THE ASSOCIAHEDRAL CHAIN COMPLEX

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`preliminary version`

Contents

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Overview

You are looking at the text "The associahedral chain complex" [pdf].

We construct the associahedral chain complex algebraically and prove its acyclicity. All details are given.

The methods seem to be new.

§1 and §2 are entirely independent.

§1 is a textbook style introduction to free multi-magmas (the set of partially parenthesized words). Some things are missing, but it contains all what is needed further on.

§2 is an important part. Proposition (2.1) describes a basic relation valid in any extended odd multi-algebra. See Remark (2.4) and section 6.1 for further comments.

§3 collects the fruits and defines the associahedral chain complex.

§4 and §5 have been added a bit later. §4 complements §1 and §5 shows acyclicity of the associahedral chain complex.

§6 mentions possible expansions.

Section 6.1 is a draft for an interpretation of the proof of Corollary (2.2) using signed coderivations. (A detailed exposition of this topic is too involved for now.)

Section 6.2 mentions the construction of the associahedron as subdivided cube. I believe that this point of view (which is not new) is basic.

Section 6.3: Another construction of the associahedral chain complex uses plane trees. It has some elegance, since it hides the sign business in the coefficient system of the graph complex. It also reveals the automorphism group (the Dieder group). There is a (well known) dictionary to polygon triangulations (mentioned in 6.4).

To be clear: I have no idea if and when I will write on these topics.

§1. Free multi-magmas

A *magma* consists of a set M and a map $M^2 \to M$, see Serre (Lie algebras and Lie groups, 1964) [5, Chap. IV, Free Lie Algebras, 1. Free magmas, p. 18].

Multi-magmas are a variant of magmas with multi-fold products.

Free multi-magmas make precise the term "partially parenthesized words".

(1.1) Definition. A set M together with a family of maps

$$m_n\colon M^n \to M \quad (n \geq 2)$$

is called a *multi-magma*. The map m_n is called the *n-ary product* of the multi-magma M.

1.1. **Free multi-magmas.** We adopt the construction of free magmas in [5].

1.1.1. **The definition.** Let X be a set. Define inductively a family of sets X_k $(k \geq 1)$ as follows:

$$\textbf{(1.2)} \qquad X_1 = X$$

$$\textbf{(1.3)} \qquad X_{k,n} = \coprod_{p_1+\cdots+p_n=k} X_{p_1} \times \cdots \times X_{p_n} \qquad (k, n \geq 2)$$

$$\textbf{(1.4)} \qquad X_k = \coprod_{n \geq 2} X_{k,n} \qquad (k \geq 2)$$

In (1.3) one understands $p_i \geq 1$. Since $n \geq 2$ one has $p_i < k$.

Put

$$M_X = \coprod_{k \geq 1} X_k$$

and let

$$\operatorname{len}\colon M_X \to \mathbf{Z}$$
$$\operatorname{len}|X_k = k \qquad (k \geq 1)$$

be the separating function. Thus $\alpha \in X_{\operatorname{len}(\alpha)}$ $(\alpha \in M_X)$.

For $\alpha \in M_X$ the integer $\operatorname{len}(\alpha)$ is called the *length* of α. An element of length 1 is called an *atom*. The atoms form the set $X \subset M_X$.

Further let

$$\operatorname{ar}\colon M_X \setminus X \to \mathbf{Z}$$
$$\operatorname{ar}|X_{k,n} = n \qquad (k, n \geq 2)$$

The integer $\operatorname{ar}(\alpha)$ is called the *arity* of α. Clearly $\operatorname{ar}(\alpha) \geq 2$. (The arity is not defined for atoms.)

The pair $(\operatorname{len}, \operatorname{ar})$ decomposes the subset of non-atomic elements into the disjoint union

$$M_X \setminus X = \coprod_{k \geq 2} \coprod_{n \geq 2} X_{k,n}$$

and a non-atomic element α can be written as $\operatorname{ar}(\alpha)$-tuple

$$\alpha = (\alpha_1, \ldots, \alpha_{\operatorname{ar}(\alpha)}) \quad (\alpha_i \in M_X)$$

Such a presentation is unique since

$$\operatorname{len}(\alpha) = \sum_{i=1}^{\operatorname{ar}(\alpha)} \operatorname{len}(\alpha_i)$$

The element α_i is called the i-th *factor* of α. Clearly $\operatorname{len}(\alpha_i) < \operatorname{len}(\alpha)$.

For $n \geq 2$ define

$$\mu_n\colon M_X^n \to M_X$$

$$\mu_n(\alpha_1, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_n) \in \prod_{i=1}^n X_{p_i} \subset X_{k,n} \subset X_k \subset M_X$$

where $p_i = \operatorname{len}(\alpha_i)$, $k = \sum_i p_i$. Thus $\mu_n(\alpha_1, \dots, \alpha_n)$ has α_i as i-th factor.

There results the arity-decomposition

$$M_X = X \amalg \coprod_{n \geq 2} M_X^n \tag{1.5}$$

with the μ_n serving as inclusions.

1.1.2. **Universal property.** The multi-magma $(M_X, (\mu_n\colon M_X^n \to M_X)_{n \geq 2})$ is called the *free multi-magma on* X.

(1.6) Lemma. *Let* $(M, (m_n\colon M^n \to M)_{n\geq 2})$ *be any multi-magma. For any map* $f\colon X \to M$ *there exists a unique multi-magma homomorphism* $F\colon M_X \to M$ *extending* f.

Proof: Define by induction on the length

$$F(\alpha) = m_n(F(\alpha_1), \dots, F(\alpha_n)), \quad \alpha = (\alpha_1, \dots, \alpha_n)$$

with $n = \operatorname{ar}(\alpha)$. □

1.1.3. **Complexity and degree.** For $\alpha \in M_X$ the *complexity* $c(\alpha)$ is defined inductively as follows: $c(x) = 0$ for atoms $x \in X$ and

$$c(\alpha) = 1 + \sum_{i=1}^{\operatorname{ar}(\alpha)} c(\alpha_i), \quad \alpha = (\alpha_1, \dots, \alpha_{\operatorname{ar}(\alpha)})$$

in the non-atomic case.

Clearly $c(\alpha) \geq 0$ with equality only for atoms. One has $c(\alpha) = 1$ if and only if all factors are atoms:

$$\alpha = (x_1, \dots, x_{\operatorname{ar}(\alpha)}) \quad (x_i \in X)$$

By definition, an element $\alpha \in M_X$ is a nested tuple (with no 1-tuples) constructed from atoms, like

$$\alpha = (x_1, (x_2, x_3, x_4), (x_5, (x_6, x_7), x_8), x_9) \qquad (x_i \in X)$$

The complexity $c(\alpha)$ is the number of paren pairs appearing in such a full expansion down to atoms.

For $\alpha \in M_X$ the *degree* $\deg(\alpha)$ is defined inductively as follows: $\deg(x) = 0$ for atoms $x \in X$ and

$$\deg(\alpha) = \operatorname{ar}(\alpha) - 2 + \sum_{i=1}^{\operatorname{ar}(\alpha)} \deg(\alpha_i), \quad \alpha = (\alpha_1, \dots, \alpha_{\operatorname{ar}(\alpha)})$$

in the non-atomic case. Obviously $\deg(\alpha) \geq 0$.

(1.7) Lemma.

$$c(\alpha) + \deg(\alpha) + 1 = \operatorname{len}(\alpha)$$

Proof: The claim is clear for atoms and follows by induction from

$$\begin{aligned} c(\alpha) &= 1 + \textstyle\sum_{i=1}^n c(\alpha_i) \\ \deg(\alpha) &= n-2+\textstyle\sum_{i=1}^n \deg(\alpha_i) \\ 1 &= 1-n+\textstyle\sum_{i=1}^n 1 \\ \operatorname{len}(\alpha) &= 0 + \textstyle\sum_{i=1}^n \operatorname{len}(\alpha_i) \end{aligned}$$

with $n = \operatorname{ar}(\alpha)$. □

1.1.4. **The free magma.** Let

$$M'_X = \{\, \alpha \in M_X \mid \deg(\alpha) = 0 \,\}$$

One has $X \subset M'_X$. The inductive description of deg shows $\operatorname{ar}(\alpha) = 2$ for $\alpha \in M'_X \setminus X$ and $\alpha_i \in M'_X$ for the two factors of α. Moreover $\mu_2({M'_X}^2) \subset M'_X$.

(1.8) Lemma. *$(M'_X, \mu_2\colon {M'_X}^2 \to M'_X)$ is the free magma on X (cf. [5]).*

Proof: One proceeds as in the proof of Lemma (1.6), this time referring to the unique presentation

$$\alpha = \mu_2(\alpha_1, \alpha_2) \quad (\alpha_i \in M'_X)$$

of $\alpha \in M'_X \setminus X$. □

1.1.5. **Notations and conventions.** In the *complete product notation* one simply drops the commas from a tuple presentation:

$$(\alpha_1 \cdots \alpha_n) = (\alpha_1, \ldots, \alpha_n) = \mu_n(\alpha_1, \ldots, \alpha_n) \quad (n \geq 2)$$

In the case of atoms no parens are written.

In this notation the example above reads as

$$(x_1(x_2x_3x_4)(x_5(x_6x_7)x_8)x_9)$$

In the *simple product notation* one drops the outer paren pair as well (in the non-atomic case) and writes

$$\alpha_1 \cdots \alpha_n = (\alpha_1, \ldots, \alpha_n)$$
$$x_1(x_2x_3x_4)(x_5(x_6x_7)x_8)x_9$$

This is often convenient for writing and reading, but when taking products one has to reinsert the omitted paren pairs before combining.

The elements of M_X are called *partially parenthesized words in X*, but often we call them simply *words*.

As already noted, the complexity of a word counts the number of paren pairs including a possible outer paren pair in a fully expanded presentation (down to atoms). Obviously it also counts the number of μ_n involved to construct the word from atoms. For example

$$(ab(cde)) = \mu_3(a, b, \mu_3(c, d, e)) \quad (a, b, c, d, e \in X)$$

has complexity 2.

A *binary word* is an element of $\mu_2(M_X^2)$, that is, a 2-tuple

$$\alpha = (\alpha_1, \alpha_2) = \alpha_1\alpha_2$$

with no extra conditions on the factors α_1, α_2. Example:

$$\alpha = (x_1x_2x_3)(x_4x_5(x_6x_7)x_8)$$

A *fully binary word* is a binary word with both factors again fully binary words or atoms. The fully binary words are the elements of the free magma M'_X. We also call them *fully parenthesized words*.

1.2. **Further generalities.** At some point I will probably discuss the opposite M^{op} of a multi-magma M. For the free multi-magma M_X this yields an involution. This involution reverses parenthesizing and order. One may also reverse parenthesizing and order separately:

$$a(bc) \leftrightarrow (ab)c \leftrightarrow (cb)a \leftrightarrow c(ba)$$

Further, it can be useful to introduce notations like

$$V^{\otimes\alpha}$$

for $\alpha \in M_X$ and a family of modules $V = (V_x)_{x\in X}$. Also, for a bifunctor $\square\colon C^2 \to C$ on a category C and an object V of C, the objects $V^{\square\alpha}$ ($\alpha \in M'_X$) are defined without an associativity constraint for $\square$.

Some of these things are mentioned here: "Notes on free alternative algebras" [pdf]. They will be expanded when needed (or dropped).

1.3. **The case of a single atom.** We are mainly interested in the free multi-magma on one element, often denoted by $*$ or $\bullet$.

For the 1-element set $X = \{*\}$, the elements of X_n for $n \leq 4$ are

$$\begin{gathered} * \\ (**) \\ (*(**)),\quad (***),\quad ((**)*) \\ (*(*(**))),\quad (**(**)),\quad ((**)(**)),\quad ((**)**),\quad (((**)*)*) \\ (*(***)),\quad (****),\quad ((***)*) \\ (*((**)*)),\quad (*(**)*),\quad ((*(**))*) \end{gathered}$$

To present elements of M_X one may of course choose any other symbol for the unique element of X. Moreover, for better readability one may want to "fill" the words with a set of say digits or letters. With this convention, the following expressions represent the same word:

$$*(*(*(**)),\quad \bullet(\bullet(\bullet(\bullet\bullet))),\quad 1(2(3(45))),\quad a(b(c(de)))$$

In the filled cases, we understand that the symbols representing the atoms are all different and in the same order for all words to be displayed.

1.4. **Free multi-algebras.** Let R be a ring (associative, commutative, unital). A *multi-algebra* over R consists of an R-module V together with a family of R-module homomorphisms

$$m_n\colon V^{\otimes n} \to V \qquad (n \geq 2)$$

For a set X, let A_X be the free R-module with basis M_X. The maps $\mu_n\colon M_X^n \to M_X$ extend to R-module homomorphisms

$$\mu_n\colon A_X^{\otimes n} \to A_X \qquad (n \geq 2)$$

which turn A_X into a multi-algebra. The multi-algebra A_X is called the *free multi-algebra on X*.

(1.9) Lemma. *Let $(V, (m_n\colon V^{\otimes n} \to V)_{n\geq 2})$ be any multi-algebra and let $f\colon X \to V$ be any map. Then there exists a unique multi-algebra homomorphism $F\colon A_X \to V$ extending f.*

Proof: The compositions of m_n with $V^n \to V^{\otimes n}$ make V a multi-magma. Extend f first to a multi-magma homomorphism on M_X (Lemma (1.6)) and then R-linearly to A_X. □

1.5. **Gradings.** A map $f\colon M \to Z$ from a set M to an abelian group Z defines a Z-grading on the free R-module A with basis M by taking for A_z the free R-submodule generated by $M_z = f^{-1}(z)$ $(z \in Z)$. For a finite family $f_i\colon M_i \to Z$ the sum $\sum_i f_i(x_i)$ $(x_i \in M_i)$ yields the natural grading on the tensor product of the free graded R-modules A_i with basis M_i. In particular, $A^{\otimes n}$ is graded via $M^n \xrightarrow{f^n} Z^n \xrightarrow{\sum} Z$.

As for A_X, one considers the following **Z**-gradings.

1.5.1. **Length grading.** The *length grading* is given by the function $\operatorname{len}\colon M_X \to \mathbf{Z}$. Here all μ_n have degree 0. In the case $|X| = 1$, the words α with $\operatorname{len}(\alpha) = k$ parameterize the cells of the $(k-2)$-dimensional associahedron $(k \geq 2)$.

1.5.2. **Degree grading.** The *degree grading* is given by the function $\deg\colon M_X \to \mathbf{Z}$. Here μ_n has degree $n-2$ as one can see from the inductive description of deg. In the case $|X| = 1$, the words α of degree k parameterize the k-cells of the associahedra.

The *shifted degree grading* is given by $1 + \deg$. Here all μ_n have degree -1. This can be easily seen directly or by noting that μ_n has degree 1 with respect to the grading given by the complexity function $c = \operatorname{len} - \deg - 1\colon M_X \to \mathbf{Z}$.

§2. The basic relation

2.1. **Involutions.** Let R be a ring (associative, commutative, unital). We consider R-modules V together with an involution, that is, an endomorphism $\varepsilon_V \in \mathrm{End}(V)$ with $\varepsilon_V^2 = 1$. In other words, the pair (V, ε_V) constitutes an $R[\mathbf{Z}/2\mathbf{Z}]$-module.

A $\mathbf{Z}/2\mathbf{Z}$-graded R-module

$$V = V_0 \oplus V_1$$

carries the involution[1]

$$\varepsilon(v) = \begin{cases} v & (v \in V_0) \\ -v & (v \in V_1) \end{cases}$$

The involutions on the module R are (R, η) with $\eta \in R$, $\eta^2 = 1$.

Let (V, ε_V), (W, ε_W) be R-modules with involution. Their tensor product is $(V \otimes W, \varepsilon_V \otimes \varepsilon_W)$. Particular cases are $(V, \varepsilon) \otimes (R, \eta) = (V, \eta\varepsilon)$ and $(V, \varepsilon)^{\otimes n} = (V^{\otimes n}, \varepsilon^{\otimes n})$.

Let $\eta \in R$, $\eta^2 = 1$. An R-module homomorphism $f \in \mathrm{Hom}(V, W)$ is called *η-homogeneous* if $f\varepsilon_V = \eta\varepsilon_W f$ (in other words, f is a homomorphism $(V, \varepsilon_V) \to (W, \varepsilon_W) \otimes (R, \eta)$ of R-modules with involution). In the cases $\eta = \pm 1$, we adopt language from $\mathbf{Z}/2\mathbf{Z}$-gradings: If $\eta = 1$, then f is called *even*, and if $\eta = -1$, then f is called *odd* (in characteristic 2 there is of course no difference between the two cases). Clearly, ε itself is even.

For an η-homogeneous endomorphism $f \in \mathrm{End}(V)$, the endomorphism f^2 is always even.

Let V be an R-module with involution ε. Then $V^{\otimes n}$ carries the involution $\varepsilon^{\otimes n}$. Let $f \in \mathrm{Hom}(V^{\otimes n}, V)$ be η-homogeneous: $f\varepsilon^{\otimes n} = \eta\varepsilon f$. Upon changing ε to $\lambda\varepsilon$ for some $\lambda \in R$, $\lambda^2 = 1$, f becomes $\lambda^{n-1}\eta$-homogeneous. In particular: If n is even, then an even/odd f is odd/even with respect to $-\varepsilon$. If n is odd, then an even/odd f remains so for $-\varepsilon$.

2.2. **The operators L_k, L_k^+.** Given an involution (V, ε), let

$$L_k, L_k^+ \colon \mathrm{Hom}(V^{\otimes n}, V) \to \mathrm{Hom}(V^{\otimes n+k}, V^{\otimes 1+k}) \qquad (k \geq 0)$$

be the linear operators

$$L_k(f) = \sum_{a,b \geq 0,\, a+b=k} (\varepsilon^{\otimes a} \otimes f \otimes 1^{\otimes b})$$

$$L_k^+(f) = \sum_{a,b \geq 0,\, a+b=k} (1^{\otimes a} \otimes f \otimes 1^{\otimes b})$$

Here $1 \in \mathrm{End}(V)$ denotes the identity map of V. (Of course L_k^+ does not depend on ε.)

If $f \in \mathrm{Hom}(V^{\otimes n}, V)$ is η-homogeneous, then $L_k(f)$, $L_k^+(f)$ are η-homogeneous as well (tensor products with even homomorphisms preserve η-homogeneity).

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[1]In scheme language, an involution is an operation of the constant group scheme $\underline{\mathbf{Z}/2\mathbf{Z}}$, while a $\mathbf{Z}/2\mathbf{Z}$-grading is an operation of $\mu_2 = \mathrm{Spec}\, R[\mathbf{Z}/2\mathbf{Z}]$. The homomorphism $\underline{\mathbf{Z}/2\mathbf{Z}} \to \mu_2$, $1 \mapsto -1$ turns a $\mathbf{Z}/2\mathbf{Z}$-grading into an involution.

2.3. **The main computation.** Let V be an R-module with involution ε and consider a family

$$\mu_n\colon V^{\otimes n} \to V \qquad (n \geq 1)$$

of *odd* homomorphisms[2]. Put

$$R_n = \sum_{k=1}^{n} \mu_k L_{k-1}(\mu_{n+1-k})\colon V^{\otimes n} \to V \qquad (n \geq 1)$$

Explicitly,

$$\begin{aligned}
R_1 &= \mu_1^2 \\
R_2 &= \mu_1\mu_2 + \mu_2(\mu_1 \otimes 1 + \varepsilon \otimes \mu_1) \\
R_3 &= \mu_1\mu_3 + \mu_3(\mu_1 \otimes 1 \otimes 1 + \varepsilon \otimes \mu_1 \otimes 1 + \varepsilon \otimes \varepsilon \otimes \mu_1) \\
&\quad + \mu_2(\mu_2 \otimes 1 + \varepsilon \otimes \mu_2) \\
R_4 &= \mu_1\mu_4 + \mu_4(\mu_1 \otimes 1^{\otimes 3} + \varepsilon \otimes \mu_1 \otimes 1^{\otimes 2} + \varepsilon^{\otimes 2} \otimes \mu_1 \otimes 1 + \varepsilon^{\otimes 3} \otimes \mu_1) \\
&\quad + \mu_3(\mu_2 \otimes 1 \otimes 1 + \varepsilon \otimes \mu_2 \otimes 1 + \varepsilon \otimes \varepsilon \otimes \mu_2) \\
&\quad + \mu_2(\mu_3 \otimes 1 + \varepsilon \otimes \mu_3)
\end{aligned}$$

A first computation: If $R_2 = 0$, then μ_1 is a derivation with respect to the product μ_2:

$$\mu_1^2\mu_2 = \mu_2(\mu_1^2 \otimes 1 + 1 \otimes \mu_1^2)$$

Namely $R_2 = 0$ yields

$$\mu_1^2\mu_2 = -\mu_1\mu_2(\mu_1 \otimes 1 + \varepsilon \otimes \mu_1) = \mu_2(\mu_1 \otimes 1 + \varepsilon \otimes \mu_1)^2$$

and

$$\begin{aligned}
(\mu_1 \otimes 1 + \varepsilon \otimes \mu_1)^2 &= \mu_1^2 \otimes 1 + \mu_1\varepsilon \otimes \mu_1 + \varepsilon\mu_1 \otimes \mu_1 + \varepsilon^2 \otimes \mu_1^2 \\
&= \mu_1^2 \otimes 1 + 1 \otimes \mu_1^2
\end{aligned}$$

since μ_1 is odd and $\varepsilon^2 = 1$.

(2.1) Proposition. *The following relation holds for $n \geq 1$ (trivially for $n = 1$):*

$$0 = \sum_{k=1}^{n} \big(\mu_{n+1-k} L^+_{n-k}(R_k) - R_k L_{k-1}(\mu_{n+1-k})\big)$$

An immediate consequence is

(2.2) Corollary. *If*

$$R_n = 0 \qquad (n \geq 2)$$

then

$$\mu_1^2\mu_n = \mu_n L^+_{n-1}(\mu_1^2) \qquad (n \geq 1)$$

The proof of the proposition needs a preparation.

Let

$$T_+V = \bigoplus_{n \geq 1} V^{\otimes n}$$

and let

$$\pi\colon T_+V \to V$$

be the projection to the first summand.

[2] Note that $n = 1$ is included. The object $(V, (\mu_n\colon V^{\otimes n} \to V)_{n\geq 1})$ could be called an (odd) *extended multi-algebra*, but I am not sure yet. One may introduce the notation $d = \mu_1$ right away.

Consider

$$B\colon T_+V \to T_+V$$

with components

$$B_{m,n}\colon V^{\otimes n} \to V^{\otimes m}$$

$$\begin{aligned} B_{m,n} &= L_{m-1}(\mu_{n+1-m}) \qquad && (1 \le m \le n) \\ B_{m,n} &= 0 && (m > n) \end{aligned}$$

Thus B is the infinite upper triangular matrix

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \cdots \\ 0 & L_1(\mu_1) & L_1(\mu_2) & \\ 0 & 0 & L_2(\mu_1) & \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

The square B^2 has components

$$R_{m,n} = (B^2)_{m,n}\colon V^{\otimes n} \to V^{\otimes m}$$

$$R_{m,n} = \sum_{m \le k \le n} L_{m-1}(\mu_{k+1-m})L_{k-1}(\mu_{n+1-k})\colon V^{\otimes n} \to V^{\otimes m} \qquad (1 \le m \le n)$$

Note that the R_n defined above form the first row of B^2:

$$R_n = R_{1,n} \qquad (n \ge 1)$$

(2.3) Lemma.

$$R_{m,n} = L^+_{m-1}(R_{n+1-m}) \qquad (1 \le m \le n)$$

Proof of Proposition (2.1): One exploits associativity: $(B^2)B = B(B^2)$.

$$\begin{aligned} (pB^3)_n = ((pB^2)(B))_n &= \sum_{k=1}^n R_k L_{k-1}(\mu_{n+1-k}) \\ (pB^3)_n = ((pB)(B^2))_n &= \sum_{k=1}^n \mu_k R_{k,n} \\ &= \sum_{k=1}^n \mu_k L^+_{k-1}(R_{n+1-k}) \end{aligned}$$

by Lemma (2.3). The claim follows by reindexing $k \leftrightarrow n+1-k$ the last sum. □

(2.4) Remark. Lemma (2.3) has a simple proof once T_+V has been set up as a coalgebra: B is an odd coderivation, so B^2 is an even coderivation. Coderivations can be reconstructed from its first row by the operators L_k resp. L^+_k, whence Lemma (2.3). See also section 6.1.

The following ad hoc proof of Lemma (2.3) is straightforward but naturally tedious. I hope I got it readable.

Clearly the indicated proof using coderivations is more satisfying. On the other hand, explaining the details of the framework of coderivations takes much more space than the proof given here. Also, it is good to see once an explicit proof, if only to appreciate the general setup.

Proof of Lemma (2.3): One has

$$R_{m,n} = \sum_{m\le k\le n} \sum_{\substack{a+1+b=m\\ c+1+d=k}} (\varepsilon^{\otimes a} \otimes \mu_{k+1-m} \otimes 1^{\otimes b})(\varepsilon^{\otimes c} \otimes \mu_{n+1-k} \otimes 1^{\otimes d})$$

Restricting the sum to the case $c \ge a$, $d \ge b$ yields with $c = a + r$, $d = b + s$

$$\sum_{m\le k\le n} \sum_{\substack{a+1+b=m\\ r+s=k-m}} (\varepsilon^{\otimes a} \otimes \mu_{k+1-m} \otimes 1^{\otimes b})(\varepsilon^{\otimes a} \otimes \varepsilon^{\otimes r} \otimes \mu_{n+1-k} \otimes 1^{\otimes s} \otimes 1^{\otimes b})$$

$$= \sum_{m\le k\le n} \sum_{\substack{a+1+b=m\\ r+s=k-m}} (1^{\otimes a} \otimes \mu_{k+1-m}(\varepsilon^{\otimes r} \otimes \mu_{n+1-k} \otimes 1^{\otimes s}) \otimes 1^{\otimes b})$$

$$= \sum_{m\le k\le n} \sum_{a+1+b=m} (1^{\otimes a} \otimes \mu_{k+1-m} L_{k-m}(\mu_{n+1-k})) \otimes 1^{\otimes b})$$

$$= \sum_{m\le k\le n} L^+_{m-1}(\mu_{k+1-m} L_{k-m}(\mu_{n+1-k}))$$

$$= \sum_{1\le j\le n-m+1} L^+_{m-1}(\mu_j L_{j-1}(\mu_{n+1-m+1-j})) = L^+_{m-1}(R_{n+1-m})$$

with $j = k + 1 - m$.

Hence the remaining terms should cancel.

Restricting to the case $a > c$ yields with $a = c + 1 + r$

$$\sum_{m\le k\le n} \sum_{\substack{c+1+r+1+b=m\\ c+1+d=k}} (\varepsilon^{\otimes c} \otimes \varepsilon \otimes \varepsilon^{\otimes r} \otimes \mu_{k+1-m} \otimes 1^{\otimes b})(\varepsilon^{\otimes c} \otimes \mu_{n+1-k} \otimes 1^{\otimes d})$$

$$= \sum_{m\le k\le n} \sum_{c+r+b+2=m} (1^{\otimes c} \otimes \varepsilon\mu_{n+1-k} \otimes \varepsilon^{r} \otimes \mu_{k+1-m} \otimes 1^{\otimes b})$$

observing $d = r + (k + 1 - m) + b$.

Restricting to the case $b > d$ yields with $b = r + 1 + d$

$$\sum_{m\le k\le n} \sum_{\substack{a+1+d+1+r=m\\ c+1+d=k}} (\varepsilon^{\otimes a} \otimes \mu_{k+1-m} \otimes 1^{\otimes r} \otimes 1 \otimes 1^{\otimes d})(\varepsilon^{\otimes c} \otimes \mu_{n+1-k} \otimes 1^{\otimes d})$$

$$= -\sum_{m\le k\le n} \sum_{a+d+r+2=m} (1^{\otimes a} \otimes \varepsilon\mu_{k+1-m} \otimes \varepsilon^{r} \otimes \mu_{n+1-k} \otimes 1^{\otimes d})$$

Here one combines as before (noting $c = a + (k + 1 - m) + r$) and uses additionally that the μ_i are odd (whence the sign).

The change of variables $k \leftrightarrow n-k+m$, $a \leftrightarrow c$, $d \leftrightarrow b$ shows that the two subsums do indeed cancel. □

§3. The associahedral chain complex

Let A be the free multi-algebra on one generator, say $A = A_{\{\bullet\}}$, with products

$$\mu_n \colon A^{\otimes n} \to A \qquad (n \geq 2)$$

We consider the involution ε given by the shifted degree grading defined by $1 + \deg$ (see section 1.5). Then all μ_n are odd.

In following, R_n, L_k, L_k^+ are as defined in section 2 (with $V = A$).

In case one likes to stay with the unshifted degree grading, one has to replace ε with $-\varepsilon$ everywhere.

Clearly, if an endomorphism $A \to A$ is homogeneous with respect to deg, then its degree doesn't change under the shift of gradings $\deg \to 1 + \deg$.

(3.1) Theorem. *There exists a unique endomorphism*

$$\mu_1 \colon A \to A$$

with

$$\mu_1(\bullet) = 0$$
$$R_n = 0 \quad (n \geq 1)$$

where the R_n are defined as in the previous section.

In particular $\mu_1^2 = 0$ (since $R_1 = \mu_1^2$).

The endomorphism μ_1 is homogeneous of degree 0 with respect to the length grading and of degree -1 with respect to the degree grading.

Proof: A has basis $M_{\{\bullet\}}$. One defines μ_1 on the basis elements by induction on the length, starting from $\mu_1(\bullet) = 0$.

For $n \geq 2$ the condition $R_n = 0$ means

$$\mu_1 \mu_n = -\mu_n L_{n-1}(\mu_1) - \sum_{k=2}^{n-1} \mu_k L_{k-1}(\mu_{n+1-k})$$

Denote the right hand side by Φ_n. Note that μ_1 appears in Φ_n only in the first term. Let

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad (\alpha_i \in M_X)$$

be a non-atomic element. Since $\operatorname{len}(\alpha_i) < \operatorname{len}(\alpha)$, one can rely on $\mu_1(\alpha_i)$. Therefore $L_{n-1}(\mu_1)(\alpha_1 \otimes \dots \otimes \alpha_n)$ is known. Define

$$\mu_1(\alpha) = \Phi_n(\alpha_1 \otimes \dots \otimes \alpha_n)$$

With this definition of μ_1 one has $R_n = 0$ ($n \geq 2$), since this holds on basis elements. By Corollary (2.2) one has

$$\mu_1^2 \mu_n = \mu_n L_{n-1}^+(\mu_1^2)$$

Again by induction on the length this shows $\mu_1^2 = 0$.

Similarly one argues for the gradings: Let d be an integer and consider a grading such that the μ_n ($n \geq 2$) have degree d. Assume that μ_1 has degree d on elements α with $\operatorname{len}(\alpha) < k$. Then $\mu_1 \mu_n = \Phi_n$ has degree $2d$ on elements $\alpha_1 \otimes \dots \otimes \alpha_n$ ($\operatorname{len}(\alpha_i) < k$) and therefore μ_1 has degree d on $\mu_n(\alpha_1, \dots, \alpha_n)$. □

Let $A_r \subset A$ be the component corresponding to len $= r+2$. For instance, A_0 has basis

$$\bullet\bullet$$

A_1 has basis

$$\bullet\bullet\bullet$$

$$\bullet(\bullet\bullet), \quad (\bullet\bullet)\bullet$$

and A_2 has basis

$$\bullet\bullet\bullet\bullet$$

$$\bullet(\bullet\bullet\bullet), \quad (\bullet\bullet)\bullet\bullet, \quad \bullet(\bullet\bullet)\bullet, \quad \bullet\bullet(\bullet\bullet), \quad (\bullet\bullet\bullet)\bullet$$

$$\bullet((\bullet\bullet)\bullet), \quad \bullet(\bullet(\bullet\bullet)), \quad (\bullet\bullet)(\bullet\bullet), \quad ((\bullet\bullet)\bullet)\bullet, \quad (\bullet(\bullet\bullet))\bullet$$

(3.2) Definition. The r-dimensional associahedral chain complex is the complex

$$A_r \xrightarrow{d} A_r$$

where d is the restriction of μ_1 in Theorem (3.1).

Decomposing with respect to the degree these complexes read as

$$A_r : \qquad 0 \to A_{r,r} \xrightarrow{d} \cdots \xrightarrow{d} A_{r,0} \to 0$$

Here $A_{r,0}$ is the free abelian group on the set of fully parenthesized words of length $r+2$ and $A_{r,r}$ is the free cyclic group generated by the unparenthesized word $\bullet\cdots\bullet$ of length $r+2$.

For $r = 1, 2$ one gets

$$\begin{aligned}
d(\bullet\bullet\bullet) &= -(\bullet\bullet)\bullet + \bullet(\bullet\bullet) \\
d(\bullet\bullet\bullet\bullet) &= -(\bullet\bullet)\bullet\bullet + \bullet(\bullet\bullet)\bullet - \bullet\bullet(\bullet\bullet) - (\bullet\bullet\bullet)\bullet + \bullet(\bullet\bullet\bullet) \\
d((\bullet\bullet)\bullet\bullet) &= -((\bullet\bullet)\bullet)\bullet + (\bullet\bullet)(\bullet\bullet) \\
d(\bullet(\bullet\bullet)\bullet) &= -(\bullet(\bullet\bullet))\bullet + \bullet((\bullet\bullet)\bullet) \\
d(\bullet\bullet(\bullet\bullet)) &= -(\bullet\bullet)(\bullet\bullet) + \bullet(\bullet(\bullet\bullet)) \\
d((\bullet\bullet\bullet)\bullet) &= +((\bullet\bullet)\bullet)\bullet - (\bullet(\bullet\bullet))\bullet \\
d(\bullet(\bullet\bullet\bullet)) &= -\bullet((\bullet\bullet)\bullet) + \bullet(\bullet(\bullet\bullet))
\end{aligned}$$

Note that $\varepsilon(\bullet) = -\bullet$ and $\varepsilon(\bullet\bullet) = -\bullet\bullet$.

The sign game here is confusing, at least to me. Not sure yet how to "fix" this. A possibility is replacing μ_n with $\varepsilon\mu_n$ $(n \geq 2)$.

§4. Free multi-magmas II

This section continues §1 and prepares §5.

4.1. **The standard embeddings.** Let

$$M_X^+ = \mu_2(X \times M_X) \subset M_X$$
$$M_X^- = \mu_2(M_X \times X) \subset M_X$$

Thus $M_X^\pm$ consists of the binary words $\alpha \in M_X$ with first resp. second factor atomic. Their intersection

$$M_X^+ \cap M_X^- = \mu_2(X, X) = X_2$$

is the set of words of length 2.

The sets $M_X^\pm$ decompose according to length into the subsets

$$\begin{aligned} X_k^+ &= \mu_2(X \times X_{k-1}) = M_X^+ \cap X_k \\ X_k^- &= \mu_2(X_{k-1} \times X) = M_X^- \cap X_k \end{aligned} \qquad (k \geq 2)$$

and μ_2 yields injective maps

$$X \times X_{k-1} \xrightarrow{\simeq} X_k^+ \subset X_k$$
$$X_{k-1} \times X \xrightarrow{\simeq} X_k^- \subset X_k$$

(If $|X| = 1$ and $k \geq 3$, the $X_k^\pm$ correspond to disjoint $(k-3)$-dimensional subassociahedra of the $(k-2)$-dimensional associahedron.)

4.2. **The graphs Γ_X.** Let

$$\Gamma_X = M_X \setminus X = \{\, \alpha \in M_X \mid \operatorname{len}(\alpha) \geq 2 \,\}$$

be the set of non-atomic elements and let

$$\operatorname{Bin}_X = \{\, \alpha \in \Gamma_X \mid \operatorname{ar}(\alpha) = 2 \,\}$$
$$\operatorname{Tin}_X = \{\, \alpha \in \Gamma_X \mid \operatorname{ar}(\alpha) \geq 3 \,\}$$

so that

$$\Gamma_X = \operatorname{Bin}_X \amalg \operatorname{Tin}_X$$

The elements of Bin_X are the binary words (cf. 1.1.5). The elements of Tin_X are called *non-binary words* (atoms are excluded).

We understand graphs and related notions as in Serre 1980 (1977, 1968/69) [4, I.2 Trees, 2.1 Graphs, pp. 13].

The maps

$$o,\, t\colon \operatorname{Tin}_X \to \operatorname{Bin}_X$$

$$\begin{aligned} o(\alpha) &= \mu_2(\alpha_1, \mu_{k-1}(\alpha_2, \ldots, \alpha_k)) \\ t(\alpha) &= \mu_2(\mu_{k-1}(\alpha_1, \ldots, \alpha_{k-1}), \alpha_k) \end{aligned} \qquad \alpha = (\alpha_1, \ldots, \alpha_k),\ k = \operatorname{ar}(\alpha)$$

establish Γ_X as *oriented* graph with Bin_X as set of vertices and Tin_X as set of oriented edges. Thus $\alpha \in \operatorname{Tin}_X$ is an edge from $o(\alpha)$ to $t(\alpha)$.

Since o, t don't change length, the graph Γ_X is the disjoint union of the finite graphs

$$\Gamma_{X,k} = \Gamma_X \cap X_k \qquad (k \geq 2)$$

(4.1) Proposition. *The maps o, t are bijections*

$$o\colon \operatorname{Tin}_X \to \operatorname{Bin}_X \setminus M_X^-$$
$$t\colon \operatorname{Tin}_X \to \operatorname{Bin}_X \setminus M_X^+$$

Proof: By uniqueness of presentation as products (cf. arity-decomposition (1.5)), α can be reconstructed from $o(\alpha)$ and from $t(\alpha)$. Moreover, a binary word is of the form $o(\alpha)$ resp. $o(\alpha)$ if and only if its second resp. first factor is non-atomic. □

By an *intervall* (as graph) *of length* n ($n \geq 0$) we understand an oriented graph isomorphic to the subdivision of the intervall $[0, n]$ by its integer points (these graphs are called Path_n in [4, p. 14]).

(4.2) Corollary. *The oriented graph Γ_X is a disjoint union of intervalls.*

The isolated vertices of Γ_X are the elements of $M_X^+ \cap M_X^- = X_2$. A connected component of Γ_X of length ≥ 1 starts in M_X^+ and ends in M_X^-.

Proof: One may consider the graphs $\Gamma_{X,k}$ separately. By Proposition (4.1), every vertex has valency (number of adjacent edges) ≤ 2. Hence $\Gamma_{X,k}$ decomposes into intervalls (using the finiteness of $\Gamma_{X,k}$). Again by Proposition (4.1), a vertex with valency 1 must be in M_X^+ (as start point) or in M_X^- (as end point). And a vertex with valency 0 is in $M_X^+ \cap M_X^-$. □

Examples:

- The graph $\Gamma_{X,2} = X_2$ consists exactly of the isolated vertices of Γ_X. These are the binary words x_1x_2 where x_1, x_2 are atoms.
- The graph $\Gamma_{X,3}$ has as edges the ternary words $x_1x_2x_3$ and as vertices the start points $x_1(x_2x_3)$ and end points $(x_1x_2)x_3$ (with atoms x_i).
- A connected component of Γ_X of length 1 has as edge an element of the form

$$\mu_n(x_1, \alpha_2, \ldots, \alpha_{n-1}, x_n) \qquad (x_1, x_n \in X,\ \alpha_i \in M_X, n \geq 3)$$

  and any such element appears this way.

§5. The homotopy

In this section we assume $|X| = 1$ and write $X = \{\omega\}$.

The standard embeddings will be considered as maps

$$i^{\pm}\colon M_{\{\omega\}} \xrightarrow{\simeq} M^{\pm}_{\{\omega\}} \subset M_{\{\omega\}}$$
$$i^{+}(\alpha) = \mu_2(\omega,\alpha)$$
$$i^{-}(\alpha) = \mu_2(\alpha,\omega)$$

The maps $i^{\pm}$ have degree 0, 1 with respect to the gradings by deg resp. len.

Recall the notation $A = A_{\{\omega\}}$ for the free R-module with basis $M_{\{\omega\}}$. We extent $i^{\pm}$ linearly to morphisms

$$i^{\pm}\colon A \to A$$

Recall also the differential

$$d = \mu_1\colon A \to A$$

from Theorem (3.1).

(5.1) Lemma.

$$di^{+} = +i^{+}d$$
$$di^{-} = -i^{-}d$$

Proof: Relation $R_2 = 0$ yields

$$d\mu_2 = -\mu_2(d\otimes 1) - \mu_2(\varepsilon\otimes d)$$

It suffices to note $d(\omega) = 0$ and $\varepsilon(\omega) = -1$. □

Lemma (5.1) says that i^{+} is a morphism of chain complexes. As for i^{-}, note that $d\bar{i}^{-} = +\bar{i}^{-}d$ with $\bar{i}^{-} = \varepsilon i^{-} = i^{-}\varepsilon$.

By Proposition (4.1) there is the decomposition

$$M_{\{\omega\}} = \{\omega\} \amalg M^{+}_{\{\omega\}} \amalg t(\mathrm{Tin}_{\{\omega\}}) \amalg \mathrm{Tin}_{\{\omega\}}$$

with t injective. Let U, T denote the free R-modules with basis $\{\omega\} \amalg M^{+}_{\{\omega\}}$ resp. $\mathrm{Tin}_{\{\omega\}}$. Then U, T are submodules of A and

$$\Phi_0\colon U\oplus T\oplus T \to A$$
$$\Phi_0(u,\alpha,\beta) = u + t(\alpha) + \beta$$

is an isomorphism.

(5.2) Proposition. *The morphism*

$$\Phi\colon U\oplus T\oplus T \to A$$
$$\Phi(u,\alpha,\beta) = u + d(\alpha) + \beta$$

is an isomorphism.

Proof: Let

$$\overline{A} = A/(U+T)$$

Since Φ_0 is bijective, the elements

$$t(\alpha) \mod (U+T) \qquad (\alpha \in \mathrm{Tin}_{\{\omega\}})$$

form a basis of $\overline{A}$ and the Lemma says that also the familiy

$$d(\alpha) \mod (U+T) \qquad (\alpha \in \mathrm{Tin}_{\{\omega\}})$$

forms a basis of $\overline{A}$.

One has

$$T = \bigoplus_I T_I$$

where I runs through the connected components of $\Gamma_{\{\omega\}}$ and T_I denotes the free R-module generated by the edges of I.

Fix a connected component I. It is an intervall by Corollary (4.2). Let h be its length and denote its edges and vertices by $\alpha_1, \dots, \alpha_h \in \mathrm{Tin}_{\{\omega\}}$ resp. $\beta_0, \dots, \beta_h \in \mathrm{Bin}_{\{\omega\}}$ in their natural order.

Since $t(\alpha_i) = \beta_i$, it suffices to show (for each I) that

(∗) $$\langle d(\alpha_1), \dots, d(\alpha_h)\rangle = \langle \beta_1, \dots, \beta_h\rangle \mod (U+T)$$

The maps o, t read as

$$\begin{aligned} o\mu_r &= \mu_2(1 \otimes \mu_{r-1}) \\ t\mu_r &= \mu_2(\mu_{r-1} \otimes 1) \end{aligned} \qquad (r \geq 3)$$

Moreover

$$0 = R_n = d\mu_n + \mu_2(\mu_{n-1} \otimes 1) + \mu_2(\mu_{n-1} \otimes \varepsilon) + \sum_{k=3}^{n} \mu_k L_{k-1}(\mu_{n+1-k})$$

It follows that

$$d(\alpha_i) = \pm\beta_{i-1} \pm \beta_i \mod T \qquad (1 \leq i \leq h)$$

Since $\beta_0 \in M^+_{\{\omega\}} \subset U$, this shows (∗). □

By Lemma (5.1), U is a subcomplex of (A, d).

(5.3) Theorem. *$(U, d|U)$ is a strong deformation retract of (A, d).*

Proof: Since

$$d\Phi(u, v, w) = d(u) + d(w) = \Phi(d(u), w, 0)$$

one has

$$d' = \Phi^{-1} d\Phi = \begin{pmatrix} d|U & 0 & 0 \\ 0 & 0 & \mathrm{id}_T \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{End}(U \oplus T \oplus T)$$

Let

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{id}_T & 0 \end{pmatrix} \in \mathrm{End}(U \oplus T \oplus T)$$

Then

$$d'H + Hd' = \mathrm{id} - p_U$$

where p_U is the projection onto the first summand U.

Hence H is a homotopy contracting $(U \oplus T \oplus T, d')$ to (U, d), constant (=trivial) on (U, d). It follows that $\Phi H \Phi^{-1}$ a homotopy contracting (A, d) to (U, d), constant on (U, d). □

(5.4) Corollary. *The complexes A_r are acyclic:* $\ker d / \operatorname{im} d = R$ *generated by the fully parenthesized word*

$$(\omega(\omega \cdots (\omega(\omega\omega)) \cdots))$$

obtained by iterated multiplication with ω from the left.

Proof: Decomposing in Theorem 5.3 according to length shows that

$$A_r \xrightarrow{i^+} A_{r+1} \qquad (r \geq -1)$$

are homotopy equvalences. □

§6. Future plans

6.1. **Coalgebras and signed coderivations.** For the associahedral applications it seems that one should restrict right away to the non-counital tensor coalgebra T_+, the positive-dimensional part of the full tensor coalgebra $T_\bullet$.

The coderivations

$$B\colon T_+V \to T_+V$$

are naturally upper triangular matrices

$$\bigoplus_{n\geq 1} V^{\otimes n} \to \bigoplus_{n\geq 1} V^{\otimes n}$$

I found it illustrative to look also at the dual notion, derivations. Note that lower triangular matrices are maps

$$\prod_{n\geq 1} V^{\otimes n} \to \prod_{n\geq 1} V^{\otimes n}$$

A standard observation: A coderivation

$$B\colon T_+V \to T_+V$$

is determined by $\pi \circ B$ where $\pi\colon T_+V \to V$ is the projection to the first summand. Given $\pi \circ B$ one may write down B using the L_k (odd case) or L_k^+ (even case).

Let

$$\pi \circ B = (\mu_1, \mu_2, \mu_3, \ldots)$$

be as in section 2. Since B is an odd coderivation, B^2 is an even coderivation.

Lemma (2.3) is just the description of B^2 in terms of $\pi \circ B^2$.

Now comes the interesting (and new?) argument.

A_∞-algebras are defined by $B^2 = 0$ which is equivalent to

$$\pi \circ B^2 = (R_1, R_2, R_3, \ldots) = 0$$

Let

$$D\colon T_+V \to T_+V$$

be the coderivation determined by

$$\pi \circ D = (\mu_1, 0, 0, \ldots)$$

Then D is the diagonal part of B.

Consider the equation:

$$B^2 = D^2$$

This equation is a weaker version of $B^2 = 0$. It is equivalent to

$$\pi \circ (B^2 - D^2) = (0, R_2, R_3, \ldots) = 0$$

One has

$$0 = (B^2)B - B(B^2) = (D^2)B - B(D^2)$$

Applying π yields Corollary (2.2):

$$\mu_1^2 \mu_n = \mu_n L_{n-1}^+(\mu_1^2)$$

(The expression is so simple, since D^2 is diagonal.)

The equation $B^2 = 0$ can't be easily resolved, mainly because of its first member $\mu_1^2 = 0$. The condition $B^2 = D^2$ drops exactly that. The catch is that it still implies that μ_1^2 is a kind of even derivation with respect to all multifold products μ_n $(n \geq 2)$.

And that is enough to define μ_1 in the case of a free multi-algebra A_X, starting from $\mu_1|X$.

Making the coderivation framework precise takes some work. I have learned a lot from [6], [1], [3], but I think the coderivation stuff needs a further detailed exposition. Of course it is mostly about signs.

Dear reader: Please tell me if you know suitable references.

6.2. **Construct the cubical associahedron.** Corollary 4.2 and Theorem 5.3 suggest a way to construct the associahedron as a cube. I am optimistic that I can work this out, but maybe first I want to digest existing constructions. (Meanwhile I realized that such constructions are more or less known since the 1990's.)

Till then: Enjoy the drawings on my home page.

6.3. **Rooted plane trees.** Goals:

Define the associahedral chain complex in terms of trees, as considered in "Notes on the associator" [pdf].

Compare the three constructions (with coderivations, cubical, trees). Signs!

6.4. **Polygon triangulations.** This would be basically about [2]. I have done some work, but nothing ready yet.

6.5. **XXX.** XXX!

References

[1] E. Getzler and J. D. S. Jones, *A_∞-algebras and the cyclic bar complex*, Illinois J. Math. **34** (1990), no. 2, 256–283. MR 1046565 20

[2] C. W. Lee, *The associahedron and triangulations of the n-gon*, European J. Combin. **10** (1989), no. 6, 551–560. MR 1022776 20

[3] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. MR 2954392 20

[4] J.-P. Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980, Translated from the French by John Stillwell. MR 607504 14, 15

[5] ———, *Lie algebras and Lie groups*, Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 2006, 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. MR 2179691 3, 5

[6] J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 293–312, **1**08 (1963), 275-292; ibid. MR 158400 20

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