NOTES ON THE ASSOCIATOR

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Contents

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Preface

We discuss the 5-term relation for associators and its relation with the associahedron.

Our main interest is to gather explanations and references.

Please comment.

§1. A relation for associators

Let

$$\mu\colon M \otimes M \to M$$
$$\mu(x, y) = xy$$

be a bilinear product on some R-module M. For μ there are no assumptions on associativity, commutativity or unitality (but the base ring R has these properties).

Let

$$(\, , \, , \,)\colon M^{\otimes 3} \to M$$
$$(x, y, z) = (xy)z - x(yz)$$

denote the associator of the algebra (M, μ).

There is the following 5-term relation

$$(*) \qquad x(y, z, t) + (x, y, z)t = (xy, z, t) - (x, yz, t) + (x, y, zt)$$

It follows easily by expanding the associator expressions (this is spelled out in [16, Proof of Lemma 1]).

Relation $(*)$ appears in Zorn 1931 [22, (2), p. 125]. It is called "Vereridentität" and used to prove Artin's theorem on alternative algebras. Other places are Schafer 1961 [17, (12), p. 10][1], Schafer 1966 [18, (2.4), p. 13] (with 3 other applications), Kurosh 1962 [5, (3), p. 265], Kurosh 1965 (1962) [4, (3), p. 244], Bourbaki 1970 [1, Lemme 1, (2), p. A III.173], Bourbaki 1974 (1970) [2, Lemma 1, (2), p. 612] (thanks to J.-P. Tignol for the hint) and Garibaldi-Petersson-Racine [3, 7.5, (2)].

Our first application of $(*)$ was related with parametrization of algebras. The relation can also be used to establish the associativity of the multiplication in the free product of groups. (To give details would lead to far here.)

§2. More general relations for the associator?

I got aware of relation $(*)$ many years ago.

Since then I was wondering occasionally whether there are more relations for the associator, valid for any M with product μ.

Only recently I realized that the question is closely related to monoidal categories (see for instance Mac Lane 1998 (1971) [11]). In this context the 5-term relation $(*)$ (and its proof) appears in the form of the pentagon axiom.

In fact, a closer look at Mac Lane 1963 [10, Theorem 3.1, p. 33] reveals that the 5-term relation $(*)$ is the main relation for the associator (in this context).

There are further relations which stem from expanding 2 associators. In the simplest case, a parenthesized expression like

$$(a, b, c) \ldots (x, y, z)$$

[1] [17] are lecture notes by Schafer from 1961 with the same title as the 1966 book [18]. On gutenberg.com there are scans, the page with the relation is p0012.png.

yields the 4-term relation

$$(ab)c \dots (x,y,z) - a(bc) \dots (x,y,z) = (a,b,c) \dots (xy)z - (a,b,c) \dots x(yz)$$

The 2 associators could also be nested, like

$$(\dots (x,y,z) \dots, b, c), \quad (a, \dots (x,y,z) \dots, c), \quad (a, b, \dots (x,y,z) \dots)$$

giving rise to further 4-term relations.

The 4-term relations are sort of obvious and less sophisticated than the 5-term relation ($*$). In the context of monoidal categories the 4-term relations are hidden in the setup.

§3. Higher relations for the associator?

Once the question for general relations for the associator is settled, one may ask for higher relations, that is, relations among the relations etc.

An answer is provided by the exactness of the chain complex of the associahedron.

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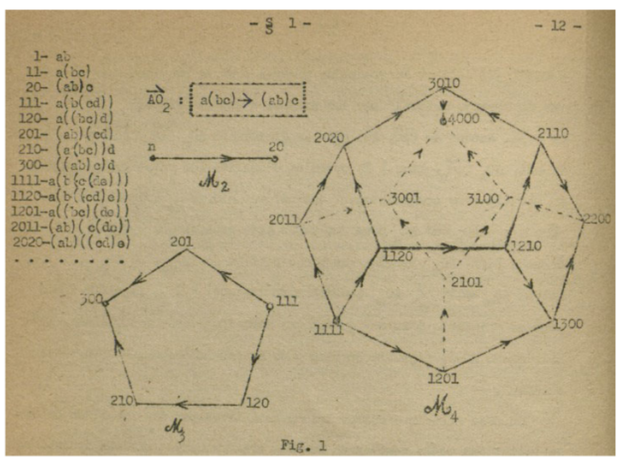

Fig. 1 Tamari's associahedron

The vertices (0-cells) of the n-dimensional associahedron T_n are the parenthesized expressions in $n+2$ variables $x_1, \dots, x_{n+2} \in M$ (each of which appears once and in the given order, as in ($*$)). The edges (1-cells) are given by associators appearing in nested ways, the 2-cells correspond to relations among these associators, etc.

The associahedron T_2 (labeled $\mathcal{M}_3$ in the picture) is a pentagon with its 2-cell representing the 5-term relation.

The associahedron T_3 (labeled $\mathcal{M}_4$) has as faces besides pentagons some quadrilaterals, each of which representing a 4-term relation.

The chain complex of the associahedron can be described in a combinatorial manner using less and less parenthesized expressions as free generators (this is not the place to give details). The crucial fact then is that this chain complex is the chain complex of a polytope (namely the associahedron). Thus the chain complex is acyclic and we know all general higher relations for the associator (in this context).

3.1. **Notes and references.** The image above of Tamari's associahedron has been taken from Stasheff 2012 [21, p. 46] in the Tamari memorial Festschrift 2012 [13]. See also Loday 2012 [8, 8 Realizing of the associahedron, p. 9–10] in the same book.

There seem to be hundreds of millions of related papers, I downloaded tens of thousands of them but read: none. Nevertheless, here are a few further references I want to mention: Stasheff 1963 [20], Shnider-Sternberg 1993 [19], Markl-Shnider-Stasheff 2002 [12], Leinster 2004 [6], Loday 2004 [7], Loday-Vallette 2012 [9], Pilaud-Santos-Ziegler 2023 [14].

§4. Questions and Remarks

- In spite of the (hopefully correct) discussion of the associahedron above, I haven't much understood about it. I don't have a grasp yet on the associahedral chain complex, despite the many papers on it.

  For monoidal/associahedral people:

  What is your answer to the question on general (higher) relations for the associator tensor $(\ ,\ ,\ )\colon M^{\otimes 3} \to M$ associated to multiplication tensors $M^{\otimes 2} \to M$?

  What is the genuine way to pin down the associahedral chain complex and settle its acyclicity? (Or at least: what is the state of the art?)
- A first problem is actually to formulate precise questions for relations for the associator tensor. We discussed here the relations among particular parenthesized expressions. One may think of other expressions involving associators. However instead of wildly speculating, one should probably rather look for concrete problems where associators are involved.
- What about relations for associators in some specific cases? I am thinking here of the cases where M is a locally free R-module of some fixed finite rank n.

  I am pretty sure that the cases $n = 2$, 3 are worthwhile to look at (the cases $n \geq 4$ might get very complicated).

  One should include here the unital cases, then of rank 3, 4.

  This touches the question of parametrizing algebras of finite rank. The set of equations for a multiplication tensor given by associativity is really a non-trivial one. It is long known that for associative, commutative and unital algebras of higher rank the etale algebras are not dense (the corresponding Hilbert scheme is not connected). See e.g. Poonen 2007 [15, Remark 1.2, p. 818].

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