Two-sided error estimates for strong Ito approximations of SODEs
Special case: stochastic theta method

Raphael Kruse

CRC 701: Spectral Structures and Topological Methods in Mathematics
Faculty of Mathematics
Bielefeld University

Spring School
Analytical and Numerical Aspects of Evolution Equations
March 30 - April 3, 2009
TU Berlin
The stochastic ordinary differential equation (SODE)

We consider for \( t \in [0, T] \)

\[
dX(t) = b^0(t, X(t))dt + \sum_{k=1}^{m} b^k(t, X(t))dW^k(t)
\]

\( X(0) = X_0, \) (SODE)

where

- \( b^k : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) measurable,
- \( W^k \) real, pairwise independent, standard Brownian motions, adapted to filtration \((\mathcal{F}_t)_{t \in [0, T]}\),
- \( X_0 \) initial value: \( \mathcal{F}_0 \)-mb. with values in \( \mathbb{R}^d \).
The stochastic theta method

For $\theta \in [0, 1]$ the STM is given by

$$0 = X_h(t_i) - X_h(t_{i-1}) - h(1 - \theta)b^0(t_{i-1}, X_h(t_{i-1}))$$

$$- h\theta b^0(t_i, X_h(t_i))$$

$$- \sum_{k=1}^{m} b^k(t_{i-1}, X_h(t_{i-1}))\Delta_h W^k(t_i), \quad 1 \leq i \leq N,$$

$$0 = X_h(t_0) - X_0,$$

where

- $h = \frac{T}{N}, \quad N \in \mathbb{N}$, equidistant step size,
- $\Delta_h W^k(t_i) = W^k(t_i) - W^k(t_{i-1})$.

Note: $\theta = 0 \Rightarrow Euler-Maruyama$ method.
Assumptions

(A1) \( \mathbb{E}(|X_0|^2) < \infty \)

(A2) There exists \( K > 0 \) such that

\[
|b^k(t, x)| \leq K(1 + |x|)
\]

and

\[
|b^k(t, x) - b^k(t, y)| \leq K|x - y|
\]

for all \( k = 0, \ldots, m \) and \( x, y \in \mathbb{R}^d, t \in [0, T] \).

(A3) There exists \( K > 0 \) such that

\[
|b^k(t, x) - b^k(s, x)| \leq K(1 + |x|)\sqrt{|t - s|}
\]

for all \( k = 0, \ldots, m \) and \( t, s \in [0, T] \), \( x \in \mathbb{R}^d \).
Existence and uniqueness result

Theorem (L. Arnold, 1974)

Under assumptions (A1) + (A2) there exists a unique solution to (SODE) with

\[ X(t) = X_0 + \int_0^t b^0(s, X(s)) \, ds + \sum_{k=1}^m \int_0^t b^k(s, X(s)) \, dW^k(s) \]

for \( t \in [0, T] \).
To solve:

\[ AX = Y \]

Numerical method:

\[ A_h X_h = r_h^F Y \]
Embedment of the STM

The operator $A : E \to F$ is given by

- $E := \{X\}$,
- $F := \{Y\} = \{(X_0, 0)\}$,
- $AX := (X(0), X(0) - \int_0^s b^0(s, X(s)) \, ds - \sum_{k=1}^m \int_0^s b^k(s, X(s)) \, dW^k(s))$.  

Discrete space

Let

$$
\tau_h := \{t_i \mid t_i = ih, \ i = 0, \ldots, N\}
$$

be the time grid. Denote by

$$
G_h := G(\tau_h, L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d))
$$

set of adapted and $L^2(\Omega)$-valued grid functions. Here $Z_h \in G_h$ is adapted if $Z_h(t_i)$ is $\mathcal{F}_{t_i}$ measurable for all $t_i \in \tau_h$. 
Metric space $E_h$

Restriction operators:

$$r_h^E : E \rightarrow \mathcal{G}_h$$

$$X \mapsto r_h^E X$$

with

$$[r_h^E X](t_i) = X(t_i).$$

We define

$$E_h := \overline{B}_\rho(r_h^E X) \subset \mathcal{G}_h$$

measured in the norm

$$\|Z_h\|_{0,h} := \max_{0 \leq i \leq N} \|Z_h(t_i)\|_{L^2(\Omega)}.$$
Metric space $F_h$

Restriction operators:

$$r^F_h : Y \mapsto r^E_h Y$$

with

$$[r^F_h Y](t_0) = X_0, \quad [r^F_h Y](t_i) = 0, \quad 1 \leq i \leq N.$$ 

We define

$$F_h := \overline{B}_\sigma(r^F_h Y) \subset G_h$$

measured in the stochastic Spijker-norm

$$\|Z_h\|_{-1,h} := \max_{0 \leq i \leq N} \|\sum_{j=0}^i Z_h(t_i)\|_{L^2(\Omega)}.$$
Residual mapping and the operator $A_h$

- Residual mapping $\text{res}_h : \mathcal{G}_h \to \mathcal{G}_h$ of the STM

\[
\text{res}_h(Z_h)(t_0) = Z_h(t_0) - X_0,
\]
\[
\text{res}_h(Z_h)(t_i) = Z_h(t_i) - Z_h(t_{i-1}) - h(1 - \theta)b^0(t_{i-1}, Z_h(t_{i-1}))
\]
\[
- h\theta b^0(t_i, Z_h(t_i))
\]
\[
- \sum_{k=1}^{m} b^k(t_{i-1}, Z_h(t_{i-1}))\Delta_h W^k(t_i), \quad 1 \leq i \leq N.
\]

- Note that if $X_h$ is a solution to STM then $\text{res}_h(X_h) = 0$.

- Operator $A_h : D(A_h) \to F_h$ is defined by

\[
A_h X_h = \text{res}_h(X_h) + r^F_h Y
\]

with domain of definition

\[
D(A_h) := \{ Z_h \in E_h \mid \|\text{res}_h(Z_h)\|_{-1,h} \leq \sigma \} \subset E_h.
\]
Discrete approximation theory

Schematic

To solve:

\[ AX = Y \]

Numerical method:

\[ A_h X_h = r_h^F Y \]

Question: Relationship between \( X \) and the numerical solutions \( X_h \)?
Consistency

Definition
A one-step method is called consistent of order $\gamma > 0$, if there exists a constant $C > 0$ and an upper step size bound $\bar{h}$, s.t.

$$\| A_h r_h^E X - r_h^F AX \|_{-1,h} \leq C h^\gamma$$

for all $h \leq \bar{h}$.

Theorem (Beyn, Kruse 2009)
Under the assumptions (A1) - (A3) the stochastic theta method is consistent of order $\gamma = \frac{1}{2}$. 


**Definition**

A one-step method is called **bistable**, if there exist constants $C, \tilde{C} > 0$ and an upper step size bound $\bar{h} > 0$ such that the operators $A_h : D(A_h) \rightarrow F_h$ are bijective and the inequality

$$C\|A_hZ_h - A_h\tilde{Z}_h\|_{-1,h} \leq \|Z_h - \tilde{Z}_h\|_{0,h} \leq \tilde{C}\|A_hZ_h - A_h\tilde{Z}_h\|_{-1,h}$$

holds for all $h < \bar{h}$ and $Z_h, \tilde{Z}_h \in D(A_h)$.

**Theorem (Beyn,Kruse 2009)**

*Under the assumptions (A1) - (A3) there exist parameter values $\rho, \sigma > 0$ such that the stochastic theta method is bistable.*
Convergence

Definition
A one-step method is called convergent with order $\gamma > 0$ if there exist an upper step size bound $\bar{h} > 0$ and a constant $C > 0$ such that the operators $A_h : D(A_h) \to F_h$ are bijective and

$$\|X_h - r^E_h X\|_{0,h} \leq Ch^\gamma$$

for all $h \leq \bar{h}$. Here $X_h$ denotes the solution to $A_h X_h = r^F_h AX$. 
Two-sided error estimate

Let $X_h$ be the solution to $A_hX_h = r_h^FAX$.

**Theorem (Beyn, Kruse 2009)**

Let the assumptions (A1) - (A3) hold and choose parameter values $L, \rho, \sigma > 0$ such that the stochastic theta method is bistable. Then the two-sided error estimate

$$C \|A_hX_h - A_hr_h^E X\|_{-1,h} \leq \|X_h - r_h^E X\|_{0,h} \leq \tilde{C} \|A_hX_h - A_hr_h^E X\|_{-1,h}$$

holds for all $h < \bar{h}$. In particular, the stochastic theta method is convergent of order $\gamma = \frac{1}{2}$.
Example (Clark, Cameron 1980)
Consider the SODE
\[
dX(t) = d \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X_1(t) \end{pmatrix} d \begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}
\]
\[
X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
For the solution
\[
X(t) = \begin{pmatrix} W^1(t) \\ \int_0^t W^1(s) dW^2(s) \end{pmatrix}
\]
one calculates
\[
\| A_h r_h^E X - r_h^F AX \|_{-1,h} = \sqrt{\frac{1}{2} Th}.
\]
Summary

- Discrete approximation theory provides “new” notion of consistency, stability and convergence.
- Stochastic Spijker-norm gives two-sided error estimate.

Outlook

- Stochastic delay equations
- SPDE (partially done for heat equation and finite difference method)
Second order estimate

Theorem (Arnold 1973)

Under assumptions (A1) and (A2) there exists $C_1 > 0$ s.t.

$$
\mathbb{E}(|X(t)|^2) \leq (1 + \mathbb{E}(|X_0|^2))e^{C_1 t}
$$

and there exists $C_2 > 0$ s.t.

$$
\mathbb{E}(|X(t) - X(s)|^2) \leq C_2 |t - s|
$$

for $t, s \in [0, T]$. 