SS 2021
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Exercises for Functional Analysis

Exercise 10
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Digital submission via the E-Learning site of the tutorial

Exercise 1.
Prove that the following functionals are in $[\ell^p]'$ and calculate their norms.

(a) $l(x) := x_1 + x_2, \qquad x = (x_1, x_2, ...), \quad p = 2$

(b) $l(x) := \sum_{k=1}^{\infty} \frac{x_k}{k}, \qquad x = (x_1, x_2, ...), \quad p = 1$

(4 Points)

Exercise 2.
Let $f \in L^2((0,1))$. Check if the following linear functional

$$g \mapsto \int_0^1 \frac{\mathrm{d}g}{\mathrm{d}x} f \ \mathrm{d}x$$

is continuous on the following spaces

(a) $H^{1,2}((0,1))$

(b) $C^1([0,1])$

(4 Points)

Exercise 3.
Let H be a real Hilbert space and $T \in L(H)$. Prove that

(a) $\|T\|_{L(H)} = \sup_{x,y \in H,\ \|x\|=\|y\|=1} (Tx, y)$

(b) *If T is symmetric (i.e. $(Tx, y) = (x, Ty)$ for all $x, y \in H$), then the following holds*

$$\|T\|_{L(H)} = \sup_{x \in H,\ \|x\|=1} |(Tx, x)|.$$

Hint: $T(x,y) = \frac{1}{4}\left((T(x+y), x+y) - (T(x-y), x-y)\right)$

(4 Points)