Two-sided error estimates for strong Ito approximations of SODEs

Special case: stochastic theta method

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The stochastic ordinary differential equation (SODE)

We consider for $t \in [0, T]$

$$dX(t) = b^0(t, X(t))dt + \sum_{k=1}^{m} b^k(t, X(t))dW^k(t) \qquad \text{(SODE)}$$
$$X(0) = X_0,$$

where

- $b^k : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ measurable,
- W^k real, pairwise independent, standard Brownian motions, adapted to filtration $(\mathcal{F}_t)_{t \in [0,T]}$,
- X_0 initial value: $\mathcal{F}_0$-mb. with values in $\mathbb{R}^d$.

The stochastic theta method

For $\theta \in [0,1]$ the STM is given by

$$\begin{aligned} 0 &= X_h(t_i) - X_h(t_{i-1}) - h(1-\theta)b^0(t_{i-1}, X_h(t_{i-1})) \\ &\quad - h\theta b^0(t_i, X_h(t_i)) \\ &\quad - \sum_{k=1}^{m} b^k(t_{i-1}, X_h(t_{i-1}))\Delta_h W^k(t_i), \quad 1 \leq i \leq N, \\ 0 &= X_h(t_0) - X_0, \end{aligned}$$

where

- $h = \frac{T}{N}$, $N \in \mathbb{N}$, equidistant step size,
- $\Delta_h W^k(t_i) = W^k(t_i) - W^k(t_{i-1})$.

Note: $\theta = 0 \Rightarrow$ *Euler-Maruyama* method.

Assumptions

(A1) $\mathbb{E}(|X_0|^2) < \infty$

(A2) There exists $K > 0$ such that

$$|b^k(t,x)| \leq K(1 + |x|)$$

and

$$|b^k(t,x) - b^k(t,y)| \leq K|x - y|$$

for all $k = 0, \ldots, m$ and $x, y \in \mathbb{R}^d$, $t \in [0, T]$.

(A3) There exists $K > 0$ such that

$$|b^k(t,x) - b^k(s,x)| \leq K(1 + |x|)\sqrt{|t - s|}$$

for all $k = 0, \ldots, m$ and $t, s \in [0, T]$, $x \in \mathbb{R}^d$.

Existence and uniqueness result

Theorem (L. Arnold, 1974)

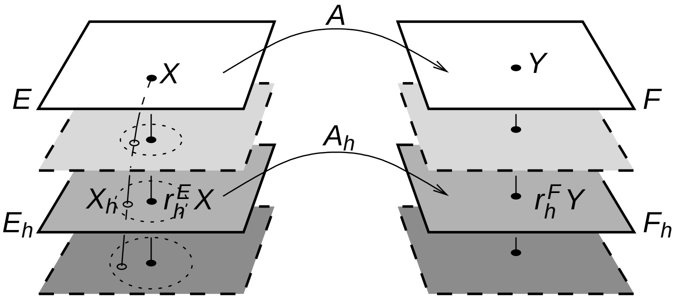
Under assumptions (A1) + (A2) there exists a unique solution to (SODE) with

$$X(t) = X_0 + \int_0^t b^0(s, X(s))ds + \sum_{k=1}^{m} \int_0^t b^k(s, X(s))dW^k(s)$$

for $t \in [0, T]$.

Discrete approximation theory

Schematic



To solve:

$$AX = Y$$

Numerical method:

$$A_h X_h = r_h^F Y$$

Embedment of the STM

The operator $A : E \to F$ is given by

- $E := \{X\}$,
- $F := \{Y\} = \{(X_0, 0)\}$,
- $AX :=$
  $\big(X(0), X(0) - \int_0^{\cdot} b^0(s, X(s))ds - \sum_{k=1}^m \int_0^{\cdot} b^k(s, X(s))dW^k(s)\big)$.

Discrete space

Let

$$\tau_h := \{t_i \mid t_i = ih,\ i = 0, \dots, N\}$$

be the time grid. Denote by

$$\mathcal{G}_h := \mathcal{G}(\tau_h, L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d))$$

set of adapted and $L^2(\Omega)$-valued grid functions.
Here $Z_h \in \mathcal{G}_h$ is adapted if $Z_h(t_i)$ is $\mathcal{F}_{t_i}$ measurable for all $t_i \in \tau_h$.

Metric space E_h

Restriction operators:

$$r_h^E : \begin{array}{l} E \to \mathcal{G}_h \\ X \mapsto r_h^E X \end{array}$$

with

$$[r_h^E X](t_i) = X(t_i).$$

We define

$$E_h := \overline{B}_\rho(r_h^E X) \subset \mathcal{G}_h$$

measured in the norm

$$\|Z_h\|_{0,h} := \max_{0 \le i \le N} \|Z_h(t_i)\|_{L^2(\Omega)}.$$

Metric space F_h

Restriction operators:

$$r_h^F : \begin{array}{l} F \to \mathcal{G}_h \\ Y \mapsto r_h^E Y \end{array}$$

with

$$[r_h^F Y](t_0) = X_0, \quad [r_h^F Y](t_i) = 0, \; 1 \le i \le N.$$

We define

$$F_h := \overline{B}_\sigma(r_h^F Y) \subset \mathcal{G}_h$$

measured in the stochastic Spijker-norm

$$\|Z_h\|_{-1,h} := \max_{0 \le i \le N} \|\textstyle\sum_{j=0}^{i} Z_h(t_i)\|_{L^2(\Omega)}.$$

Residual mapping and the operator A_h

- Residual mapping $\mathrm{res}_h : \mathcal{G}_h \to \mathcal{G}_h$ of the STM

$$\begin{aligned}
\mathrm{res}_h(Z_h)(t_0) &= Z_h(t_0) - X_0, \\
\mathrm{res}_h(Z_h)(t_i) &= Z_h(t_i) - Z_h(t_{i-1}) - h(1-\theta)b^0(t_{i-1}, Z_h(t_{i-1})) \\
&\quad - h\theta b^0(t_i, Z_h(t_i)) \\
&\quad - \textstyle\sum_{k=1}^m b^k(t_{i-1}, Z_h(t_{i-1}))\Delta_h W^k(t_i), \ 1 \le i \le N.
\end{aligned}$$

- Note that if X_h is a solution to STM then $\mathrm{res}_h(X_h) = 0$.
- Operator $A_h : \ D(A_h) \to F_h$ is defined by

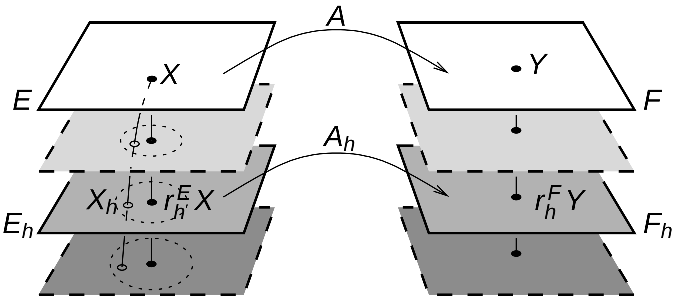
$$A_h X_h = \mathrm{res}_h(X_h) + r_h^F Y$$

with domain of definition

$$D(A_h) := \left\{ Z_h \in E_h \mid \|\mathrm{res}_h(Z_h)\|_{-1,h} \le \sigma \right\} \subset E_h.$$

Discrete approximation theory

Schematic



To solve:

$$AX = Y$$

Numerical method:

$$A_h X_h = r_h^F Y$$

Question: Relationship between X and the numerical solutions X_h?

Consistency

Definition

A one-step method is called consistent of order $\gamma > 0$, if there exists a constant $C > 0$ and an upper step size bound $\overline{h}$, s.t.

$$\left\| A_h r_h^E X - r_h^F A X \right\|_{-1,h} \leq C h^\gamma$$

for all $h \leq \overline{h}$.

Theorem (Beyn, Kruse 2009)

Under the assumptions (A1) - (A3) the stochastic theta method is consistent of order $\gamma = \frac{1}{2}$.

Bistability

Definition

A one-step method is called bistable, if there exist constants $C, \tilde{C} > 0$ and an upper step size bound $\overline{h} > 0$ such that the operators $A_h : D(A_h) \to F_h$ are bijective and the inequality

$$C \|A_h Z_h - A_h \tilde{Z}_h\|_{-1,h} \leq \|Z_h - \tilde{Z}_h\|_{0,h} \leq \tilde{C} \|A_h Z_h - A_h \tilde{Z}_h\|_{-1,h}$$

holds for all $h < \overline{h}$ and $Z_h, \tilde{Z}_h \in D(A_h)$.

Theorem (Beyn,Kruse 2009)

Under the assumptions (A1) - (A3) there exist parameter values $\rho, \sigma > 0$ such that the stochastic theta method is bistable.

Convergence

Definition

A one-step method is called convergent with order $\gamma > 0$ if there exist an upper step size bound $\bar{h} > 0$ and a constant $C > 0$ such that the operators $A_h : D(A_h) \to F_h$ are bijective and

$$\|X_h - r_h^E X\|_{0,h} \leq Ch^\gamma$$

for all $h \leq \bar{h}$. Here X_h denotes the solution to $A_h X_h = r_h^F AX$.

Two-sided error estimate

Let X_h be the solution to $A_h X_h = r_h^F AX$.

Theorem (Beyn, Kruse 2009)

Let the assumptions (A1) - (A3) hold and choose parameter values $L, \rho, \sigma > 0$ *such that the stochastic theta method is bistable. Then the two-sided error estimate*

$$C \| A_h X_h - A_h r_h^E X \|_{-1,h} \leq \| X_h - r_h^E X \|_{0,h} \leq \tilde{C} \| A_h X_h - A_h r_h^E X \|_{-1,h}$$

holds for all $h < \overline{h}$. *In particular, the stochastic theta method is convergent of order* $\gamma = \frac{1}{2}$.

Maximum order of convergence

Example (Clark, Cameron 1980)

Consider the SODE

$$dX(t) = d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X_1(t) \end{pmatrix} d\begin{pmatrix} W^1(t) \\ W^2(t) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the solution

$$X(t) = \begin{pmatrix} W^1(t) \\ \int_0^t W^1(s)dW^2(s) \end{pmatrix}$$

one calculates

$$\|A_h r_h^E X - r_h^F AX\|_{-1,h} = \sqrt{\frac{1}{2}Th}.$$

Summary

- Discrete approximation theory provides “new” notion of consistency, stability and convergence.
- Stochastic Spijker-norm gives two-sided error estimate.

- Outlook
  - Stochastic delay equations
  - SPDE (partially done for heat equation and finite difference method)

Second order estimate

Theorem (Arnold 1973)

Under assumptions (A1) and (A2) there exists $C_1 > 0$ *s.t.*

$$\mathbb{E}(|X(t)|^2) \leq (1 + \mathbb{E}(|X_0|^2))e^{C_1 t}$$

and there exists $C_2 > 0$ *s.t.*

$$\mathbb{E}(|X(t) - X(s)|^2) \leq C_2 |t - s|$$

for $t, s \in [0, T]$.