Exercises to Stochastic Analysis

Sheet13
Total points: 16
Submission before: Friday, 27.01.2023, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Weak solutions for constant-diffusion SDE by Girsanov). (1+2+2+1+2 Points)

In this exercise, you learn how to apply the Girsanov theorem to obtain probabilistically weak solutions to SDES with constant diffusion, i.e. (comparing with the notation of Ch.6) $\sigma \equiv 1$. Notably, this works for very irregular drift (in fact, no regularity assumption beyond measurability is needed at all, as you see below!). Recall that the corresponding ODE $\dot{x}_t = b(t, x_t)$ might not have a solution if b is only bounded and measurable! Hence, we see here an instance of the very important concept of regularization by noise.

Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded and product measurable wrt. the usual Euclidean Borel σ-algebras. Consider, for arbitrary but fixed $x \in \mathbb{R}$, the SDE

$$dX_t = b(t, X_t)dt + dB_t, \quad t \geqslant 0, \quad X_0 = x, \tag{SDE}$$

i.e. we consider the "Markov case", meaning that b (and, of course, $\sigma \equiv= 1$ as well) depend only on X_t at time t.

(i) Prove that any probabilistically weak solution process $X = (X_t)_{t\geqslant 0}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has $\mathbb{P}$-almost surely continuous sample paths. Moreover, make an "educated guess" whether in general one can expect the sample paths to be differentiable in $t \geqslant 0$.

(ii) Use the Girsanov theorem for Brownian motion to prove the existence of a probabilistically weak solution process X^T to (SDE) on each $[0, T]$ (i.e., for fixed $T > 0$, one considers (SDE) only for $t \in [0, T]$).

(iii) Use (ii) and the Girsanov theory to show the existence of a probabilistically weak solution process $X = X^\infty$ to (SDE) on $[0, \infty)$.

(iv) For X as in (iii), show $\mathbb{P}(X_t \geqslant M) > 0$ for all $M < \infty$ and $t > 0$, where $\mathbb{P}$ denotes the probability measure of the probability space on which X is defined.

(v) Now let $b(t, x) = -r$ for a constant $r > 0$. Prove that in this case for X as in (iii), one has $X_t \longrightarrow -\infty$ as $t \to \infty$, $\mathbb{P}$-a.s.

Problem 2 (Assumption on the Brownian motion in the definition of weak solutions to SDEs). (3 Points)

Consider the definition of weak solutions to an SDE. One can ask the question why the definition requires an $(\mathcal{F}_t)$-Brownian motion, i.e. a Brownian motion $B = (B_t)_{t\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, which is $(\mathcal{F}_t)$-adapted and for which

$$B_t - B_s \text{ is independent of } \mathcal{F}_s \text{ for all } 0 \leqslant s \leqslant t,$$

since this is not required for the well-definedness of the stochastic integral. Recall that by the theory of Ch.2.3., in general the integrator of a stochastic integral needs to be a martingale wrt. the given filtration in order to obtain that the stochastic integral is well-defined and a martingale (actually, one could drop the assumption that the integrator is a martingale and only require that it is adapted to the given filtration. Then the stochastic integral can still be defined, but will not be a martingale. But in this case, weak solution processes to SDEs will not be semi-martingales, which is, however very desirable). So, the natural question arises whether the a priori stronger assumption that B is an $(\mathcal{F}_t)$-Brownian motion is more restrictive (i.e. whether it is a strictly stronger assumption). This exercise shows that this is not the case. Consequently, the required properties of B in the definition of a weak solution are natural, if one wants weak solutions to be semi-martingales.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space with a standard Brownian motion B. Prove that B is an $(\mathcal{F}_t)$-martingale if and only if B is an $(\mathcal{F}_t)$-Brownian motion.

Problem 3 (Unique representation of Itô-processes). (5 Points)

An important class of stochastic processes with regard to SDEs are the so-called Itô-processes. *An Itô-process is a process X of type* (3.1). *An important subclass of these are solutions to SDEs (then σ and b depend on X), but here we consider the general case and prove that the representation of an Itô-process in terms of an initial value (a random variable), the "drift" b and the "diffusion term" σ is unique. Itô-processes are not to be confused with the Itô-representation theorem from Sect.2.5, although there is an obvious connection: Every stochastic integral is in particular an Itô-process without drift and with initial value* 0.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space and, for $i \in \{1,2\}$, $\sigma^{(i)}, b^{(i)} : [0,\infty) \times \Omega \to \mathbb{R}$ be $(\mathcal{F}_t)$-adapted such that $\mathbb{E}[\int_0^t |b_s^{(i)}| ds] < \infty$ and $\mathbb{E}\big[\int_0^t |\sigma_s|^2 ds\big] < \infty$ for all $t \geqslant 0$. Moreover, let $\tilde{X}^{(i)} : \Omega \to \mathbb{R}$ be $\mathcal{F}_0$-measurable and assume $X = (X_t)_{t\geqslant 0}$ is an $(\mathcal{F}_t)$-adapted process on Ω such that for $i \in \{1,2\}$

$$X_t = \tilde{X}^{(i)} + \int_0^t b_s^{(i)} ds + \int_0^t \sigma_s^{(i)} dB_s, \quad t \geqslant 0, \quad \mathbb{P}-\text{a.s.} \tag{3.1}$$

Prove the following assertions:

(i) $\tilde{X}^{(1)} = \tilde{X}^{(2)}$, $\sigma^{(1)} = \sigma^{(2)}$ and $b^{(1)} = b^{(2)}$, where the first identity holds $\mathbb{P}$-a.s. in the sense of paths, and the second and third $\mathbb{P} \otimes dt$-a.s.

(ii) Now we drop the superindices from the notation and simply write $\tilde{X}, \sigma_s$ and b_s. Prove: X is deterministic (i.e. there is a zero set $N \in \mathcal{F}$ such that for all $\omega_1, \omega_2 \in N^c$, the paths $t \mapsto X_t(\omega_1)$ and $t \mapsto X_t(\omega_2)$ coincide) if and only if $\sigma \equiv 0$ and $\tilde{X}$ and b are deterministic.

Please note: There will be a further exercise sheet 14, published on Friday, 27.01. after the lecture and due on Thursday, 02.02. at noon. Since there is no time for regular tutorials, you can collect your solutions at my office after the lecture on Friday, 03.02. (V4-128). Alternatively, I will offer a tutorial for this sheet directly after the lecture on Friday, 03.02., if sufficiently many of you attend. Therefore, please let me know via mail (mrehmeier@math.uni-bielefeld.de) until Monday, 30.01., whether you would like such a tutorial.
The reason is that without an additional sheet, you would see no exercises on SDEs (and the martingale problem), which are, however, among the most important parts of the lecture. The sheet will be shorter than average and partially consist of bonus points.