NOTES ON FREE ALTERNATIVE ALGEBRAS

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Summary

We compute the free alternative algebra up to degree 4.

Introduction

Let R be a ground ring (associative, commutative and unital) and let V be a R-module.

Consider the universal object for R-linear morphisms $V \to A$ to R-algebras of some type. The algebra types to be considered are associative commutative, associative non-commutative, non-associative, alternative, respectively. We assume unitality. (The prefix "non-" stands for "not required".)

In the first resp. second case the universal algebra is the symmetric resp. tensor algebra of V:

$$S^\bullet V = R \oplus V \oplus S^2V \oplus S^3V \oplus \cdots$$
$$T^\bullet V = R \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$

Next consider the case of non-associative algebras.

If S is a set and V is the free R-module with basis S, the universal object is the free R-algebra on S together with an extra term $1 \cdot R$ since we assume unitality. See Serre [6, Chap. IV, Free Lie Algebras, p. 18], Bourbaki [1, Chap. II, Algèbres de Lie libre].

More generally, for any R-module V there exists the universal R-linear morphism to (unital) non-associative R-algebras. (This is straightforward, but I don't know a reference.) The corresponding universal algebra looks as follows:

$$\begin{aligned}
M^\bullet V &= M^0V \oplus M^1V \oplus M^2V \oplus M^3V \oplus \cdots \\
&= R \oplus V \oplus [V \otimes V] \oplus M^3V \oplus \cdots \\
M^3V &= [(V \otimes V) \otimes V] \oplus [V \otimes (V \otimes V)] \\
M^4V &= [(V^{\otimes 2} \otimes V) \otimes V] \oplus [(V \otimes V^{\otimes 2}) \otimes V] \\
&\qquad \oplus [V \otimes (V^{\otimes 2} \otimes V)] \oplus [V \otimes (V \otimes V^{\otimes 2})] \\
&\qquad\qquad \oplus [V^{\otimes 2} \otimes V^{\otimes 2}] \\
M^nV &= [V^{\otimes n}]^{\oplus C_n}
\end{aligned}$$

where C_n is the number of parenthesized expressions of length n (the Catalan number) with first values $1, 1, 2, 5, 14, 42$. (The parentheses indicate the product in $M^\bullet$, the square brackets are added for readability.)

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The summands of M^nV $(n \geq 1)$ are parameterized by the elements of the free magma $X(\{*\})$ on one element,

$$*,\ **,\ (**)*,\ *(**),$$
$$((**)*)*,\ (*(**))*,\ *((**)*),\ *(*(**)),\ (**)(**),$$
$$(((**)*)*)*,\ \cdots$$

and one has

$$M^nV = \mathbf{Z}[X_n] \otimes V^{\otimes n}$$

where X_n is the subset of elements of length n of $X(\{*\})$.

Let

$$\sigma\colon X(\{*\}) \to \big(X(\{*\})\big)^{\mathrm{op}}$$

be the op-involution, the unique magma homomorphism with $* \mapsto *$. It can be described as a nested transpose. Examples are $*(**) \leftrightarrow (**)*$, $((**)*)* \leftrightarrow *(*(**))$.

Extended by the identity maps on $V^{\otimes n}$, it yields a module automorphism

$$\sigma\colon M^\bullet V \to M^\bullet V$$

We call this map the *paren involution* (or is there another name in the literature?). Thus the paren involution just permutes the C_n components of M^nV.

The op-involution on $M^\bullet$,

$$\iota\colon M^\bullet V \to (M^\bullet V)^{\mathrm{op}}$$

is the algebra homomorphism defined by the universal property extending the identity on $M^1V = V$. It commutes with σ and the composition $\tau = \sigma \circ \iota = \iota \circ \sigma$ acts on $\mathbf{Z}[X_n]$ as identity and on $V^{\otimes n}$ by the transpose

$$\tau_n\colon V^{\otimes n} \to V^{\otimes n}$$
$$\tau_n(x_1 \otimes \cdots \otimes x_n) = x_n \otimes \cdots \otimes x_1$$

(Passing to $T^\bullet V$, the paren involution becomes the identity and we are left with the remark that the op-involution on $T^\bullet V$ is given by the τ_n.)

Now we turn to alternative R-algebras. (A fitting reference for this text is [8, Chap. 13, Free Alternative Algebras, p. 258]). It is easy to guess a construction of the universal R-linear morphism $V \to A$ to (unital) alternative R-algebras. It is given by the quotient

$$B^\bullet V = M^\bullet V \,/\, \text{alternativity}$$

of $M^\bullet V$ by the alternative rules

$$(\alpha\alpha)\beta = \alpha(\alpha\beta)$$
$$(\alpha\beta)\beta = \alpha(\beta\beta)$$

for $\alpha, \beta \in M^\bullet V$. If we add the linearized alternative rules

$$(\alpha\gamma + \gamma\alpha)\beta = \alpha(\gamma\beta) + \gamma(\alpha\beta)$$
$$(\alpha\beta)\gamma + (\alpha\gamma)\beta = \alpha(\beta\gamma + \gamma\beta)$$

we may assume that α, β, γ are homogeneous, that is $\alpha \in M^aV$, $\beta \in M^bV$, $\gamma \in M^cV$ for some integers $a, b, c \geq 1$. It is therefore clear that $B^\bullet V$ inherits the grading and there is a natural decomposition

$$B^\bullet V = B^0V \oplus B^1V \oplus B^2V \oplus B^3V \oplus B^4V \oplus \cdots$$

with each B^nV a quotient of M^nV.

Evidently there are the obvious epimorphisms of R-algebras

$$M^\bullet V \to B^\bullet V \to T^\bullet V \to S^\bullet V$$

given by the respective strengthenings of algebraic structures.

The algebra $B^\bullet V$ inherits the paren involution, denoted by

$$\sigma\colon B^\bullet \to B^\bullet$$

as well.

In the very first degrees there are the bijections

$$\begin{aligned} M^{\le 1}V &= B^{\le 1}V = T^{\le 1}V = S^{\le 1}V \\ M^{\le 2}V &= B^{\le 2}V = T^{\le 2}V \end{aligned}$$

To compute further, let

$$\begin{aligned} M'^n V &= \ker(M^n V \to T^n V) \\ B'^n V &= \ker(B^n V \to T^n V) \\ K^n V &= \ker(M^n V \to B^n V) \end{aligned}$$

($K^n V \subset M^n V$ is given by homogeneous alternativity rules).

In other words, there is the commutative exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K^n V & = & K^n V & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & M'^n V & \longrightarrow & M^n V & \longrightarrow & T^n V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B'^n V & \longrightarrow & B^n V & \longrightarrow & T^n V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

The submodule $M'^n V$ of $M^n V$ is generated by the various n-linear expressions involving an associator. For instance

$$M'^3 V = V^{\otimes 3} \subset M^3 V = (V^{\otimes 2} \otimes V) \oplus (V \otimes V^{\otimes 2})$$

is generated by the associators

$$A(x, y, z) = \big((xy)z, -x(yz)\big)$$

Similarly, $M'^4 V$ is generated by expressions of the form $A(x, y, z)t$, $A(xy, z, t)$, etc.

Clearly $K^nV \subset M'^nV$—after all, alternativity is a condition on the associator (namely that it is alternating). The first non-trivial case is

$$K^3V \subset M'^3V = V^{\otimes 3}$$

which is generated by the 3-tensors xxy, xyy. It follows that $B'^3V = \Lambda^3V$ and one gets the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^{\otimes 3} & \longrightarrow & M^3V & \longrightarrow & T^3V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Lambda^3V & \longrightarrow & B^3V & \longrightarrow & T^3V & \longrightarrow & 0 \end{array}$$

This computation of B^3V shows that there exists a not-associative alternative algebra. Further, it encodes the definition of an alternative algebra A in the form that the associator is alternating, i.e., a map

$$\Lambda^3A \to A$$

Namely, if x_1, x_2, x_3 are elements in A and $B^\bullet V \to A$, $V = \langle e_1, e_2, e_3\rangle_R$ is the corresponding homomorphism, then

$$\Lambda^3V \subset B^3V \to A$$

maps $e_1 \wedge e_2 \wedge e_3$ to the associator $(x_1x_2)x_3 - x_1(x_2x_3)$.

So far things were simple enough. What happens in higher degrees? The main purpose of this text is to present a computation of B^4V for locally free V.

As for degrees ≥ 5, we don't know much. Is B^nV a locally free R-module for locally free R-modules V? (This question reduces to the case $R = \mathbf{Z}$, $V = \mathbf{Z}^N$.)

As for a computation of B^5V: One has to look at a quotient of $M^5V = [V^{\otimes 5}]^{\oplus 14}$ and an ad hoc computation quickly gets tiring. It seems one should first write down the chain complex of the 3-dimensional associahedron, if only to get all signs right. See [4] (Notes on the associator, April 2024, [pdf]) for a related discussion.

In the following we assume that V is locally free. The letter V will often be dropped from statements. In parts for brevity, but it should be noted anyway that the functors M, B, T, S are polynomial functors, and the morphisms we consider are morphisms of polynomial functors.

Here is the main result about B^4:

Proposition. *There is an isomorphism*

$$\Phi\colon B'^4 \to \big(T^1 \otimes \Lambda^3\big)^{\oplus 2}$$

Clearly, if $\operatorname{rank} V = 2$, it follows that $B'^4V = 0$. This is a reflection of Artin's theorem (an alternative algebra with 2 generators is associative).

For another illustration, let

$$\delta\colon \Lambda^4 \to T^1 \otimes \Lambda^3$$

$$\delta(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = \sum_i (-1)^{i-1} v_i \otimes \wedge\widehat{v_i}$$

be the standard map and put

$$f = -\Phi^{-1} \circ (\delta, \delta)\colon \Lambda^4 \to B'^4$$

By the universality of $B^{\bullet}V$, it follows that for any alternative algebra A there is a certain 4-alternating map

$$f\colon \Lambda^4 A \to A$$

This is the function f considered in Bruck and Kleinfeld 1951 [2, p. 880], see my text [5, Corollary 4, p. 5] (Notes on associator identities, May 2024, [pdf]) for details.

§1. The computation

1.1. The complex $C_\bullet$. The first step is to set up an exact sequence

$$0 \to C_2 \xrightarrow{R} C_1 \xrightarrow{A} C_0 \xrightarrow{\varepsilon} T^4 \to 0 \tag{1}$$

with the terms

$$\begin{aligned}
T^4V &= V^{\otimes 4}\\
C_0V = M^4V &= [V^{\otimes 4}]^{\oplus 5}\\
&= [(V^{\otimes 2} \otimes V) \otimes V] \oplus [(V \otimes V^{\otimes 2}) \otimes V]\\
&\quad \oplus [V \otimes (V^{\otimes 2} \otimes V)] \oplus [V \otimes (V \otimes V^{\otimes 2})]\\
&\quad\quad \oplus [V^{\otimes 2} \otimes V^{\otimes 2}]\\
C_1V &= [V^{\otimes 4}]^{\oplus 5}\\
&= [V^{\otimes 3} \otimes V] \oplus [V \otimes V^{\otimes 2} \otimes V] \oplus [V \otimes V^{\otimes 3}]\\
&\quad\quad \oplus [V \otimes V \otimes V^{\otimes 2}] \oplus [V^{\otimes 2} \otimes V \otimes V]\\
C_2V &= V^{\otimes 4}
\end{aligned}$$

Let C_5 be the cyclic group of order 5 with generator ζ and consider the standard exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\sum_i \zeta^i} \mathbf{Z}[C_5] \xrightarrow{1-\zeta} \mathbf{Z}[C_5] \xrightarrow{\varepsilon} \mathbf{Z} \to 0 \tag{2}$$

of C_5-modules.

The sequence (1) is defined exactly as (2) tensored (over $\mathbf{Z}$) with T^4: If one takes $1, \zeta, \zeta^2, \zeta^3, \zeta^4$ as basis for $\mathbf{Z}[C_5]$, then the maps in (1) have the same matrices as the maps in (2), with respect to the indicated decompositions of C_0 and C_1. For instance the map $A\colon C_1 \to C_2$ is on the first component given by

$$\begin{gathered}
[V^{\otimes 3} \otimes V] \to [(V^{\otimes 2} \otimes V) \otimes V] \oplus [(V \otimes V^{\otimes 2}) \otimes V] \oplus 0 \oplus 0 \oplus 0\\
(xyz)t \mapsto \big(((xy)z)t, -(x(yz))t, 0, 0, 0\big)
\end{gathered}$$

Moreover, R is just the diagonal and ε is the sum.

See also [5, Lemma 1, p. 4] (Notes on associator identities, May 2024, [pdf]). The sequence (2) is the augmented chain complex of the 2-dimensional associahedron (the pentagon). However we do not refer here to associahedra, but have set up (1) from scratch.

Since $M'^4 = \ker \varepsilon$, we have a resolution

$$0 \to C_2 \xrightarrow{R} C_1 \xrightarrow{A} M'^4 \to 0$$

In order to compute $B'^4 = M'^4/K^4$, we will describe a lift $\widetilde{K}^4$ (of an extension) of K^4 to C_1.

1.2. The module $X(V)$. But first we need some further notations. There is the exact sequence

$$0 \to S_3V \to S_2V \otimes V \oplus V \otimes S_2V \to V^{\otimes 3} \to \Lambda^3 V \to 0$$

where $S_kV = \big(V^{\otimes k}\big)^{\Sigma_k}$ denotes the module of symmetric tensors.

Put

$$X(V) = \ker(V^{\otimes 3} \to \Lambda^3 V) = \frac{S_2V \otimes V \oplus V \otimes S_2V}{(i, -i)(S_3V)}$$

1.3. **The submodule $\widetilde{K}^4V$.** Let

$$\widetilde{K}^4V \subset C_1V$$

be the submodule generated by

$$\big(X(V)\otimes V, 0, 0, 0, 0\big)$$
$$\big(0, 0, V\otimes X(V), 0, 0\big)$$
$$\big(0, x(yz)x, 0, 0, 0\big)$$
$$\big(0, x(yz)t, 0, xt(yz), 0\big)$$
$$\big(0, x(yz)t, 0, 0, (yz)xt\big)$$

Observe that $\widetilde{K}^4$ contains the elements

$$\big(0, 0, 0, xx(yz), 0\big)$$
$$\big(0, 0, 0, 0, (yz)xx\big)$$
$$\big(0, 0, 0, xy(zt), (zt)yx\big)$$

Using this, it not difficult to check that A maps $\widetilde{K}^4$ onto K^4. Hint: Spelling out the homogeneous alternativity rules of degree 4 yields an epimorphism

$$(X(V)\otimes V)^{\oplus 2} \oplus (S_2\otimes V^{\otimes 2})^{\oplus 3} \oplus (V^{\otimes 4})^{\oplus 3} \to K^4$$

Now eliminate some redundant terms.

Hence there is an exact sequence

$$C_2 \oplus \widetilde{K}^4 \to C_1 \to B'^4 \to 0$$

1.4. **More notations.** We abbreviate

$$P = T^1 \otimes \Lambda^3$$

Let

$$\mu\colon P \to \Lambda^4$$

be the multiplication in the exterior algebra and let

$$p\colon T^4 \to \Lambda^4$$
$$p(x_1x_2x_3x_4) = x_1\wedge x_2\wedge x_3\wedge x_4$$

be the projection.

Put

$$\rho_i, \bar\rho\colon T^4 \to P$$
$$\rho_1(x_1x_2x_3x_4) = +x_1\otimes(x_2\wedge x_3\wedge x_4)$$
$$\rho_2(x_1x_2x_3x_4) = -x_2\otimes(x_1\wedge x_3\wedge x_4)$$
$$\rho_3(x_1x_2x_3x_4) = +x_3\otimes(x_1\wedge x_2\wedge x_4)$$
$$\rho_4(x_1x_2x_3x_4) = -x_4\otimes(x_1\wedge x_2\wedge x_3)$$
$$\bar\rho = \rho_1+\rho_2+\rho_3+\rho_4$$

The signs ensure that the $\mu\circ\rho_i$ are all equal to the projection p:

$$\mu\circ\rho_i = p \qquad (i = 1, 2, 3, 4)$$

Moreover, $\bar\rho$ is alternating and factors as

$$\bar\rho\colon T^4 \xrightarrow{p} \Lambda^4 \xrightarrow{\delta} P$$

where δ is the standard map (the natural inclusion via $\Lambda^n V \subset V^{\otimes n}$).

1.5. **The map Φ.** Define

$$\widehat\Phi\colon C_1 \to P \oplus P$$
$$\widehat\Phi = \begin{pmatrix} \rho_4 & \rho_2 & 0-\bar\rho & \rho_3 & \rho_1 \\ 0 & \rho_3 & \rho_1-\bar\rho & \rho_4 & \rho_2 \end{pmatrix}$$

where the matrix notation corresponds to the definition of C_1 above, copied here for convenience:

$$\begin{aligned} C_1V = [V^{\otimes 3} \otimes V] \oplus [V \otimes V^{\otimes 2} \otimes V] \oplus [V \otimes V^{\otimes 3}] \\ \oplus [V \otimes V \otimes V^{\otimes 2}] \oplus [V^{\otimes 2} \otimes V \otimes V] \end{aligned}$$

Note that $\widehat\Phi$ is an epimorphism (already the first 2 columns are epimorphic as the ρ_i are epimorphisms).

Clearly $\widehat\Phi$ vanishes on the image of C_2 (the row sums are trivial).

It vanishes also on $\widetilde K^4$: Namely, ρ_4 vanishes obviously on $X \otimes T^1 = \ker\rho_4$. Similarly, ρ_1 vanishes on $T^1 \otimes X = \ker\rho_1$. Hence $\bar\rho = \delta\mu\rho_1$ vanishes as well on $T^1 \otimes X$. Next note that ρ_2, ρ_3 are alternating in x_1, x_4 and so vanish on the elements $x(yz)x$. Finally, the elements

$$\begin{gathered} \big(0, x(yz)t, 0, xt(yz), 0\big) \\ \big(0, x(yz)t, 0, 0, (yz)xt\big) \end{gathered}$$

map to

$$\begin{pmatrix} -y \otimes xzt + y \otimes xzt \\ +z \otimes xyt - z \otimes xyt \end{pmatrix} = 0, \qquad \begin{pmatrix} -y \otimes xzt + y \otimes xzt \\ +z \otimes xyt - z \otimes xyt \end{pmatrix} = 0$$

respectively.

Hence $\widehat\Phi$ factors through B'^4, inducing an epimorphism

$$\Phi\colon B'^4 \to P \oplus P$$

Here is an explicit description of Φ in terms of the generators of M'^4 (the associators):

$$\begin{aligned} A(x,y,z)t &\mapsto \begin{pmatrix} -t \otimes xyz \\ 0 \end{pmatrix} \\ A(x,yz,t) &\mapsto \begin{pmatrix} -y \otimes xzt \\ z \otimes xyt \end{pmatrix} \\ xA(y,z,t) &\mapsto \begin{pmatrix} -x \otimes yzt + y \otimes xzt - z \otimes xyt + t \otimes xyz \\ y \otimes xzt - z \otimes xyt + t \otimes xyz \end{pmatrix} \\ -A(x,y,zt) &\mapsto \begin{pmatrix} z \otimes xyt \\ -t \otimes xyz \end{pmatrix} \\ -A(xy,z,t) &\mapsto \begin{pmatrix} x \otimes yzt \\ -y \otimes xzt \end{pmatrix} \end{aligned}$$

1.6. **Injectivity of Φ.** One defines a left inverse to Φ. Consider

$$\begin{aligned}\Psi\colon P \oplus P &\to B'^4\\ (t \otimes xyz, 0) &\mapsto -A(x,y,z)t\\ (0, z \otimes xyt) &\mapsto A(x,yz,t) - A(x,z,t)y\end{aligned}$$

We first show that Ψ is well defined. For the first component this is obvious as A is alternating. For the second component one needs additionally that the right hand side vanishes for $t = y$:

$$A(x,yz,y) = A(x,z,y)y$$

This is a basic relation in alternative algebras and can be shown in one way or the other. We refer here to [5, Corollary 5, (2.3), p. 5]. As mentioned there, it appears in Zorn [9, p. 142] and was used by Moufang [3, p. 419] to derive what are called the Moufang identities.

One finds $\Psi \circ \Phi = \mathrm{id}$ by the formulas just formulated.

It remains to show that Ψ is surjective. It catches the first 2 summands of C_1. The 3rd summand of C_1 can be eliminated using $R(C_2)$ and the 4th and 5th summands can be reduced to the 2nd using $\widetilde{K}^4$.

Remark. In a future version of this text I might start out with Ψ (including a proof that it is well defined) and then establish its inverse Φ.

§2. More considerations

2.1. **A resolution of B'^4.** Consider

$$\overline{C}_1 = C_1/\widetilde{K}^4$$

There is the induced exact sequence

$$C_2 \xrightarrow{\overline{R}} \overline{C}_1 \to B'^4 \to 0$$

One may compute the kernel of $\overline{R}$. This needs some further work. In the end one gets the following:

Let

$$\begin{aligned}\eta\colon T \otimes S_2 \otimes T &\to T^4\\ \eta(x \otimes yy \otimes t) &= (xy+yx)(yt+ty) - xyyt\\ &= xyty + yxyt + yxty\end{aligned}$$

PropositionXL. *There exists a natural exact sequence*

$$\begin{aligned}0 \to S_4 \xrightarrow{(3,i)} S_4 \oplus \big(T \otimes S_2 \otimes T\big) &\xrightarrow{(i,-\eta)}\\ T^4 \xrightarrow{\overline{R}'} (\Lambda^3 \otimes T^1) \oplus (T^1 &\otimes \Lambda^2 \otimes T^1) \oplus (T^1 \otimes \Lambda^3)\\ &\xrightarrow{\Phi'} \big(\Lambda^3 \otimes T^1\big) \oplus \big(T^1 \otimes \Lambda^3\big) \to 0\end{aligned}$$

with

$$B'^4 = \operatorname{coker} \overline{R}'$$

On the way the following exact sequence is useful:

$$0 \to S_4 \to S_2 \otimes S_2 \to T^1 \otimes \Lambda^2 \otimes T^1 \to (\Lambda^3 \otimes T^1) \oplus (T^1 \otimes \Lambda^3) \to \Lambda^4 \to 0$$

Here the components are the obvious maps, decorated with signs. Exactness follows for instance from the exactness of the Koszul complexes $(S_i \otimes \Lambda^{N-i})_i$ for $N \leq 4$.

2.2. **The Kleinfeld function.** As for the Kleinfeld function

$$\begin{aligned} f\colon \Lambda^4 V &\to B'^4 V \subset B^4 V \\ f(x,z,y,t) &= A(x,yz,t) - A(x,z,t)y - zA(x,y,t) \\ &= A(x,z,[y,t]) + A([x,z],y,t) \end{aligned}$$

see [5, Corollary 4, p. 5] and also [8, Chap. 7, Simple alternative algebras, p. 139]: one finds indeed

$$\Phi \circ f = (0,-\delta,-\delta)$$

2.3. **Endomorphisms.** The endomorphism algebra of $P = T^1 \otimes \Lambda^3$ as a polynomial functor over $\mathbf{Z}$ is

$$\operatorname{End}(P) = \mathbf{Z}[\alpha]/(\alpha^2 - 4\alpha)$$

with $\alpha = \delta \circ \mu$.

An interesting involution of $P^{\oplus 2}$ is

$$\omega = \begin{pmatrix} -\alpha & 1+\alpha \\ 1-\alpha & \alpha \end{pmatrix} \in \mathrm{GL}_2\big(\operatorname{End}(P)\big) = \operatorname{Aut}(P \oplus P)$$

It is related with the paren involution on B^4.

One finds (τ is the switch involution)

$$\omega \circ \widehat{\Phi} = \tau \circ \widehat{\Phi}'$$

with

$$\widehat{\Phi}' = \begin{pmatrix} \rho_4 - \bar\rho & \rho_2 & 0 & \rho_3 & \rho_1 \\ 0 - \bar\rho & \rho_3 & \rho_1 & \rho_4 & \rho_2 \end{pmatrix}$$

or

$$\widehat{\Phi}' - \widehat{\Phi} = \begin{pmatrix} -\bar\rho & 0 & \bar\rho & 0 & 0 \\ -\bar\rho & 0 & \bar\rho & 0 & 0 \end{pmatrix}$$

2.4. **A variant of Φ.** The formulas for $\widehat{\Phi}$, $\widehat{\Phi}'$ have an apparent asymmetry because of the $\bar\rho$-column (which is essentially the Kleinfeld function). As we have just seen, using ω one can move the $\bar\rho$-column to the other possible slot.

An alternative is the following: There is the split exact sequence

$$\begin{gathered} 0 \to P \oplus P \xrightarrow{i} \Lambda^4 \oplus P \oplus P \xrightarrow{\pi} \Lambda^4 \to 0 \\ 0 \to P \oplus P \xleftarrow{j} \Lambda^4 \oplus P \oplus P \xleftarrow{s} \Lambda^4 \to 0 \\ \pi(\eta,\beta,\gamma) = 3\eta + \mu(\beta) + \mu(\gamma) \\ s(\eta) = \big(-\eta, \delta(\eta), 0\big) \\ j(\eta,\beta,\gamma) = \big(\beta + \delta(\eta), -\gamma\big) \\ i(\beta,\gamma) = \big(\mu(\beta-\gamma), \beta - \delta\mu(\beta-\gamma), -\gamma\big) \end{gathered}$$

Note the factor 3 in the definition of $\pi = (3,\mu,\mu)$.

For the composition with $\widehat{\Phi}$ one finds

$$i \circ \widehat{\Phi} = \begin{pmatrix} p & 0 & -p & 0 & 0 \\ \rho_4 - \bar{\rho} & +\rho_2 & 0 & +\rho_3 & +\rho_1 \\ 0 & -\rho_3 & -\rho_1 + \bar{\rho} & -\rho_4 & -\rho_2 \end{pmatrix}$$

which has a slightly more symmetric form (the extra term $\bar{\rho}$ appears in both columns).

Under this map, the paren involution corresponds to

$$(\eta, \beta, \gamma) \mapsto (-\eta, -\gamma, -\beta)$$

Remark. The resulting exact sequence

$$0 \to B'^4 \xrightarrow{i \circ \Phi} \Lambda^4 \oplus P \oplus P \xrightarrow{(3,\mu,\mu)} \Lambda^4 \to 0$$

has the shape of our first computation of B'^4. It somehow appeared naturally.

After eliminating the Λ^4-terms using the section s (and some sign changes) we obtained the formula for $\widehat{\Phi}$. The latter is perhaps a bit more transparent and seems to be more convenient for proofs. However it breaks a symmetry caused by the choice of the section s. The other obvious choice $s(\eta) = (-\eta, 0, \delta(\eta))$ gives rise to $\widehat{\Phi}'$.

Remark. Finally let us note that

$$T^4 \to B^4$$
$$xyzt \mapsto (xy)(zt)$$

is a section to $B^4 \to T^4$ which invariant under the paren involution.

This section is a particular feature in degree 4, as $(**)(**)$ is the only fixed point of the paren involution acting on X_4.

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