Summer term 2022
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Exercises to Probability Theory II

Sheet 6
Submission before: Friday, 20.05.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 15. (Stopping Times and Filtrations)
Let T and R be stopping times in the sense of Definition 7.5.1 with $\Omega := S^{\mathbb{N}_0}$ in the situation of Section 7.1. Show the following:

(i) If $T = n$ for an $n \in \mathbb{N}_0$, then $\mathcal{A}_T = \mathcal{A}_n$. (1 point)

(ii) If $R \leqslant T$, then $\mathcal{A}_R \subseteq \mathcal{A}_T$. (1.5 points)

(iii) T is $\mathcal{A}_T$-measurable. (1.5 points)

Problem 16. (Random walk on $\mathbb{Z}$)
Let (X_n) be a random walk on $\mathbb{Z}$ in the sense of Def. 7.2.2 (i).

(i) Let $(X_n)_{n\in\mathbb{N}}$ be a general (i.e. not necessarily symmetric) random walk with parameters $r, u \in [0,1]$, $r + u = 1$. Show that $Y_i := X_i - X_{i-1}$, $i \in \mathbb{N}$, are independent. (2 points)

Hint: Use Remark 2.5.6!

(ii) Now consider a symmetric random walk (i.e. $r = u = 1/2$). Consider the **Dirichlet problem** for this random walk as in Ex. 7.6.2, i.e. $A =]0, \infty[$, $f = 1_{]-\infty,0[}$. Show that for any $c \geqslant 0$

$$h(x) = \begin{cases} 1 + cx, & \text{for } x > 0, \\ 1, & \text{for } x \leqslant 0 \end{cases}$$

is a solution of the Dirichlet problem for A and f. (2 points)

Problem 17. (Reversibility)
Consider the situation of Section 7.1. Let S be countable. Then, a measure $\mu\colon \mathcal{A} \to [0,1]$ is reversible if and only if[1]

$$\mu(x)p(x,y) = \mu(y)p(y,x), \qquad (**)$$

for all $x, y \in S$. (4 points)

[1] As usual in the discrete case, we set $\mu(x) := \mu(\{x\})$, $p(x,y) := p(x,\{y\})$.