Injectivity of $K_2D \to K_2F$ for quaternion algebras

by Markus Rost

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Let F be field of characteristic different from 2 and let D be a quaternion algebra over F. The purpose of this paper is to show that the reduced norm $\mathrm{Nrd} : K_2D \to K_2F$ is injective. (for definition and properties of Nrd see [MS; § 6, § 7]). This result is essential for the proof of Hilbert 90 for K_3 for degree-two extensions in Milnor K-Theory of fields ([R1]). Our method of proof is in some sense similar to the proof of Hilbert 90 for K_2 by Merkur'ev and Suslin. However the important role of Severi-Brauer varieties in the work of Merkur'ev and Suslin has now to be played by a certain type of three-dimensional nonsingular quadrics X_c, for which we show that $H^1(X_c, \mathcal{K}_3) = K_2F$. This result is based on the computation of the K-Theory of nonsingular quadrics in [Sw] and the more elementary determination of $SK_1(X_c)$ in [R2].

§ 0 The results

Let F be a field, Char $F \neq 2$. Every quaternion algebra over F is isomorphic to

$$D = D(a, b) = \langle A, B | A^2 = a, B^2 = b, AB = -BA \rangle$$

for some $a, b \in F^*$.

Theorem 1
The reduced norm

$$\mathrm{Nrd} : K_2D \to K_2F$$

is injective.

Let $\det : D \to F$ be the norm of D. In coordinates we have

$$\det(x_1 + x_2A + x_3B + x_4AB) = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$$

If D is trivial, i.e. $D = M_2(F)$, then det is the usual determinant.
It is not difficult to prove Theorem 1 if det is surjective, see § 5. Therefore we study field extensions which enlarge the image of det. To D and a fixed element $x \in F^*$ we associate a nonsingular three-dimensional quadric $X = X_c$ as follows. Let $q : D \times F \to F$ be the quadratic form $q(x, y) = \det(x) - cy^2$, and define $X \subset \mathbb{P}(D \times F) \simeq \mathbb{P}^4$ by $q = 0$. X is a smooth irreducible variety over F. Its function field is denoted by $F(X)$. The important role of X in the proof of Theorem 1 relies on the fact that $c \in \det(D \otimes_F F(X))$.

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Theorem 2
The homomorphism
$$K_2D \to K_2(D \otimes_F F(X))$$
induced by inclusion is injective.

Let $K_i(X) = K_i(X)^0 \supset K_i(X)^1 \supset K_1(X)^2 \supset K_i(X)^3$ be the filtration given by codimension of support. We put $K_i(X)^{n/m} = K_i(X)^n / K_i(X)^m$ for $m \geq n$.

Theorem 3

For $i = 0, 1, 2$ there are natural isomorphisms
$$K_i(X)^{0/1} = K_iF \quad \text{and} \quad K_i(X)^{1/2} = K_iF.$$
Consider the sequence
$$\bigoplus_{v \in X} K_3\kappa(v) \xrightarrow{d} \bigoplus_{v \in X^{(1)}} K_2\kappa(v) \xrightarrow{d'} \bigoplus_{v \in X^{(2)}} K_1\kappa(v)$$
given by the localization sequence in K-Theory [Q; § 5].

Theorem 4
There is a natural isomorphism
$$\text{Ker } d'/\text{Im } d = K_2F.$$

§ 1 Preliminaries

For a splitting field L of D, finite over F, there is a natural homomorphism $\theta_L : K_iL \to K_iD$ by composing the transfer $K_i(D \otimes_F L) \to K_iD$ and the isomorphism $K_iL = K_i(M_2(L)) = K_i(D \otimes_F L)$ (See [MS; § 1] for functorial properties). For the following proposition see [MS; Theorem 5.2] or [RS; § 4].

Proposition 1.1.
For $i = 0, 1, 2$ the map
$$\theta = (\theta_L) : \bigoplus_L K_iL \to K_iD$$
is surjective. Here L runs over all splitting fields of D, finite over F.

Let Y be the Severi-Brauer variety associated to D and denote by $F(Y)$ its function field. Y is isomorphic to the quadric in $\mathbb{P}^2$ given by $x_1^2 - ax_2^2 - bx_3^2 = 0$. The residue fields of the points of Y are splitting fields of D. The K-Cohomology of Y was studied in [MS]. We need the following results.

Proposition 1.2.
The following sequences are exact

$$0 \to K_2F \to K_2F(Y) \xrightarrow{d} \bigoplus_{v \in Y^{(1)}} K_1\kappa(v) \xrightarrow{\theta} K_1D \to 0 \tag{1.2.1}$$

$$K_3F(Y) \xrightarrow{d} \bigoplus_{v \in Y^{(1)}} K_2\kappa(v) \xrightarrow{\theta} K_2D \to 0 \tag{1.2.2}$$

Here $\theta = (\theta_{\kappa(v)})$ as above. For (1.2.1) see also [So; Proposition 3]. (1.2.2) is a consequence of (1.2.1), the long exact localization sequence for Y and the isomorphism $K_i(Y) = K_iF \oplus K_iD$ ([Q; Theorem 4.1]).

Now let $X = X_c = X(q)$ be the quadric defined in § 1. For the following analogue of the exactness of the right part of (1.2.1) see [R2].

Proposition 1.3.
The sequence

$$\bigoplus_{v \in X^{(2)}} K_2\kappa(v) \xrightarrow{d} \bigoplus_{v \in X^{(3)}} K_1\kappa(v) \xrightarrow{\mathcal{N}} K_1F$$

is exact.
Here $\mathcal{N}$ is induced by the usual norm for finite extensions.

For trivial D, X is isomorphic to the canonical quadric $\hat{X}$ in $\mathbb{P}^4$ defined by

$$x_1x_2 + x_3x_4 - x_5^2 = 0.$$

Let $Z \subset \hat{X}$ be the irreducible hyperplane section given by $x_1 = 0$ and let $v_Z \in \hat{X}^{(1)}$ be the point corresponding to Z. The birational correspondence

$$\phi : \hat{X} \to \mathbb{P}^3, \quad [x_1 : x_2 : x_3 : x_4 : x_5] \to [x_1 : x_3 : x_4 : x_5]$$

induces an isomorphism $\hat{X} \setminus Z \to \mathbb{P}^3 \setminus \{x_1 = 0\} = \mathbb{A}^3$. From this it is clear that $\mathrm{Pic}(\hat{X}) = \mathbb{Z}$, generated by a hyperplane section.

Proposition 1.4.:
The sequence

$$0 \to K_iF \to K_iF(\hat{X}) \xrightarrow{d} \bigoplus_{\substack{v \in \hat{X}^{(1)} \\ v \neq v_Z}} K_{i-1}\kappa(v)$$

is exact.

Proof
ϕ induces via $\hat{X} \leftarrow \hat{X} \setminus Z \simeq \mathbb{A}^3 \to \mathbb{P}^3$ an equivalence of the sequence in question with the sequence

$$0 \to K_iF \to K_iF(\mathbb{P}^n) \xrightarrow{d} \bigoplus_{v \in (\mathbb{A}^n)^{(1)}} K_{i-1}\kappa(v)$$

for $n = 3$. The exactness of this sequence is known for $n = 1$ and induction yields the result.

Corollary 1.5.
The natural homomorphism

$$K_2F \to K_2F(X)$$

is injective.

This follows from a result of Suslin, since F is algebraically closed in $F(X)$. It can be also derived from Proposition 1.2. and Proposition 1.4. and the commutative diagram

$$\begin{array}{ccc} K_2F & \longrightarrow & K_2F(X) \\ \downarrow & & \downarrow \\ K_2F(Y) & \longrightarrow & K_2F(Y)(X) \end{array}$$

since X is isomorphic to $\hat{X}$ over $F(Y)$.

Let $\mu : K_iF \otimes K_0(X) \to K_i(X)$ be the multiplication in K-Theory. μ respects filtration, i.e. $\mu(K_iF \otimes K_0(X)^n) \subset K_i(X)^n$. Let $C = C_0(q)$ be the even part of the Clifford algebra of q. For the following theorem see [Sw; Theorem 9.1].

Theorem 1.6.
There is a natural isomorphism

$$(u_0, u_1, u_2, w) : (K_iF)^3 \oplus K_iC \to K_i(X).$$

The definition of w ($= u'$ in the notation of [Sw]) shows that the following diagram commutes

$$\begin{array}{ccc} K_iF \otimes K_0C & \longrightarrow & K_iC \\ \downarrow{\scriptstyle 1\otimes w} & & \downarrow{\scriptstyle w} \\ K_iF \otimes K_0(X) & \longrightarrow & K_i(X) \end{array}$$

Moreover we have

$$u_i(\alpha) = \mu(\alpha, [\mathrm{O}_X(-i)]) \quad \text{for} \quad \alpha \in K_iF.$$

§ 2 Proof of Theorem 3

Lemma 2.1

$$C = M_2(D)$$

Proof
Let $q' = cx_1^2 - acx_2^2 - bcx_3^2 + abcx_4^2$. Then $q = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2 - cx_5^2$ is equivalent to $c(q' \oplus \langle -1 \rangle)$. [Sw; Lemma 4.4 and Lemma 4.5] yields $C = C_0(q) = C_0(q' \oplus \langle -1 \rangle) = C(q')$ Hence

$$\begin{aligned} C = \langle e_1, e_2, e_3, e_4 \mid e_1^2 = c, e_2^2 = -ac, e_3^2 = -bc, e_4^2 = abc, \\ e_i e_j + e_j e_i = 0 \quad \text{for} \quad i \neq j \rangle. \\ = \langle f_1, f_2, g_1, g_2 \mid f_1^2 = a, f_2^2 = b, g_1^2 = 1, g_2^2 = -abc; \\ f_1 f_2 + f_2 f_1 = 0, g_1 g_2 + g_2 g_1 = 0, [f_i, g_j] = 0 \rangle \end{aligned}$$

where $f_1 = e_1^{-1} e_2, f_2 = e_1^{-1} e_3, g_1 = (e_2 e_3)^{-1} e_1 e_4, g_2 = e_1^{-1} e_2 e_3$. Therefore $C = D(a,b) \otimes D(1, -abc) \simeq D(a,b) \otimes M_2(F)$. qed

Lemma 2.1 and Proposition 1.1 imply

Corollary 2.2
For $i = 0, 1, 2$ the map

$$\bigoplus_L K_i(C \otimes_F L) \to K_i C$$

induced by transfer is surjective. Here L runs over all splitting fields of D, finite over F.

Lemma 2.3
Let $i = 0, 1, 2$. Then
i) $K_i(X) = u_1(K_i F) + K_i(X)^1$.
ii) $K_i(X) = u_0(K_i F) + u_1(K_i F) + K_i(X)^2$.

Proof
Let $\gamma = [O_X] - [O_X(-1)] \in K_0(X)$. Since γ is defined by a hyperplane section we have $\gamma \in K(X)^1$ and $\gamma^2 \in K(X)^2$. Hence

$$u_0(\alpha) - u_1(\alpha) = \mu(\alpha, \gamma) \in K_i(X)^1$$

which shows ii) $\Rightarrow$ i). For ii) first note

$$u_0(\alpha) - 2u_1(\alpha) + u_2(\alpha) = \mu(\alpha, \gamma^2) \in K_i(X)^2.$$

By Theorem 1.6 it remains to show that

$$(*) \qquad \underline{w(K_i C) \subset u_0(K_i F) + u_1(K_i F) + K_i(X)^2}$$

Since the transfer $K_i(X \times_F L) \to K_i(X)$ respects filtration, Corollary 2.2 shows that we may assume $D = M_2 F$; in particular we may assume that X is the canonical quadric $\hat{X}$.

Let us first assume $i = 0$. The Gersten spectral sequence $E_2^{p,q} \Rightarrow K_{-p-q}$ yields isomorphisms $\mathrm{Ch}^0(X) = E_2^{0,0} = E_\infty^{0,0} = K_0(X)^{0/1}$ and $\mathrm{Ch}^1(X) = E_2^{1,-1} = E_\infty^{1,-1} = K_0(X)^{1/2}$. Therefore $K_0(X) = [O_X]\mathbb{Z} \oplus K_0(X)^1$ and $K_0(X)^1 = \gamma\mathbb{Z} \oplus K_0(X)^2$, since $\mathrm{Ch}^1(X) = \mathrm{Pic}(X)$ is generated by a hyperplane section.

Now let i be arbitrary. Let $\varepsilon \in K_0C$ be the unit of the ring $K_*C = K_*D = K_*F$ and let $k, l \in \mathbb{Z}$ such that

$$w(\varepsilon) = k[\mathrm{O}_X] + l[\mathrm{O}_X(-1)] \quad \mod \ K_0(X)^2$$

Then, for $\alpha \in K_iF = K_iC$:

$$\begin{aligned} w(\alpha) &= w(\alpha\varepsilon) = \mu(\alpha, w(\varepsilon)) \\ &= ku_0(\alpha) + lu_1(\alpha) + \mu(\alpha, w(\varepsilon) - k[\mathrm{O}_X] - l[\mathrm{O}_X(-1)]) \\ &\in u_0(K_iF) + u_1(K_iF) + K_i(X)^2 \end{aligned}$$

This proves (*).

To prove Theorem 3 it remains to show that the sums in Lemma 2.3 are direct.

Proposition 2.4
Let $i = 0, 1, 2$. Then
 i) $u_0(K_iF) \cap K_i(X)^1 = 0$
 ii) $\tilde{u}(K_iF) \cap K_i(X)^2 = 0$ where $\tilde{u} = u_0 - u_1$.

Let Y be the Severi-Brauer variety associated to D. To prove 2.4 we may replace F by $F(Y)$, because $K_iF \to K_iF(Y)$ is injective (Proposition 1.2). Then again $D = M_2(F)$ and $X = \hat{X}$.

Proof of i)
$K_i(X)^1$ is the kernel of the natural map res : $K_i(X) \to K_iF(X)$. res$\circ\, u_0 : K_iF \to K_iF(X)$ is the homomorphism induced by inclusion, hence injective (Proposition 1.4). qed

Proof of ii)
We have to look closer to the Gersten spectral sequence. Let $\mathfrak{M}^n$ be the category of coherent O_X-modules M with $\operatorname{cod}\operatorname{supp} M \geq n$. Let $Z \subset X = \hat{X}$ be the hyperplane section as in § 1.
We have a commutative diagram

$$\begin{array}{ccccccc} K_iF & \xrightarrow{\cdot[\mathrm{O}_Z]} & K_i(Z) & \xrightarrow{f} & K_i(\mathfrak{M}^1) & \xrightarrow{j} & K_i(\mathfrak{M}^0) = K_i(X) \\ \Big\downarrow{\scriptstyle \mathrm{res}_{F(Z)|F}} & & & & \Big\downarrow{\scriptstyle g} & & \\ & & & & K_i(\mathfrak{M}^1/\mathfrak{M}^2) & & \\ & & & & \Big\| & & \\ K_iF(Z) = K_i\kappa(v_z) & & \hookrightarrow & & \bigoplus_{v \in X^{(1)}} K_i\kappa(v) & & \end{array}$$

Here f, j, g are the obvious maps. The composition of the maps in the upper row is just $\tilde{u}$. Now suppose $\tilde{u}(\alpha) \in K_i(X)^2$ for some $\alpha \in K_iF$. Then

$$f(\alpha \cdot [\mathrm{O}_Z]) \in \operatorname{Ker} j + \operatorname{Im}(K_i(\mathfrak{M}^2) \to K_i(\mathfrak{M}^1)) =$$
$$= \operatorname{Im}(K_{i+1}(\mathfrak{M}^0/\mathfrak{M}^1) \to K_i(\mathfrak{M}^1)) + \operatorname{Im}(K_i(\mathfrak{M}^2) \to K_i(\mathfrak{M}^1)),$$

hence

$$gf(\alpha\cdot[\mathrm{O}_Z]) \in \mathrm{Im}(K_{i+1}(\mathfrak{M}^0/\mathfrak{M}^1) \to K_i(\mathfrak{M}^1/\mathfrak{M}^2)) = d(K_{i+1}F(X))$$

On the other hand $gf(\alpha\cdot[\mathrm{O}_Z]) \in K_i\kappa(v_Z)$. Proposition 1.4. yields $K_i\kappa(v_Z) \cap d(K_{i+1}F(X)) = 0$ and therefore $\mathrm{res}_{F(Z)|F}(\alpha) = gf(\alpha\cdot[\mathrm{O}_Z]) = 0$. Since $F(Z)$ is pure transcendental over F, we have $\alpha = 0$. qed

§ 3 Proof of Theorem 4

Consider the Gersten spectral sequence $E_2^{p,q} \Rightarrow K_{-p-q}(X)$. By definition we have $\mathrm{Ker} d'/\mathrm{Im}\, d = E_2^{1,-3}$ and $E_2^{3,-4}$ is the cokernel of the differential in Proposition 1.3. The spectral sequence yields the complex

$$0 \to E_\infty^{1,-3} \to E_2^{1,-3} \to E_2^{3,-4} \to E_\infty^{3,-4}$$

which is exact with possible exception at $E_2^{3,-4}$. (Note that $E_2^{p,q} = 0$ for $p+q > 0$, $p < 0$ and $p > \dim X = 3$). Since $E_\infty^{1,-3} = K_2(X)^{1/2} = K_2F$ by Theorem 3, it is enough to show that $E_2^{3,-4} \to E_\infty^{3,-4} = K_1(X)^3$ is injective.

The inclusion $X \subset \mathbb{P}^4$ induces the commutative diagram

$$\begin{array}{ccccc}
\bigoplus\limits_{v\in X^{(2)}} K_2\kappa(v) & \xrightarrow{d_X} & \bigoplus\limits_{v\in X^{(3)}} K_1\kappa(v) & \xrightarrow{\mathcal{N}} & K_1F \\
\downarrow & & \downarrow & & \| \\
\bigoplus\limits_{v\in(\mathbb{P}^4)^{(3)}} K_2\kappa(v) & \xrightarrow{d_{\mathbb{P}^4}} & \bigoplus\limits_{v\in(\mathbb{P}^4)^{(4)}} K_1\kappa(v) & \xrightarrow{\mathcal{N}} & K_1F
\end{array}$$

The upper row is exact by Proposition 1.3, hence coker $d_X \to \mathrm{coker} d_{\mathbb{P}^4}$ is injective. Now the injectivity of $E_2^{3,-4} \to E_\infty^{3,-4}$ follows from the diagram

$$\begin{array}{ccc}
\mathrm{coker} d_X = E_2^{3,-4}(X) & \longrightarrow & E_\infty^{3,-4}(X) = K_1(X)^3 \\
\downarrow & & \downarrow \\
\mathrm{coker} d_{\mathbb{P}^4} = E_2^{4,-5}(\mathbb{P}^4) & \longrightarrow & E_\infty^{4,-5}(\mathbb{P}^4) = K_1(\mathbb{P}^4)^4
\end{array}$$

and the fact that $E_2(\mathbb{P}^4) \to E_\infty(\mathbb{P}^4)$ is an isomorphism.

§ 4 Proof of Theorem 2

Put $\bar{Y} = Y \times_F F(X)$, $\bar{D} = D \otimes_F F(X)$, $\bar{X} = X \times_F F(Y)$. One has natural identifications $F(\bar{X}) = F(Y) = F(X \times Y)$.

Consider the commutative diagram

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & K_3F(Y) & \xrightarrow{d_F} & \bigoplus_{v\in Y^{(1)}} K_2\kappa(v) & \xrightarrow{\theta} & K_2D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_3F(X) & \longrightarrow & K_3F(X\times Y) & \xrightarrow{d_{F(X)}} & \bigoplus_{v\in \bar{Y}^{(1)}} K_2\kappa(v) & \xrightarrow{\bar{\theta}} & K_2\bar{D} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d'_F} & & \downarrow{\scriptstyle d'_{F(Y)}} & & & & & & \\
 & & \bigoplus_{v\in X^{(1)}} K_2\kappa(v) & \longrightarrow & \bigoplus_{v\in \bar{X}^{(1)}} K_2\kappa(v) & & & & & & \\
 & & \downarrow{\scriptstyle d''_F} & & \downarrow{\scriptstyle d''_{F(Y)}} & & & & & & \\
0 & \longrightarrow & \bigoplus_{v\in X^{(2)}} K_1\kappa(v) & \longrightarrow & \bigoplus_{v\in \bar{X}^{(2)}} K_1\kappa(v) & & & & & &
\end{array}
$$

The differentials here are the differentials of spectral sequences for Y and $\bar{Y}$ (horizontal), X and $\bar{X}$ (vertical) respectively. According to Theorem 4 and Proposition 1.2 the homomorphism Ker $d''_F/\text{Im } d'_F \to$ Ker $d_{F(Y)}/\text{Im } d'_{F(Y)}$ is injective. Since $D \otimes_F F(Y)$ is trivial, the vertical sequence associated to $\bar{X}$ is exact at $K_3F(Y)$ and $K_3F(X \times Y)$ by Proposition 1.4. Moreover $f = \text{res}_{F(X)|F}$ is injective by Corollary 1.5.

Lemma 4.1

Let $\alpha \in \bigoplus_{v\in Y^{(1)}} K_2\kappa(v)$ and $\beta \in K_3F(X \times Y)$ such that $d_{F(X)}(\beta) = f(\alpha)$. Then there exist $\gamma \in \bigoplus_{v\in X^{(1)}} K_2\kappa(v)$ such that $\text{res}_{F(Y)|F}(\gamma) = d'_{F(Y)(\beta)}$.

Let us first assume the lemma is true. To prove Theorem 2, i.e. the injectivity of coker $d_F \to$ coker $d_{F(X)}$, we have to find for a given α as in Lemma 4.1 an element

$\delta \in K_3F(Y)$ such that $d_F(\delta) = \alpha$. This is done by a pure diagram chasing, using Lemma 4.1 and the above remarks.

Proof of Lemma 4.1

Let $F'|F$ be isomorphic to $F(Y)|F$. We consider the above diagram after the base extension $F \to F'$. $D' = D \otimes_F F'$ is trivial, hence $Y' = Y \times_F F' \simeq \mathbb{P}^1_{F'}$. Therefore, over F', both horizontal sequences are exact. Put $\alpha' = \mathrm{res}_{F'|F}(\alpha)$ and $\beta' = \mathrm{res}_{F'|F}(\beta)$. The injectivity of $K_2F' \to K_2F'(X)$ in the commutative diagram

$$\begin{array}{ccc} K_2D & \longrightarrow & K_2D' = K_2F' \\ \downarrow & & \downarrow \\ K_2\bar{D} & \longrightarrow & K_2\bar{D}' = K_2F'(X) \end{array}$$

yields $\theta'(\alpha') = 0$. By exactness there is an element $\delta' \in K_3F'(Y)$ such that $d_{F'}(\delta') = \alpha'$. Put $\tilde{\beta} = \beta' - \mathrm{res}_{F'(X\times Y)|F'[Y)}(\delta')$. Since $d'_{F'(X)}(\tilde{\beta}) = 0$, we have $\tilde{\beta} \in K_3F'(X)$. Put $\lambda = \mathrm{res}_{F'(Y)|F(Y)}(d'_{F(Y)}(\beta))$. Since $\lambda = d_{F'(Y)}(\beta') = \mathrm{res}_{F'(Y)|F'}(d'_{F'}(\tilde{\beta}))$ we have

$$\lambda \in \mathrm{res}_{F'(Y)|F(Y)}\Big(\bigoplus_{v\in\bar{X}^{(1)}} K_2\kappa(v)\Big) \cap \mathrm{res}_{F'(Y)|F'}\Big(\bigoplus_{v\in X'^{(1)}} K_2\kappa(v)\Big).$$

Therefore

$$\lambda \in \bigoplus_{v\in X^{(1)}} [\mathrm{res}_{F'(Y)|F(Y)}(K_2\kappa(v_{F(Y)})) \cap \mathrm{res}_{F'(X)|F'}(K_2\kappa(v_{F'}))]$$

Proposition 1.2 yields

$$\lambda \in \bigoplus_{v\in X^{(1)}} \mathrm{res}_{F'(Y)|F}(K_2\kappa(v))$$

Now let γ be the unique element such that $\mathrm{res}_{F'(Y)|F}(\gamma) = \lambda$. Then

$$\mathrm{res}_{F'(Y)|F(Y)}(\mathrm{res}_{F(X)|F}(\gamma) - d'_{F(Y)}(\beta)) = 0,$$

hence $\mathrm{res}_{F(Y)|F}(\gamma) = d'_{F(Y)}(\beta)$. qed

§ 5 Proof of Theorem 1

Let $\mu : K_1F \otimes K_1D \to K_2D$ be the multiplication in K-Theory. For a splitting field L of D, finite over F, the following diagram is commutative:

$$
\begin{array}{ccc}
K_1F \otimes K_1L & \longrightarrow & K_2L \\
\big\downarrow {\scriptstyle 1\otimes\theta_L} & & \big\downarrow {\scriptstyle \theta_L} \\
K_1F \otimes K_1D & \xrightarrow{\mu} & K_2D \\
\big\downarrow {\scriptstyle 1\otimes \mathrm{Nrd}} & & \big\downarrow {\scriptstyle \mathrm{Nrd}} \\
K_1F \otimes K_1F & \longrightarrow & K_2F
\end{array}
\qquad (N_{L|F} : K_2L \to K_2F)
$$

μ is surjective ([RS; Theorem 4.3]).

Lemma 5.1
μ is a symbol, that is $\mu(1 - Nrd(\alpha), \alpha) = 0$ for $\alpha \in K_1D$, $\mathrm{Nrd}(\alpha) \neq 1$.

Proof
Let $L \subset D$ be a maximal commutative subfield and let $u \in L^*$ such that $\alpha = \theta_L(u)$. Then $\mathrm{Nrd}(\alpha) = N_{L|F}(u)$, hence $\mu(1 - \mathrm{Nrd}(\alpha), \alpha) = \theta_L(\{1 - N_{L|F}(u), u\})$. Let σ be the generator of $\mathrm{Gal}(L|F) = \mathbb{Z}/2$. By Skolem-Noether $L \to D$ is equivariant with respect to σ and an inner automorphism of D. Therefore $\theta_L \circ = \sigma = \theta_L$ and the claim follows since $\{1 - N_{L|F}(u), u\} \in (1 - \sigma)(K_2L)$, see [Me, Lemma 4].

Lemma 5.2
If $\mathrm{Nrd} : K_1D \to K_1F$ is surjective, then $\mathrm{Nrd} : K_2D \to K_2F$ is an isomorphism.

Proof
Since $K_1D \to K_1F$ is always injective, we have $K_1D = K_1F$. Lemma 5.1 shows that $\mu : K_1F \otimes K_1F \to K_2D$ induces an inverse $K_2F \to K_2D$ to Nrd.

Proof of Theorem 1
For $c \in F^*$ let X_c be the quadric of § 0. Then $c \in \det(D \otimes_F F(X_c))$. Let $\hat{F}$ be the compositum of the fields $F(X_c)$, $c \in F^*$ and put $F_0 = F, F_1 = \hat{F}, F_{n+1} = \hat{F}_n, \bar{F} = \bigcup_{n\geq 0} F_n$ and $\bar{D} = D \otimes_F \bar{F}$. Then $\det : \bar{D} \to \bar{F}$ is surjective and therefore the same is true for $\mathrm{Nrd} : K_1\bar{D} \to K_1\bar{F}$. The composition of $K_2D \to K_2\bar{D} \to K_2\bar{F}$ is injective by Theorem 2 and Lemma 5.2; this clearly implies the injectivity of $K_2D \to K_2F$. qed.

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Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
D-93040 Regensburg