TWO-DIMENSIONAL SEMI-LOG-CANONICAL HYPERSURFACES

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Abstract. We derive explicit equations for all two-dimensional, semi-log-canonical hypersurface singularities by an elemetary method.

1. Introduction

One of the milestones of the theory of normal surfaces singularities is the classification of what are now called canonical surface singularities by Du Val [4]. Their importance stems from the fact that these are exactly the singularities that appear on canonical models of surfaces of general type; from this point of view they were defined and studied in all dimensions by Reid [15].

While a modular compactification of the moduli space of curves had been constructed by Deligne, Mumford, and Knudson in the sixties, it was only 20 years later that Kollár and Shepherd–Barron made the first step for surfaces by considering the following question: "Which singular surfaces do we have to allow to get a modular compactification of the moduli space of (smooth) canonically polarised surfaces?" Inspired by results from the minimal model program this lead to the definition of *semi-log-canonical* singularities in [11]. The name was chosen to indicate that these are non-normal analogues for log-canonical singularities, which had been defined previously in minimal model theory.

Over the past decades, major developments in these areas have, among other results, led to a construction of the moduli space of stable surfaces (in characteristic 0), which is compact and contains the moduli space of surfaces of general type as an open subset [6].

Semi-log-canonical surface singularities had been classified in terms of their minimal semi-resolutions already in [11] extending the classification in the normal case. However, general log-canonical and even more so semi-log-canonical singularities can be quite complicated.

For example, while canonical surfaces singularities are equivalently characterised as ordinary hypersurface singularities or as rational double points, a general log-canonical singularity is not rational and can have arbitrarily high embedding dimension. In particular, a classification of such singularities up to local analytic isomorphism is out of reach.

The aim of the present article is to understand the hypersurface case over the complex numbers.

Theorem — *Every complex semi-log-canonical hypersurface singularity of dimension two is locally analytically isomorphic*[1] *to one of the singularities* $0 \in S \subset \mathbb{C}^3$ *given in Table 1.*

In the normal case, compiling the list was a matter of collecting the relevant results for simple elliptic singularities from [10, Thm. 4.57] and for cusps from [7, Thm. 3], see also [18, 14, 12, 13, 17]. The case of du Val singularities is classical and can be found in [10, Ch. 4] together with much more information on general log-canonical surface singularities. So our contribution consists in the classification of non-normal semi-log-canonical hypersurface singularities in dimension two by elemetary means. Some examples are shown in Figure 1 on page 6 and Figure 2 on page 8.

After the completion of this article we noticed that the singularities we considered had been classified in work of Stevens, Shepherd-Barron and others [19, 20] under different names

2010 *Mathematics Subject Classification.* 32S25; 14J17, 14B05.

[1]Considering formal instead of analytic neighbourhoods the result also holds for algebraic varieties over an algebraically closed field of characteristic 0.

Table 1. Semi-log-canonical hypersurface singularities in dimension two

type*	name	symbol	equation $f \in \mathbb{C}[x,y,z]$		$\mathrm{mult}_0(f)$
terminal	smooth	(A_0)	x		1
canonical	du Val	A_n	$x^2+y^2+z^{n+1}$	$n \geq 1$	2
		D_n	$x^2+z(y^2+z^{n-2})$	$n \geq 4$	2
		E_6	$x^2+y^3+z^4$		2
		E_7	$x^2+y^3+yz^3$		2
		E_8	$x^2+y^3+z^5$		2
log-canonical	simple elliptic	$X_{1,0}$	$x^2+y^4+z^4+\lambda xyz$	$\lambda^4 \neq 64$	2
		$J_{2,0}$	$x^2+y^3+z^6+\lambda xyz$	$\lambda^6 \neq 432$	2
		$T_{3,3,3}$	$x^3+y^3+z^3+\lambda xyz$	$\lambda^3+27 \neq 0$	3
	cusp	$T_{p,q,r}$	$xyz+x^p+y^q+z^r$	$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$	2 or 3
semi-log-canonical	normal crossing	A_∞	x^2+y^2		2
	pinch point	D_∞	x^2+y^2z		2
	degenerate cusp	$T_{2,\infty,\infty}$	$x^2+y^2z^2$		2
		$T_{2,q,\infty}$	$x^2+y^2(z^2+y^{q-2})$	$q \geq 3$	2
		$T_{\infty,\infty,\infty}$	xyz		3
		$T_{p,\infty,\infty}$	$xyz+x^p$	$p \geq 3$	3
		$T_{p,q,\infty}$	$xyz+x^p+y^q$	$q \geq p \geq 3$	3

* The distinction in the first column is understood to be inclusive, that is, terminal ⇒ canonical ⇒ log-canonical ⇒ semi-log-canonical.

(see also [21, Lemma 2.6]). We believe however, that with our focus on semi-log-canonical hypersurfaces the classification becomes a bit more transparent and accessible.

The paper is organised as follows. In Section 2 we recall some basic facts about semi-log-canonical singularities and from local analytic geometry. Then in Sections 3 and 4 we classify non-normal semi-log-canonical double points respectively triple points. Our methods are quite elementary, using little more than the Weierstrass Preparation Theorem and blow-ups; in the triple point case our approach is inspired by Arnold [1].

Besides giving concrete examples of semi-log-canonical singularities our classification has further consequences. For example, if $S \subset \mathbb{P}^3$ is a stable surface, that is, a surface of degree at least 5 with semi-log-canonical singularities, then the singular locus of S consists of isolated points together with a curve that has at most ordinary double points or ordinary triple points of embedding dimension 3 as singularities.

Acknowledgements. Both authors were supported by the DFG via the second authors Emmy–Noether project and partly by SFB/TR 45 "Periods, moduli spaces and arithmetic of algebraic varieties" and SFB 701 "Spectral Structures and Topological Methods in Mathematics". We would like to thank the referee for a careful reading of the manuscript and many useful suggestions.

2. Preparations

2.1. **Semi-log-canonical singularities.** We now start to define semi-log-canonical singularities and related notions from the minimal model program. See [10] for a general introduction to this circle of ideas. We adopt the convention that a variety is a scheme of finite type over $\mathbb{C}$ or a complex space which is reduced and pure-dimensional but not necessarily irreducible.

Let us consider a simple example as a motivation. If C is a reduced curve then one way to understand the singularities of C is to consider the normalisation $\nu\colon \tilde{C} \to C$ and on $\tilde{C}$ the divisor D defined by the conductor ideal $\mathcal{H}om_{\mathcal{O}_C}(\nu_*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C)$. It is easy to see that C has only ordinary nodes if and only if D is a sum of distinct points.

So we will define the class of singularities of possibly non-normal varieties we are interested via the singularities of a normalisation together with a boundary divisor.

Definition 2.1 — Let X be a normal variety and $\Delta = \sum a_i D_i \subset X$ a (possibly empty or non-effective) $\mathbb{Q}$-Weil-divisor such that the log-canonical divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier, that is, some multiple of $K_X + \Delta$ is a Cartier divisor. Let $\pi\colon \tilde{X} \to X$ be a log-resolution of singularities[2], E_i the exceptional divisors and $\tilde{D}_i$ the strict transform of D_i. Then there exist unique rational numbers b_i such that

$$K_{\tilde{X}} + \sum_i a_i \tilde{D}_i + \sum_i b_i E_i \equiv \pi^*(K_X + \Delta),$$

where $\equiv$ denotes *numerical equivalence* of $\mathbb{Q}$-Weil-divisors. The pair (X, Δ) is called log-canonical respectively canonical if all $a_i, b_i \leq 1$ respectively $a_i, b_i \leq 0$.

In applications one usually assumes Δ to be effective but negative coefficients appear naturally in some statements. For an example, apply the next lemma to the blow-up in a smooth point not contained in the support of Δ.

Lemma 2.2 — [10, Lemma 2.30] *Let (X, Δ) be a pair with X a normal variety and Δ a $\mathbb{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi_1 : X_1 \to X$ be a proper birational morphism. If Δ_1 is a divisor such that*

$$K_{X_1} + \Delta_1 \equiv \pi_1^*(K_X + \Delta) \text{ and } \pi_*\Delta_1 = \Delta,$$

then (X_1, Δ_1) is log-canonical if and only if (X, Δ) is log-canonical.

Morally, semi-log-canonical varieties consist of log-canonical pairs glued along their boundary divisors. More precisely, they are defined in the following way.

Definition 2.3 — A variety X is said to have semi-log-canonical (slc) singularities if

(*i*) X satisfies Serre's condition S_2,
(*ii*) X has at most normal crossing singularities in codimension 1,
(*iii*) the Weil divisor class K_X is $\mathbb{Q}$-Cartier,
(*iv*) denoting by $\nu\colon \tilde{X} \to X$ the normalisation, and by $D \subset \tilde{X}$ the conductor divisor, that is the reduced preimage of the codimension 1 singular locus, the pair $(\tilde{X}, D)$ is log canonical.

A projective variety X is called stable if it has semi-log-canonical singularities and K_X is ample.

Remark 2.4 — The conditions (*i*) and (*iii*) seem technical at first glance, however they are essential for the theory. Since we are dealing with hypersurfaces in a smooth ambient space in this article they are automatically satisfied, because every local complete intersection is Gorenstein and the canonical divisor is Cartier by the adjunction formula.

In general note that by (*ii*) on an slc variety X we can find an open subset U with complement of codimension at least two, such that U has at most normal crossing singularities. Thus the dualising sheaf ω_U is a line bundle and any canonical divisor on U extends uniquely to a canonical divisor on X.

For a nice discussion of Serre's condition S_k and the canonical divisor see [16] or [9, Ch. 1] for the general case.

Remark 2.5 — Stable varieties are exactly the ones needed for the compactification of the moduli space of canonically polarised varieties with canonical singularities. Much more information on this kind of singularities can be found in [9].

[2] In other words, $\tilde{X}$ is smooth and the union of the strict transform of Δ and the exceptional divisor is a simple normal crossing divisor.

The original approach to such non-normal singularities is not via the normalisation but via so-called semi-resolutions, where one finds a partial resolution $\hat{X} \to X$ which has only normal-crossing and pinch points (see [11, 22]).

Remark 2.6 — We will later use the following easy observation: if X is a smooth surface and D a reduced curve on X then (X, D) is log-canonical if and only if D has at most ordinary nodes, that is, it is a normal crossing divisor.

2.2. **Semi-log-canonical hypersurfaces.** We will now restrict to the case of hypersurfaces which makes life considerably easier: everything is Gorenstein and adjunction works as expected.

Lemma 2.7 — *Let X be a smooth variety of dimension $n+1$ containing an n-dimensional variety S.*

- (i) *The variety S is slc in 0 if and only if the pair (X, S) is lc in 0.*
- (ii) *Let $\pi_1 \colon X_1 \to X$ be a blow-up in a smooth centre Z contained in S. Let S_1 be the strict transform of S and E_1 the exceptional divisor. Then*

$$K_{X_1} + S_1 + (\operatorname{mult}_Z(S) - \operatorname{codim}(Z, S))E_1 = \pi_1^*(K_X + S).$$

where $\operatorname{mult}_Z(S)$ is the multiplicity of S at the generic point of Z. In particular, if S is a semi-log-canonical surface then it has at most triple points. (By definition it has only normal crossing points in codimension one.)

Proof. Both items are straightforward computations using adjunction and blow-ups so we refer to [11, Thm. 5.1] for (i) and to [10, Lem. 2.29] for (ii). □

2.3. **Local analytic geometry.** Assume $S \subset \mathbb{C}^3$ is the germ of a surface defined by a convergent power series $f \in \mathbb{C}\{x, y, z\}$. We will now recall some basic tools that help to bring f into a normal form. First of all we will need the Weierstrass Preparation Theorem [5, Thm. 1.6].

Theorem 2.8 — *Assume that $f \in \mathbb{C}\{x, y, z\}$ and*

$$f(x, 0, 0) = \lambda \cdot x^d + (\textit{higher degree terms in } x), 0 \neq \lambda \in \mathbb{C}.$$

Then there is a unit u such that $f = u(x^d + a_1 x^{d-1} + \cdots + a_d)$ with $a_i \in \mathbb{C}\{y, z\}$.

Clearly the germ S does not change upon multiplying f by a unit.

Given a Weierstrass polynomial $f = x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{C}\{y, z\}[x]$, the so-called *Tschirnhaus tranformation* $x \mapsto x - \frac{a_1}{d}$ eliminates the degree $d-1$ term from f, that is,

$$f(x - \frac{a_1}{d}, y, z) = x^d + b_2 x^{d-2} + \cdots + b_d \text{ with } b_i \in \mathbb{C}\{y, z\}.$$

We will also use the following notions.

Definition 2.9 — Let f be a power series and $\mathfrak{m}$ the maximal ideal in $\mathbb{C}\{x, y, z\}$. Then the n-jet $j_n f$ of f is the image of f in $\mathbb{C}\{x, y, z\}/\mathfrak{m}^{n+1}$.

By abuse of notation we will usually treat $j_n f$ as if it were an element of $\mathbb{C}[x, y, z]$.

Definition 2.10 — For any convergent (respectively formal) power series $f \neq 0$ in $\mathbb{C}\{x, y, z\}$ (respectively $\mathbb{C}[\![x, y, z]\!]$), we define the *initial form* $In(f)$ to be the lowest degree part of f. Given a non-zero ideal I, the *initial ideal* $In(I)$ is the ideal generated by all $In(f)$ for $0 \neq f \in I$.

2.4. **Notation for blow-ups.** Let $p \in S \subset X$ be a hypersurface singularity of dimension two and multiplicity d, defined by a single local equation $f(x, y, z)$, where x, y, z are local coordinates. To analyse the singularities, we will either blow up X in 0 or in one of the coordinate axis; we explain our notation in the first case, the second case being similar.

Let $\pi_1 \colon X_1 \to X$ be the blow up of X in 0 given as

$$X_1 = \left\{ ((x, y, z), (\tilde{x} : \tilde{y} : \tilde{z})) \in X \times \mathbb{P}^2 \mid \operatorname{rk} \begin{pmatrix} x & y & z \\ \tilde{x} & \tilde{y} & \tilde{z} \end{pmatrix} \leq 1 \right\}$$

It is covered by three standard charts, for example in $U_1(x) = \{\tilde{x} \neq 0\}$ we have coordinates

$$(x, \frac{\tilde{y}}{\tilde{x}}, \frac{\tilde{z}}{\tilde{x}}) \text{ so that } y = \frac{x\tilde{y}}{\tilde{x}}, z = \frac{x\tilde{z}}{\tilde{x}}.$$

By abuse of notation we will again denote the local coordinates on $U_1(x)$ with $(x, y, z) = (x, \frac{\tilde{y}}{\tilde{x}}, \frac{\tilde{z}}{\tilde{x}})$. The relation to the previous coordinates is indicated by the chart.

Now the surface S comes into play. Since we are mostly interested in discrepancies we will consider on X_1 the divisor given by Lemma 2.7(ii), which in general differs both from the strict transform and from the total transform. We give an example to fix the notation.

$$\begin{array}{ccc}
(X, S) & \text{———} & z^3 + zxy + x^2y \\
\uparrow {\scriptstyle 0\in X} & & \\
(X_1, S_1 + E_1) & \text{———} & \begin{cases} U_1(x)\colon & (z^3 + zy + y)x \\ U_1(y)\colon & (z^3 + zx + x^2)y \\ U_1(z)\colon & \text{smooth} \end{cases}
\end{array}$$

where S_1 is the strict transform of S and $E_1 \cong \mathbb{P}^2$ is the exceptional divisor. It is not hard to see that in $U_1(x)$ and $U_1(y)$ we have a normal crossing divisor so the pair is lc and S is slc. This is a special case of the results in Section 4.

Later the following Lemma will be useful.

Lemma 2.11 — *Let $\hat{S}$ be a normal surface defined by $\hat{f} = t^2 + y^2 + g(z, y, t)$ and either* $\operatorname{mult}_0(g) \geq 3$ *or $j_2(g)$ does not contain t^2 or y^2. Then the pair $(\hat{S}, D = \{z = 0\})$ is log-canonical in the origin.*

Proof. If $\operatorname{mult}_0(g) = 1$ the pair is log-canonical so we may assume $\operatorname{mult}_0(g) \geq 2$. We first write $g = t^2g_1 + tg_2 + g_3(y, z)$ and thus $\hat{f} = t^2(1 + g_1) + tg_2 + y^2 + g_3$. By assumption $1 + g_1$ is a unit so we may change the t-coordinate and apply a Tschirnhaus transformation to reduce to an equation $\hat{f} = t^2 + y^2 + g(y, z)$. Repeating the same for the y coordinate we arrive at

$$t^2 + y^2 + vz^k, \quad v \text{ a unit.}$$

Dividing by v and changing the t, y coordinates once again we may assume that $v = 1$. In that case the assertion is easy to check by induction on k or a weighted blow-up. □

3. List of non-normal slc double points

We now give an explicit classification of all non-normal semi-log-canonical double points. Some examples are shown in Figure 1.

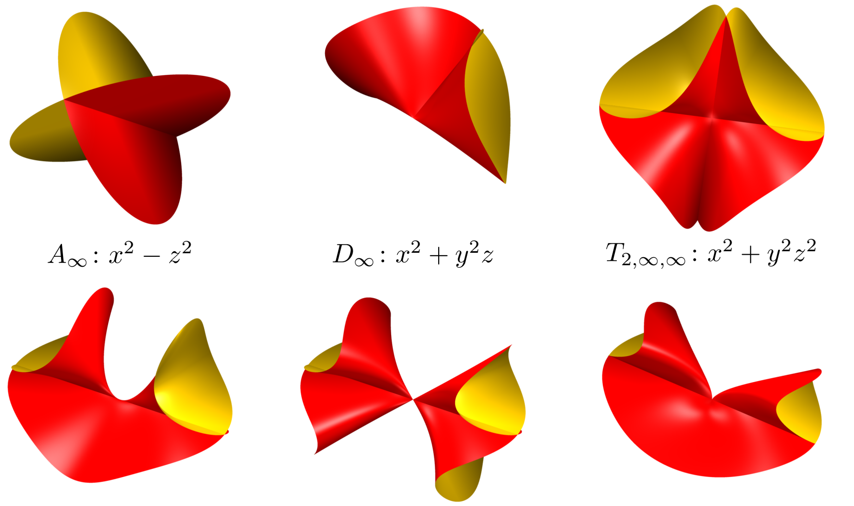
Proposition 3.1 — *Consider the the hypersurfaces in $\mathbb{C}^3$ defined by the following equations:*

$$\begin{aligned}
&A_\infty\colon\ x^2 + y^2 = 0 \ \textit{(Normal crossing)} \\
&D_\infty\colon\ x^2 + y^2z = 0 \ \textit{(Pinch point)} \\
&T_{2,\infty,\infty}\colon\ x^2 + y^2z^2 = 0 \\
&T_{2,q,\infty}\colon\ x^2 + y^2(z^2 + y^{q-2}) = 0 \ \textit{for all } q \geq 3
\end{aligned}$$

Then a two-dimensional, non-normal hypersurface double point is semi-log-canonical if and only if it is locally analytically isomorphic to the origin in one of the above hypersurfaces.

Remark 3.2 — The name $T_{2,\infty,\infty}$ usually refers to the equation $x^2 + xyz = (x + \frac{yz}{2})^2 - \frac{y^2z^2}{4}$, so we recover the above equation after a coordinate change. We prefer our choice of coordinates because it allows to read of the singular locus easily. The standard equation for $T_{2,q,\infty}$ is obtained from ours by a similar transformation.

FIGURE 1. Some non-normal slc double points



$A_\infty : x^2 - z^2$ $D_\infty : x^2 + y^2 z$ $T_{2,\infty,\infty} : x^2 + y^2 z^2$

$T_{2,3,\infty} : x^2 + y^2(y - z^2)$ $T_{2,4,\infty} : x^2 - y^2(z^2 - y^2)$ $T_{2,5,\infty} : x^2 - y^2(z^2 - y^3)$

3.1. **Proof of Proposition 3.1.** It is not hard to see that the listed singularities are actually slc. For example we can normalise the singularity $0 \in S$ of type $T_{2,q,\infty}$ by adding the rational function $t := \frac{x}{y}$ and the normalisation turns out to be $\hat{S} := \{t^2 + z^2 + y^{q-2} = 0\} \subset \mathbb{C}^3$ with conductor divisor $D := \{y = 0\} \subset \hat{S}$ (see also Equation (1) below); then Lemma 2.11 says that $(\hat{S}, D)$ is lc, hence S is slc. The argument for the other types of singularities is similar.

Next we prove the other implication of the proposition, which is more demanding.

3.1.1. *Set-up.* We work in a small neighbourhood of 0 in $\mathbb{C}^3$ where the non-normal slc surface S is given by one equation f. As $0 \in S$ is a double point the 2-jet is non-zero. Using Theorem 2.8 and a linear change of coordinates we may assume $f = u(x^2 + xf_1(y,z) + f_2(y,z))$ for some $f_1, f_2 \in \mathbb{C}\{y,z\}$ and a unit u. By a division by u and a Tschirnhaus tranformation, the equation takes the form

$$f = x^2 + b(y,z)$$

where the $\mathrm{mult}_0(b) \geq 2$. Let g_i be the irreducible factors of b in $\mathbb{C}\{y,z\}$ which we order in such a way that

$$b = g_1^{\lambda_1} \cdot \dots \cdot g_r^{\lambda_r} \text{ with } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r.$$

We denote by $B = \{x = b = 0\}$ the coefficient curve, by $B_i := \{x = g_i = 0\}$ its components, and let

$$\mu = (\mu_1, \dots, \mu_r) = (\mathrm{mult}_0(g_1), \dots, \mathrm{mult}_0(g_r)) \text{ and } \bar{\mu} = \sum \lambda_i \mu_i = \mathrm{mult}_0(b).$$

3.1.2. *Bounding the multiplicity of b.* To reduce the number of cases to consider we first bound the multiplicity of b.

Lemma 3.3 — *With the above notation the following holds.*

(*i*) *We have $\lambda_1 = 2$ and hence $1 \leq \lambda_i \leq 2$ for all i.*

(*ii*) *The multiplicity of b in 0 is at most 4.*

Proof. We first prove (*i*). By calculating the gradient we see that the singular locus of f is exactly the locus of points where b has multiplicity at least 2 (compare [8, Claim 2.59.1]). As we assumed S to be non-normal this implies $\lambda_1 \geq 2$.

So assume that $\lambda_1 \geq 3$. We will show that in this case S cannot be slc. Indeed, then B_1 defines a 1-dimensional component of the singular locus. Pick a general point p in this component (near the origin) where B_1 is smooth and all other g_i become units in $\mathcal{O}_{S,p}$. Then, taking $t = g_1 \sqrt[\lambda_1]{g_2^{\lambda_2} \cdot \dots \cdot g_r^{\lambda_r}}$ as our new coordinate in p the equation of S near p becomes $x^2 + t^{\lambda_1}$ which is not normal crossing if $\lambda_1 \geq 3$, so S is not slc in this case.

For (ii) note that if we blow-up the origin we get

$$\begin{array}{ccc} (X, S) & \text{———} & x^2 + b(y,z) \\ \uparrow {\scriptstyle 0\in X} & & \\ (X_1, S_1) & \text{——} & U_1(y)\colon x^2 + y^{-2}b(y, yz) = x^2 + y^{\bar\mu - 2}\tilde b(y,z). \end{array}$$

Thus as soon as $\bar\mu \geq 5$ infer from (i) that S_1 is not slc and hence S is not slc by Lemma 2.2 and Lemma 2.7. □

3.1.3. *The case b a square.* If we assume that b is a square, then $f = x^2 - c^2$ for some $c(y,z)$ with multiplicity at most 2 in the origin. Geometrically S is the union of two smooth hypersurfaces $S_+ = \{x + c(y,z) = 0\}$ and $S_- = \{x - c(y,z) = 0\}$ glued along the curve $C = \{x = c(y,z) = 0\}$. Hence, by definition, S is slc if and only if the pair (S_+, C) (resp. (S_-, C)) is lc in the origin which is the case if and only if C is either a smooth or a reduced normal crossing curve in the origin (cf. Remark 2.6). Then we may choose coordinates such that

$$f = x^2 + y^2 \text{ or } f = x^2 + y^2z^2,$$

thus obtaining the cases A_∞ and $T_{2,\infty,\infty}$.

3.1.4. *The case where b is not a square and $\bar\mu = 3$.* By Lemma 3.3 we have $\mu_1 = 1$ and in appropriate coordinates we can write $f = x^2 + y^2h(y,z)$, where $\mathrm{mult}_0(h) = 1$, $y \nmid h$.

Then the normalisation $\hat S$ of S is algebraically given by

$$\mathbb{C}\{x,y,z\}/(f) \to \mathbb{C}\{t,y,z\}/(t^2 + h(y,z)), \quad x \mapsto yt, \tag{1}$$

and by definition S is slc if and only if the pair $(\hat S, D = \{y = 0\})$ is lc.

Since $\mathrm{mult}_0(h) = 1$ the normalisation is smooth and the pair $(\hat S, D)$ is lc if and only if D is smooth or has an ordinary node in the origin (cf. Remark 2.6), which is equivalent to $h(0,z)$ having multiplicity 1 or 2 in the origin. By the means of a Tschirnhaus transformation and a change of the z-coordinate we may assume $h = z$ or $h = y(\mathrm{unit}) + z^2$. After another coordinate change a normal form for f is

$$f = x^2 + y^2z \text{ or } f = x^2 + y^2(y + z^2).$$

These are the cases D_∞ and $T_{2,3,\infty}$ on the list.

3.1.5. *The case where b is not a square and $\bar\mu = 4$.* The remaining case is where $f = x^2 + y^2h(y,z)$ such that $y \nmid h$, h is not a square, and $\mathrm{mult}_0(h) = 2$.

We first assume that y^2 divides the 2-jet of h. Then in appropriate coordinates S is defined by $x^2 + y^2(y^2 + z^d)$ with $d \geq 3$ and if we blow up the singular locus $L = \{x = y = 0\}$ we get

$$\begin{array}{ccc} (X, S) & \text{———} & x^2 + y^2(y^2 + z^d) \\ \uparrow {\scriptstyle L\subset X} & & \\ (X_1, S_1 + E_1) & \text{——} & \Big\{ U_1(x)\colon \quad (x^2 + (y^2 + y^{d-2}z^d))y\,. \end{array}$$

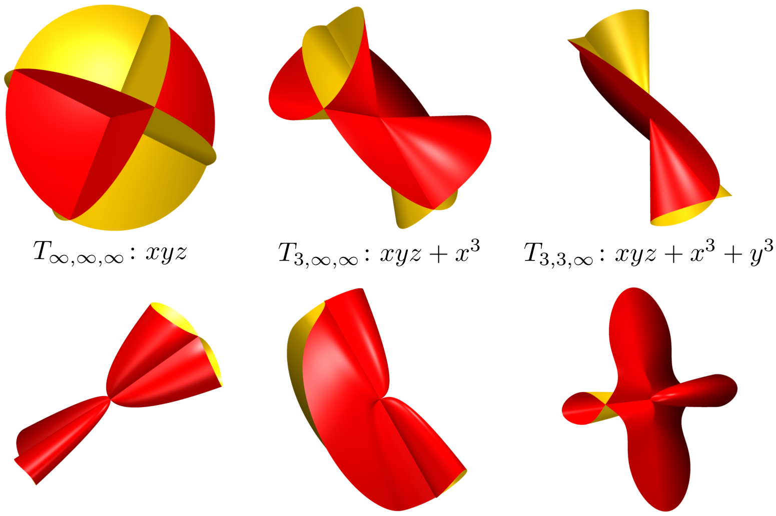
We see that E_1 and S_1 are tangent at a general point of their intersection, thus $(X_1, S_1 + E_1)$ is not log canonical and S cannot be slc by Lemma 2.7.

Therefore we may assume that y^2 does not divide the 2-jet of h and consider the normalisation $\hat S$ as in Equation (1). In appropriate coordinates $\hat S$ is defined by $t^2 + z^2 + y^d$ with $d \geq 2$ and then $(\hat S, \{y = 0\})$ is a log-canonical pair by Lemma 2.11. Now S is defined by

$$f = x^2 + y^2(z^2 + y^d) \qquad d \geq 2$$

yielding the remaining cases $T_{2,q,\infty}$ with $q = d + 2 \geq 4$ in the list. □

Figure 2. Some non-normal slc triple points



$T_{\infty,\infty,\infty}\colon xyz$ $T_{3,\infty,\infty}\colon xyz+x^3$ $T_{3,3,\infty}\colon xyz+x^3+y^3$

$T_{4,4,\infty}\colon xyz+x^4+y^4$ $T_{4,5,\infty}\colon xyz+x^4+y^5$ $T_{5,5,\infty}\colon xyz+x^5+y^5$

4. List of non-normal slc triple points

The aim of this section is to classify non-normal semi-log-canonical hypersurface triple points up to analytic isomorphism. Some examples are shown in Figure 2.

Proposition 4.1 — *Consider the the hypersurfaces in $\mathbb{C}^3$ defined by the following equations:*

$$\begin{aligned} T_{\infty,\infty,\infty}\colon&\ xyz=0\\ T_{p,\infty,\infty}\colon&\ xyz+x^p=0 \quad (p\geq 3)\\ T_{p,q,\infty}\colon&\ xyz+x^p+y^q=0 \quad (q\geq p\geq 3)\end{aligned}$$

Then a two-dimensional, non-normal hypersurface triple point is semi-log-canonical if and only if it is locally analytically isomorphic to the origin in one of the above hypersurfaces.

4.1. **Proof of Proposition 4.1.** We start by proving that all equations given in Proposition 4.1 define non-normal slc triple points at the origin. Consider the blow-up in the origin, which we give here only for the case $T_{p,q,\infty}$, the other two cases being similar:

$$\begin{array}{ccl} (X,S) & \text{———} & zxy+x^p+y^q\\ \uparrow {\scriptstyle 0\in X} & & \\ (X_1,S_1+E_1) & \text{——} & \begin{cases} U_1(x)\colon & (zy+x^{p-3}(1+y^qx^{q-p}))x\\ U_1(y)\colon & (zx+x^py^{p-3}+y^{q-3})y\\ U_1(z)\colon & (xy+z^{p-3}(x^p-y^qz^{q-p}))z\end{cases}\end{array}.$$

After normalisation one gets two pairs (S_1,D) and (E_1,D). The exceptional surface E_1 is isomorphic to $\mathbb{P}^2$ and D is the union of the coordinate axis, so the second pair is slc (cf. Remark 2.6). The first pair can be checked to be slc as well by applying Lemma 2.11 to the singular points in each of the three charts.

In the rest of this section we will show that every hypersurface slc triple point of dimension two can be brought into one of the normal forms given above. Our approach is very much inspired by the English translation of [1] that appeared for example in [2].

4.1.1. *Setup.* Let $(0\in S)$ be a non-normal slc hypersurface triple point of dimension two. As in the double point case we assume that $0\in S\subset\mathbb{C}^3$, defined by an equation f in the local ring

$\mathbb{C}\{x, y, z\}$ of convergent power series in variables x, y, z. As $(0 \in S)$ is a triple point the 3-jet $j_3 f$ does not vanish.

4.1.2. *Restrictions on the 3-jet.* We blow up the origin $(\mathbb{C}^3, S) \leftarrow (\widetilde{\mathbb{C}^3}, S_1 \cup E)$ and normalise the non-normal surface $S_1 \cup E$. Then, by definition and Lemma 2.2, S is slc in 0 if and only if the pairs $(\tilde{S}_1, D_1)$ and (E, D_2), where D_i is the preimage of the double locus, are both log-canonical.

To restrict the possible 3-jets we only look at the second pair which by construction is $(E, D_2) = (\mathbb{P}^2, \{j_3 f = 0\})$. This is slc if and only if $j_3 f$ defines a reduced plane curve with at most nodes (Remark 2.6), so up to a coordinate change the 3-jet is one of the following: xyz, $xyz + x^3$, $xyz + x^3 + y^3$, $\lambda xyz + x^3 + y^3 + z^3$ $(\lambda^3 + 27 \neq 0)$. However, using the finite determinacy theorem [5, Thm. I.2.23] it is straight forward to see that the last equation is 3-determined, that is, every equation with this 3-jet defines a cone over a plane elliptic curve, in particular a normal singularity. Thus the only possible 3-jets of non–normal, semi-log-canonical triple points are up to a linear coordinate change

$$xyz,\ xyz + x^3,\ xyz + x^3 + y^3. \tag{2}$$

All subsequent coordinate changes will be chosen such that the 3-jet is preserved.

4.1.3. *Normalising the equation.* We will now conclude the proof of Proposition 4.1 by showing that an equation that starts with a 3-jet as above can be brought into one of the normal forms in Proposition 4.1.

Lemma 4.2 — *Suppose that $f \in \mathbb{C}\{x, y, z\}$ defines a non-normal semi-log-canonical singularity in the origin, with a 3-jet as in Equation (2). Then there are integers $q \geq p \geq 3$, an automorphism φ of $\mathbb{C}\{x, y, z\}$ which preserves the 3-jet of f, and a unit u such that*

$$\varphi(f) = u(xyz + \delta_1 x^p + \delta_2 y^q),$$

where each $\delta_i \in \{0, 1\}$.

Proof. As an intermediate step we show that there are formal power series in x, y, z

$$\overline{\psi}_x = x + \cdots,\ \overline{\psi}_y = y + \cdots,\ \overline{\psi}_z = z + \cdots,$$

where "$\cdots$" means higher order terms, and formal power series in single variables $\overline{a}(x), \overline{b}(y), \overline{c}(z)$ such that

$$f(\overline{\psi}_x, \overline{\psi}_y, \overline{\psi}_z) = xyz + \overline{a}(x) + \overline{b}(y) + \overline{c}(z).$$

Obviously $j_3 f = j_3(f(\overline{\psi}_x, \overline{\psi}_y, \overline{\psi}_z))$.

In any case we can write $f = j_3 f + f_1$ where $f_1 = a_1(x) + b_1(y) + c_1(z) + g$ and the degrees of the polynomials $a_1(x), b_1(y), c_1(z)$ are smaller than $k = \mathrm{mult}_0(g)$. To construct the formal coordinate change we inductively construct coordinate transformations that preserve the 3-jet and increase the multiplicity of g. The induction step has to be adapted according to the possible 3-jets given in (2).

Case 1: $j_3 f = xyz$. The gradient of f is

$$\nabla f = \left(yz + \frac{\partial f_1}{\partial x},\ xz + \frac{\partial f_1}{\partial y},\ xy + \frac{\partial f_1}{\partial z} \right).$$

As $k \geq 4$ there are $\lambda_i \in \mathbb{C}$ and homogeneous polynomials h_x, h_y, h_z of degree $k - 2$ such that the lowest degree part of g decomposes into

$$j_k g = \lambda_1 x^k + \lambda_2 y^k + \lambda_3 z^k + xyh_z + xzh_y + yzh_x.$$

We now apply the coordinate transformation

$$\psi^{(k)}\colon\ x \mapsto \psi_x^{(k)} = x - h_x, \quad y \mapsto \psi_y^{(k)} = y - h_y, \quad z \mapsto \psi_z^{(k)} = z - h_z,$$

so that

$$\psi^{(k)}(f) = f(\psi_x^{(k)}, \psi_y^{(k)}, \psi_z^{(k)}) = xyz + (a_1(x) + \lambda_1 x^k) + (b_1(y) + \lambda_2 y^k) + (c_1(z) + \lambda_3 z^k) + g'$$

with $\mathrm{mult}_0(g') > k$ which finishes the induction step.

Case 2: $j_3 f = xyz + x^3$. The gradient of f is

$$\nabla f = \left(yz + 3x^2 + \frac{\partial f_1}{\partial x},\; xz + \frac{\partial f_1}{\partial y},\; xy + \frac{\partial f_1}{\partial z} \right).$$

The lowest degree parts are $3x^2 + yz, xz, xy$ respectively and thus the degree three part of the initial ideal $\mathrm{In}(Jac(f))$ is a vector space spanned by $x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^2z, yz^2$. As $k \geq 4$ there are $\lambda_i \in \mathbb{C}$ and $h_i \in \mathbb{C}\{x, y, z\}$ of multiplicity at least $k-2 \geq 2$ such that the lowest degree part of g decomposes into

$$j_k(g) = \lambda_1 x^k + \lambda_2 y^k + \lambda_3 z^k + j_k \left(h_1 \frac{\partial f}{\partial x} + h_2 \frac{\partial f}{\partial y} + h_3 \frac{\partial f}{\partial z} \right)$$

because $j_k(g) - \lambda_1 x^k - \lambda_2 y^k - \lambda_3 z^k$ is in $\mathrm{In}(Jac(f))$. We now apply the coordinate transformation

$$\psi^{(k)} \colon\; x \mapsto \psi_x^{(k)} = x - h_x, \quad y \mapsto \psi_y^{(k)} = y - h_y, \quad z \mapsto \psi_z^{(k)} = z - h_z,$$

so that

$$\psi^{(k)}(f) = f(\psi_x^{(k)}, \psi_y^{(k)}, \psi_z^{(k)}) = xyz + x^3 + (a_1(x) + \lambda_1 x^k) + (b_1(y) + \lambda_2 y^k) + (c_1(z) + \lambda_3 z^k) + g'$$

with $\mathrm{mult}_0(g') > k$ which finishes the induction step.
Case 3: $j_3 f = xyz + x^3 + y^3$. We only note that, in this case $\mathrm{In}(Jac(f))$ contains the monomials $x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2$ and proceeding along similar lines as above gives the induction step.

Note that in the coordinate transformations

$$\psi^{(k)}(x, y, z) = (x - h_x, y - h_y, z - h_z)$$

constructed in each of the cases above the multiplicities of h_x, h_y, h_z are at least $k-2$. This guarantees that we can compose the coordinate changes in the induction steps to obtain a formal coordinate transform $\overline{\psi}$ preserving the 3-jet such that

$$\overline{\psi}(f) = xyz + \overline{a}(x) + \overline{b}(y) + \overline{c}(z).$$

Extracting the lowest degree terms of $\overline{a}(x), \overline{b}(y), \overline{c}(z)$, we have

$$\overline{\psi}(f) = xyz + v_1 x^p + v_2 y^q + v_3 z^r$$

where $p, q \geq 3$, $r \geq 4$, and v_i is either zero or a unit. We now make an Ansatz to determine units $\overline{u}$, $\overline{u}_1$, $\overline{u}_2$, $\overline{u}_3 \in \mathbb{C}[\![x, y, z]\!]^*$ such that

$$\overline{\psi}(f)(\overline{u}_1 x, \overline{u}_2 y, \overline{u}_3 z) = \overline{u}(xyz + \delta_1 x^p + \delta_2 y^q + \delta_3 z^r)$$

with $\delta_i \in \{0, 1\}$. This can easily be solved since $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Including appropriate roots of these units in the coordinate transformation we find a formal coordinate change $\overline{\varphi}$ such that

$$\overline{\varphi}(f) = \overline{u}(xyz + \delta_1 x^p + \delta_2 y^q + \delta_3 z^r), \qquad \overline{u} \in \mathbb{C}[\![x, y, z]\!]^*, \delta_i \in \{0, 1\}.$$

By Artin's approximation theorem [3] we can then also solve the equation the the ring of convergent power series, i.e., there exists an coordinate change φ of $\mathbb{C}\{x, y, z\}$ and a unit $u \in \mathbb{C}\{x, y, z\}^*$, such that

$$\varphi(f)(x, y, z) = u(xyz + \delta_1 x^p + \delta_2 y^q + \delta_3 z^r).$$

If all $\delta_i = 1$ then S has a cusp singularity as given in Table 1, in particular it is normal contradicting our assumptions. Thus, up to permutation of the coordinates $\delta_3 = 0$ which concludes the proof. □

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