$, $a_k \neq 0$. *Then*

$$(A_h)_{h>0} \text{ is bistable}$$

if and only if

$$(A_h)_{h>0} \text{ satisfies the strong root condition.}$$

Two-sided error estimate

Let X_h be the solution to $A_h X_h = r_h^F AX = 0$.

Theorem

Assume that the multistep method $(A_h)_{h>0}$ is bistable. Then

$$(A_h)_{h>0} \text{ is consistent of order } \gamma$$

if and only if

$$(A_h)_{h>0} \text{ is convergent of order } \gamma.$$

Moreover the two-sided error estimate

$$C\|A_h X_h - A_h r_h^E X\|_{-1} \leq \|X_h - r_h^E X\|_0 \leq \tilde{C}\|A_h X_h - A_h r_h^E X\|_{-1}$$

holds for all $h < \overline{h}$.

Stability – assumptions

(S1) There exists $L > 0$ such that for all $j = k, \ldots, N$

$$\mathbb{E}\left(\max_{k\le i\le j}\left|\sum_{\eta=k}^{i}\left[\Psi_h(t_\eta, Y_h(t_{\eta-k}), \ldots, Y_h(t_\eta), (I_\alpha^{t_{\eta+n-k}})_{\alpha,n})\right.\right.\right.$$
$$\left.\left.\left.-\Psi_h(t_\eta, Z_h(t_{\eta-k}), \ldots, Z_h(t_\eta), (I_\alpha^{t_{\eta+n-k}})_{\alpha,n})\right]\right|^2\right)$$
$$\le Lh\sum_{i=0}^{j}\mathbb{E}\left(\max_{0\le\eta\le i}|Y_h(t_\eta) - Z_h(t_\eta)|^2\right)$$

and

$$\mathbb{E}\left(\left|\Psi_h(t_j, Y_h(t_{j-k}), \ldots, Y_h(t_j), (I_\alpha^{t_{j+i-k}})_{\alpha,i})\right.\right.$$
$$\left.\left.-\Psi_h(t_j, Z_h(t_{j-k}), \ldots, Z_h(t_j), (I_\alpha^{t_{j+i-k}})_{\alpha,i})\right|^2\right)$$
$$\le Lh\sum_{i=0}^{k}\mathbb{E}\left(\left|Y_h(t_{j+i-k}) - Z_h(t_{j+i-k})\right|^2\right)$$

for all $Y_h, Z_h \in \mathcal{G}_h$.

Stability – assumptions

(S2) The characteristic polynomial ρ of the multistep method satisfies $a_k \neq 0$, $\rho(1) = 0$ and Dahlquist's strong root condition.

Lemma

Under (A1) and (A2) the stability assumptions (S1) and (S2) are satisfied by the stochastic theta method and BDF2-Maruyama.

Stability - Itô-Taylor scheme

(S3) For Itô-Taylor scheme of order γ:
Assume that $b^r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ are sufficiently smooth, such that the operators $(A_h)_{h>0}$ are well-defined and there exists $K > 0$ such that

$$|f_\alpha(t,x)) - f_\alpha(t,y)| < K|x-y|$$

for all $\alpha \in \mathcal{A}_\gamma$, $t \in [0, T]$ and $x, y \in \mathbb{R}^d$.

Lemma

Under (A1), (A2) and (S3) the stability assumptions (S1) and (S2) are satisfied by the Itô-Taylor scheme of order γ.

Bistability theorem - sketch of proof

3 steps (cf. Grigorieff 1977)

1. (S1) $\Rightarrow$ invertibility of A_h.
2. $A_h = L_h + T_h$, A_h bistable iff L_h bistable.
3. L_h bistable iff strong root condition.

Lemma

Assume a multistep method $(A_h)_{h>0}$ satisfies $a_k \neq 0$ and (S1). Then there exists $\overline{h} > 0$ such that A_h is bijective for all $h < \overline{h}$.

$A_h = L_h + T_h$

Write $A_h = L_h + T_h$ with the linear part

$$[L_h V_h](t_i) = \begin{cases} V_h(t_i), & \text{for } i = 0, \ldots, k-1, \\ \sum_{j=0}^{k} a_j V_h(t_{i+j-k}), & \text{for } k \le i \le N. \end{cases}$$

Lemma

Assume a multistep method $(A_h)_{h>0}$ satisfies $a_k \neq 0$ and (S1). Then

$$(A_h)_{h>0} \text{ is bistable}$$

if and only if

$$(L_h)_{h>0} \text{ is bistable.}$$

Stability of the linear part L_h

Lemma

Assume $a_k \neq 0$ and $\rho(1) = 0$. Then

$$(L_h)_{h>0} \text{ is bistable}$$

if and only if

$$\rho \text{ satisfies the strong root condition.}$$

Consistency – Assumptions

(C1) There exists $K > 0$ such that

$$|b^r(t,x) - b^r(s,x)| \leq K(1+|x|)\sqrt{|t-s|}$$

for all $x \in \mathbb{R}^d$, $t, s \in [0,T]$.

(C2) For Itô-Taylor schemes of order γ:
Assume that $b^r : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are sufficiently smooth, such that the operators $(A_h)_{h>0}$ are well-defined and

$$\int_0^T \mathbb{E}\left(|f_\alpha(s,X(s))|^2\right) ds < \infty$$

for all $\alpha \in \mathcal{B}(\mathcal{A}_\gamma)$.

Consistency

Theorem

- *(A1), (A2) and (C1)*
  $\Rightarrow$ *STM and BDF2-Maruyama is consistent with order* $\frac{1}{2}$*.*
- *(A1), (A2) and (C2)*
  $\Rightarrow$ *The Itô-Taylor method of order* γ *is consistent of order* γ*.*

Tools for the proof:

- Doob's martingale inequality
- martingale property and Spijker's norm
- Itô-isometry

Maximum order of convergence

Example (Clark, Cameron 1980)

Consider the SODE

$$dX(t) = d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X_1(t) \end{pmatrix} d\begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the solution

$$X(t) = \begin{pmatrix} W^1(t) \\ \int_0^t W^1(s)dW^2(s) \end{pmatrix}$$

and the Euler-Maruyama scheme one computes

$$\sqrt{\frac{1}{2}Th} \leq \|A_h r_h^E X - r_h^F AX\|_{-1} \leq \sqrt{2Th}.$$

Proof – Lower bound

One finds by the martingale property and the Itô-isometry

$$
\begin{aligned}
&\|A_h r_h^E X - r_h^F AX\|_{-1}^2 \\
&= \mathbb{E}\left(\max_{1\le i\le N}\left|\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\left(W^1(s)-W^1(t_{j-1})\right)dW^2(s)\right|^2\right) \\
&\ge \max_{1\le i\le N}\mathbb{E}\left(\left|\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\left(W^1(s)-W^1(t_{j-1})\right)dW^2(s)\right|^2\right) \\
&= \max_{1\le i\le N}\left[\sum_{j=1}^{i}\mathbb{E}\left(\left|\int_{t_{j-1}}^{t_j}\left(W^1(s)-W^1(t_{j-1})\right)dW^2(s)\right|^2\right)\right] \\
&= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\mathbb{E}\left(\left|W^1(s)-W^1(t_{j-1})\right|^2\right)ds \\
&= \frac{1}{2}Th.
\end{aligned}
$$

Proof – Upper bound

One finds by Doob's martingale inequality

$$
\begin{aligned}
&\|A_h r_h^E X - r_h^F AX\|_{-1}^2 \\
&= \mathbb{E}\left(\max_{1\le i\le N}\left|\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\left(W^1(s)-W^1(t_{j-1})\right)dW^2(s)\right|^2\right) \\
&\le \left(\frac{2}{2-1}\right)^2 \mathbb{E}\left(\left|\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\left(W^1(s)-W^1(t_{j-1})\right)dW^2(s)\right|^2\right) \\
&= \dots \\
&= 2Th.
\end{aligned}
$$

Milstein scheme

Consider

$$dX(t) = \begin{pmatrix}1\\0\\0\end{pmatrix} dW^1(t) + \begin{pmatrix}0\\X_1\\0\end{pmatrix} dW^2(t) + \begin{pmatrix}0\\0\\X_2\end{pmatrix} dW^3(t),$$
$$X(0) = (0,0,0)^T,$$

and apply the Milstein scheme:

$$X_h(t_i) = X_h(t_{i-1}) + \begin{pmatrix} I_{(1)}^{t_i} \\ X_h^1(t_{i-1}) I_{(2)}^{t_i} + I_{(1,2)}^{t_i} \\ X_h^2(t_{i-1}) I_{(3)}^{t_i} + X_h^1(t_{i-1}) I_{(2,3)}^{t_i} \end{pmatrix}.$$

Then

$$\sqrt{\frac{1}{6}Th^2} \leq \|A_h r_h^E X - r_h^F A X\|_{-1} \leq \sqrt{\frac{2}{3}Th^2}.$$

Summary

- Discrete approximation theory provides “new” notion of consistency, stability and convergence.
- Dahlquist’s strong root condition $\Leftrightarrow$ bistability
- Stochastic Spijker-norm gives two-sided error estimate.
- works for a broad range of stochastic onestep and multistep methods

- Outlook
  - stoch. delay equations
  - SPDE

Stability of L_h

Main idea:

- Write $\rho(z) = (z-1)\rho^*(z)$ with $\rho^*(z) = \sum_{j=0}^{k-1} a_j^* z^j$
- Consider the operator $L_h^* : E_h \to F_h$ with

$$[L_h^* V_h](t_i) = \begin{cases} V_h(t_i), & \text{for } i = 0, \dots, k-2, \\ \sum_{j=0}^{k-1} a_j^* V_h(t_{i+j-k+1}), & \text{for } k-1 \leq i \leq N. \end{cases}$$

- Note that

$$\begin{aligned} \|L_h V_h\|_{-1} &= \sum_{j=0}^{k-1} \|V_h(t_j)\|_{L^2} + \left(\mathbb{E} \left(\max_{k \leq i \leq N} \left|\textstyle\sum_{j=k}^{i} L_h V_h(t_j)\right|^2 \right) \right)^{\frac{1}{2}} \\ &\leq \sum_{j=0}^{k-1} \|V_h(t_j)\|_{L^2} + 2 \left(\mathbb{E} \left(\max_{k-1 \leq i \leq N} |L_h^* V_h(t_i)|^2 \right) \right)^{\frac{1}{2}} \end{aligned}$$