REMARKS ON JORDAN ALGEBRAS (DIM 9, DEG 3), CUBIC SURFACES, AND DEL PEZZO SURFACES (DEG 6)

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The purpose of these notes is to record some formulas and remarks. Everything deserves a check.

1. A construction of Jordan algebras (deg 3, dim 9)

The base field $F$ has char $\neq 3$.

Let us ask for a functorial construction $B : (L, K) \mapsto B(L, K)$ which associates to an ordered pair of separable degree 3 extensions a 9-dimensional Jordan algebra of degree 3. Consider the split cases $L = K = F \oplus F \oplus F$ and let $\tilde{B} = B(L, K)$. The functoriality of $B$ then yields a homomorphism $\Psi_B : S_3 \times S_3 = \text{Aut}(L) \times \text{Aut}(K) \to \text{Aut}(\tilde{B})$.

Clearly $B$ is determined by $\tilde{B}$ and $\Psi_B$.

Here is an example: Let $Z = F[x]/(x^2 + x + 1)$, $\sigma : Z \to Z, \sigma(x) = x^2$

and let $A = M_3(Z), \tau : A \to A, \tau(a) = \sigma(a)^t$.

Then $(A, \tau)$ is an algebra with involution of second kind. Put $\tilde{B} = A^\tau$.

There are the (split) subalgebras

$L = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$

and

$K = \frac{1}{3}(1 + \beta + \beta^2)F \oplus \frac{1}{3}(1 + x\beta + x^2\beta^2)F \oplus \frac{1}{3}(1 + x^2\beta + x\beta^2)F$

where

$\beta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Let $G$ be the subgroup of $\text{Aut}(\tilde{B})$ (as Jordan algebra) leaving the subalgebras $L$ and $K$ invariant. The natural homomorphism $G \to \text{Aut}(L) \times \text{Aut}(K)$ turns out to be an isomorphism.

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Lemma 1. Let $B$ be as above and suppose that $\Psi_B$ is injective. Then $B$ is isomorphic to the functor given by the example. □

Lemma 2. The last lemma can be extended to unordered pairs of cubic extensions, that is to say to a cubic extension over a quadratic extension. The underlying group is then $(S_3 \times S_3) \rtimes \mathbb{Z}/2$.

One would like to see a rational description of $B(L,K)$ for arbitrary $K$ and $L$. Here it is for $\text{char } F \neq 2, 3$:

Put $B = B(L,K) = L \otimes K$.

Let $L_0 \subset L, K_0 \subset K$ be the subspaces of trace 0 elements. Define a Jordan product $\cdot$ on $B$ by the following formulas with $\alpha \in L_0$ and $\beta \in K_0$.

\[
(1 \otimes 1)^2 = 1 \otimes 1 \\
(\alpha \otimes 1)^2 = \alpha^2 \otimes 1 \\
(1 \otimes \beta)^2 = 1 \otimes \beta^2 \\
(\alpha \otimes \beta) \cdot (\alpha \otimes 1) = \frac{1}{4} \left( \text{trace}(\alpha^2) - 2\alpha^2 \right) \otimes \beta \\
(\alpha \otimes \beta) \cdot (1 \otimes \beta)^2 = \frac{1}{4} \alpha \otimes \left( \text{trace}(\beta^2) - 2\beta^2 \right) \\
(\alpha \otimes \beta)^2 = -\frac{1}{2} \alpha^2 \otimes \beta^2 + \frac{1}{8} \left( \text{trace}(\alpha^2) \otimes \beta^2 + \alpha^2 \otimes \text{trace}(\beta^2) \right)
\]

(One could clean these formulas a bit, by using the adjoint $\alpha^\# = \alpha - \frac{1}{2} \text{trace}(\alpha^2)$.)

If $L$ is cyclic and $K$ is a Kummer extension, then $B(L,K) = A^+$ where $A$ is the usual crossed product.

From this it is not difficult to see that the $H^2$-$\text{mod}3$ invariant of $B$ is the cup product of the $H^1$-$\text{mod}3$ invariants of $L$ and $K$ (all of these invariants are defined only up to sign).

Also concerning the “mod2-part” of $B(L,K)$ there is a sort of product.

Lemma 3.

\[\text{trace}_{B(L,K)/F} \simeq 3 \text{trace}_{L/F} \otimes \text{trace}_{K/F}\]

Proof. This follows from the above formulas. There might be a better proof. □

Note that the trace form of a cubic extension with discriminant $\delta$ is $(1, 2, 2\delta)$. The associated 2-fold Pfister form is $\langle -2, -\delta \rangle$.

Lemma 4. Let $\delta_L, \delta_K$ be the discriminants of $L$ and $K$, respectively. Then the $H^3(\mathbb{Z}/2)$-invariant of $B$ is

\[(-2, -\delta_L, -\delta_K) \in H^3(F, \mathbb{Z}/2)\]

This follows from Lemma 3. Before I was aware of Lemma 3 I used the following arguments.

Proof. To check this one looks at our split example $\tilde{B}$ (which has $H^3(\mathbb{Z}/2)$-invariant $(-3, -1, -1)$) and restricts to a $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup of $G$.

To be specific, introduce the following coordinates

\[\tilde{B} = \left\{ \begin{pmatrix} a & \bar{u} & \bar{w} \\ u & b & \bar{v} \\ w & v & c \end{pmatrix} \middle| a, b, c \in F, u, v, w \in \mathbb{Z} \right\}\]
in $\tilde{BB}$ (with $\tilde{} = \sigma$). Moreover let

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\epsilon, \theta : B \rightarrow B, \quad \epsilon = \text{ad}_\alpha, \quad \theta = t.$$ 

Then $\epsilon \theta$ is an element of order 2 in $\text{Aut}(L) \subset G$ and $\theta$ is an element of order 2 in $\text{Aut}(K) \subset G$.

The trace form of $B$ has the diagonal form

$$\text{trace } B = (1, 1) \perp 2(1, 3) \perp 2(1, 3) \otimes (1, 1)$$

with respect to the coordinates

$$(a, b, c, u, (v, w)) \in F^2 \oplus F \oplus Z \oplus Z^2$$

and

$$Z = F \oplus \sqrt{-3} F.$$ 

After twisting one has

$$\text{trace } B = 2(1, \delta_L) \perp (1) \perp 2(1, 3\delta_K) \perp 2(1, 3\delta_L \delta_K) \otimes 2(1, \delta_L).$$

This gives

$$\text{trace } B = (1, 1, 1) \perp 2\langle -3\delta_K \delta_L \rangle \otimes (2, 2\delta_L, \delta_L)$$

$$= (1, 1, 1) \perp 2\langle -3\delta_K \delta_L \rangle \otimes \langle -2, -\delta_L \rangle.$$ 

Finally note $\langle -3\delta_K \delta_L, -2, -\delta_L \rangle = \langle -2, -\delta_L, -\delta_K \rangle$. □

2. Twisting sums of four cubes

Consider the cubic form

$$\Phi : L_0 \otimes K_0 \rightarrow F, \quad \Phi = (N_{L/F} | L_0) \otimes (N_{K/F} | K_0).$$

It turns out that $\Phi$ is also given by the norm form of $B(L, K)$:

$$\Phi = N_{B(L,K)/F} | (L_0 \otimes K_0).$$

Let

$$C = C(L, K) = \{ \Phi = 0 \} \subset \mathbb{P}(L_0 \otimes K_0)$$

be the associated cubic surface.

Suppose that $K = F[x]/(x^3 - b)$. Then

$$\Phi(\alpha \otimes x + \alpha' \otimes x^2) = N_{L/F}(\alpha)b + N_{L/F}(\alpha')b^2$$

In particular, for $L = F[x]/(x^3 - a)$ this gives the diagonal cubic form

$$\Phi = ab(1, a, b, ab)$$

If $b = 1$ and $L = F \oplus F \oplus F$, then $\Phi$ has the form

$$uv(u + v) + st(s + t)$$

**Lemma 5.** The surface $C(L, K)$ has a rational point if and only if the $H^2$-mod3-invariant of $B(L, K)$ is trivial (i.e., $B(L, K)$ has zero divisors).
Proof. A cubic form is isotropic if and only it is isotropic over a quadratic extension. We may therefore assume that \( L = F[x]/(x^3 - a) \) and \( K = F[x]/(x^3 - b) \). In this case the algebras is \((L, b)\) and the cubic form is
\[
\Phi = ab(1, a, b, ab)
\]
which is isotropic if and only if the equation
\[
b = N_{L/F}(\frac{u + vx}{w + lx})
\]
has a solution. But any element in \( L = F[x]/(x^3 - a) \) is of the form \( u + vx + w + lx \). So the cubic form is isotropic if and only if \( b \in N_{L/F}(L^\times) \), i.e., the algebra has zero divisors. □

Lemma 6. The surface \( C(L, K) \) has a rational point if and only if it is rational.

Proof. If \( L \) is split, \( L = e_1 F \oplus e_2 F \oplus e_3 F \), then \((e_i - e_j) \otimes K_0 \) give 3 disjoint lines in the cubic surface. Hence in general there is a set of three lines in \( C \) defined over \( F \). As Colliot-Thelene informed me, in this case \( C \) is rational if and only if \( C \) has a rational point. The reference is:


This paper is not available to me till now. □

Corollary 7. The stable birational equivalence class of \( C(L, K) \) depends only on the \( H^2\)-mod3-invariant of \( B(L, K) \) (defined up to sign).

Proof. Indeed, if \( C \) is rational over \( F(C') \), and vice versa, then \( C \times C' \) is stable birational to \( C \) and \( C' \). □

Question 1. What about birational equivalence?

3. A construction of (all) del Pezzo surface of degree 6

A del Pezzo surface of degree 6 is a form of \( \mathbb{P}^2 \) blown up in 3 points in general position. They may be constructed as follows. Let \( B \) be a Jordan algebra (of the type as above) and let \( L \subset B \) be a separable associative subalgebra of degree 3. Define
\[
Y(B, L) = \{ [b] \in \mathbb{P}(B) \mid \{b, L_0, b\} = 0 \}
\]
Here \( \{b, \lambda, b\} \) denotes the Jordan triple product.

In the split situation \( B = M_3^+ \) and \( L = \Delta \) (diagonal matrices) this gives
\[
Y(M_3, \Delta) = \{ [X] \in \mathbb{P}(M_3) \mid X\Delta_0 X = 0 \}
\]
Any matrix \( X \) with \([X] \in Y(M_3, \Delta)\) has rank 1, so that \( X = v \cdot w^t \) for some 3-vectors \( v, w \). This gives an identification
\[
Y(M_3, \Delta) = \{ ([v], [w]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid v_1w_1 = v_2w_2 = v_3w_3 \}
\]
In other words, \( Y(M_3, \Delta) \) is the “quadratic correspondence of \( \mathbb{P}^2 \), as described in Hartshorne’s book.

Let’s discuss the corresponding automorphism groups.
On $M_3$ the group $\text{PGL}_3 \rtimes \mathbb{Z}/2$ acts by conjugation and transposition. The subgroup leaving $\Delta_0$ invariant is of the form

$$H = T^2 \rtimes (S_3 \times \mathbb{Z}/2)$$

with $T^2$ a 2-dimensional torus (=projective diagonal matrices).

So $H$ acts on $Y = Y(M_3, \Delta)$, and one finds that $H = \text{Aut}(Y)$, since on such a del Pezzo surface the hexagon consisting of the 6 exceptional lines is left invariant under all automorphisms of $Y$—and so $\text{Aut}(Y)$ consists of the automorphisms of the toric structure on $Y$.

**Corollary 8.** There is a bijection between (isomorphism classes of) pairs $(B, L)$ and del Pezzo surfaces of degree 6.

**Question 2.** What about the (stable) birational classification of the $Y$’s?

The stable question this is not difficult to answer by using the toric structure. There are classifying $H^2\text{-mod}3$ and $H^2\text{-mod}2$ invariants.

Let $(\mathcal{A}, \tau)$ be an algebra with involution of second kind with center $\mathbb{Z}$ such that $B = \mathcal{A}^\tau$. Then the imbedding of $Y$ to $\mathbb{P}^2 \times \mathbb{P}^2$ twists to an embedding $Y(B, L) \subset R_{\mathbb{P}^1}(\text{SB}(\mathcal{A}))$

If $Z$ is split, i.e., $B = \mathcal{A}^+$ for a central simple algebra $\mathcal{A}'$, then

$$Y(B, L) \subset \text{SB}(\mathcal{A}') \times \text{SB}(\mathcal{A}'^\text{op})$$

The projection to any of the factors is the blow down of 3 lines $R_{\mathbb{P}^1}(\mathbb{P}^1) \subset Y(B, L)$.

Let still $B = \mathcal{A}'$ and let

$$S = \{ [b] \in \mathbb{P}(B) \mid \text{rank } b = 1, b^2 = 0 \}$$

If $B = M_3^+$, then

$$S = \{ ([v], [w]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \}$$

The intersection $S \cap Y$ is exactly the hexagon on $Y$.

4. **Blowing down the cubic surface**

We return to the case $L = F \oplus F \oplus F$ and $K = F[x]/(x^3 - 1)$. Note that then $B(L, K) = M_3^+$. Then the cubic surface is given by

$$C = \{ uw(u + v) + st(s + t) = 0 \}$$

Consider the map

$$C \rightarrow \mathbb{P}^2 \times \mathbb{P}^2, \quad (u, v, s, t) \mapsto (-vs, st, uv, -ut, uv, st)$$

If I am not mistaken, this map is everywhere defined, maps to $Y$ and the map $C \rightarrow Y$ is a blow down of 3 lines. The map $C \rightarrow \mathbb{P}^2$ (given by any of the two projections) should be the blow up in the 6 points

$$[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1], [1, \zeta, \zeta^2], [1, \zeta^2, \zeta]$$

with $\zeta$ a primitive 3rd of unity.

**Question 3.** How to describe these blow downs in the non split situation?