Summer term 2022
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory II

Sheet 8
Submission before: Friday, 03.06.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 18. (Doob's L^p maximal inequality for $p = 1$)
Let $(X_t)_{t\geqslant 0}$ be a right-continuous submartingale with $X_t \geqslant 0$ for all $t \geqslant 0$, and let $X_t^* := \sup_{0\leqslant s\leqslant t} X_s$.

(a) Show that

$$\mathbb{E}[X_t^* 1_{\{X_t^*\geqslant 1\}}] \leqslant \mathbb{E}[X_t \log(X_t^*) 1_{\{X_t^*\geqslant 1\}}] + P(X_t^* \geqslant 1).$$

Hint: Calculate $\mathbb{E}[X_t^* 1_{\{X_t^*\geqslant 1\}}] - P(X_t^* \geqslant 1)$ using the trick of in the proof of Proposition 8.2.1 (ii), Fubini, and Proposition 8.2.1 (i). (4 points)

(b) Conclude from (a) and the inequality $x - 1 \leqslant x\log(x)$ that (2 points)

$$\mathbb{E}[X_t^*] \leqslant \mathbb{E}[X_t \cdot \log(X_t^*)] + 1.$$

(c) Show that for $a, b > 0$ (1 point)

$$a\log(b) \leqslant a\log(a) + \frac{b}{e}$$

(d) Show using (b) and (c) that (1 point)

$$\mathbb{E}[X_t^*] \leqslant \frac{e}{e-1}\left(\mathbb{E}[X_t \cdot \log(X_t)] + 1\right)$$

Problem* 19. (Roulette and supermartingales)
Assume that when playing Roulette, we bet RED over and over (either we lose our bet, or we get twice as much back; the winning probability is $\frac{18}{37}$). We start with 1 Euro, and every time we lose, we double our bet. If we win once, we stop. Show that the Optional Stopping Theorem does not hold for unbounded stopping times by giving a supermartingale $(M_n)_{n\in\mathbb{N}_0}$ and a stopping time T such that

$$\mathbb{E}[M_T] > \mathbb{E}[M_0].$$

(4 points)

Problem 20. (Remark 8.4.5)
Let $I = \mathbb{R}_+$ and $(\mathcal{A}_t)_{t\geqslant 0}$ be a filtration. Define

$$\mathcal{A}_t^+ := \mathcal{A}_{t+} := \bigcap_{s>t} \mathcal{A}_s.$$

Let $(X_t)_{t\geqslant 0}$ be a right-continuous, $(\mathcal{A}_t^+)_{t\geqslant 0}$-adapted family of (real-valued) random variables and $U \subset \mathbb{R}$ be an open subset. We define the first entry time in U by

$$T := \inf\{t \geqslant 0 \mid X_t \in U\}.$$

(a) Show that T is an $(\mathcal{A}_t^+)$-stopping time. (3 points)
Hint: Show that if $T(\omega) < t$, there exists a rational number $q < t$ such that $X_q(\omega) \in U$.

(b) Give a concrete example for Ω (with two elements!), $(X_t)_{t \geqslant 0}$, $(\mathcal{A}_t)_{t \geqslant 0}$ and U for which T is not an $(\mathcal{A}_t)$-stopping time. (1 point)