ON PERMUTATION BRAIDS

MARKUS ROST

`preliminary version`

Contents

Date: August 7, 2026.

You are looking at the text "On permutation braids" [pdf].

Introduction

Let

$$p\colon B_n \to S_n$$

be the obvious epimorphism from the braid group B_n to the symmetric group S_n.

The purpose of this text to establish the "permutation braid" section

$$\mathcal{S}\colon S_n \to B_n$$

The image of $\mathcal{S}$ is known as the set of *permutation braids*. So for any permutation there is a canonical choice of a braid inducing the given permutation. Needless to say that $\mathcal{S}$ is not a homomorphism.

Until July 2026 I was not aware of the notion of "permutation braids" and the map $\mathcal{S}$. When looking for references I first found Birman, Ko and Lee 1998 [1, 1. Introduction, p. 324] and then El-Rifai and Morton 1994 [2, Lemma 2.3, p. 484]. In fact, Lemma 2.3 in [2] says it all. I fully agree with the following subsequent remark in [2]:

> The great benefit of this lemma is however that it is quite unnecessary to remember these braids as braid words; the permutation is quite enough, and makes for much greater ease in comparing such braids.

Dear reader: I would be grateful for more references.

Besides the geometric description in [2, Lemma 2.3, p. 484] I know of the following characterizations of the map $\mathcal{S}$:

$$\mathcal{S}(tt_i) = \mathcal{S}(t)s_i^{[t,x_i]} \qquad (t \in S_n,\ 1 \le i \le n-1) \tag{0.1}$$

$$\mathcal{S}(t_it) = s_i^{[t^{-1},x_i]}\mathcal{S}(t) \qquad (t \in S_n,\ 1 \le i \le n-1) \tag{0.2}$$

$$\mathcal{S}(t_{i_1}\cdots t_{i_r}) = s_{i_1}\cdots s_{i_r} \qquad t_{i_1}\cdots t_{i_r} \text{ a shortest word} \tag{0.3}$$

Here $s_i \in B_n$ are the standard generators, $t_i = \tau_{i,i+1} \in S_n$ is the corresponding transposition and

$$[t, x_i] = \begin{cases} +1 & t \text{ is order preserving on } x_i = \{i, i+1\} \\ -1 & t \text{ is order reversing on } x_i = \{i, i+1\} \end{cases}$$

Description (0.3) is more or less immediate from the description in [2]. (I hope to provide details in a later extension of this text).

Description (0.1) is perhaps the quickest way to write down a characterization of the map $\mathcal{S}$. In this text we establish the map $\mathcal{S}$ based on (0.1). See Theorem (1.7).

Description (0.3) follows the easily. (Again: I hope to provide details in a later extension of this text).

One could equally use the variant (0.2) to define $\mathcal{S}$. However, surprisingly, the equivalence of (0.1) and (0.2) seems not to be immediate in terms of generators and relations (at least I don't know a quick argument). One the other hand, it is not difficult to derive (0.1) and (0.2) from (0.3).

Currently this text has two parts.

The first part (§1) defines $\mathcal{S}$ purely in terms of generators and relations (the way of "combinatorial group theory"). That got much longer than I thought at the beginning. However all the steps look very natural to me. Maybe one can streamline some parts to get a shorter exposition.

The second part (§2) describes how I "met" the section $\mathcal{S}$. Not sure whether it is worthwile to present all details in a future version of this text.

Reminder to the reader: I would be grateful for references.

§1. The section $S_n \to B_n$

1.1. **Inversions (Fehlstellungen).** Let $n \geq 1$ be an integer and put

$$[n] = \{1, \dots, n\}$$

The group of permutations of $[n]$ (the symmetric group) is denoted by Σ_n.

For 2-element subsets of $[n]$ we use the notations

$$X = \{\, x_i \mid 1 \leq i \leq n-1 \,\} \subset Y = \binom{[n]}{2} = \{\, y \subset [n] \mid |y| = 2 \,\}$$

where

$$x_i = \{i, i+1\} \qquad (1 \leq i \leq n-1)$$

For $y \in Y$ let $\tau_y \in \Sigma_n$ be the transposition at y (swapping the 2 elements of y and leaving $k \in [n] \setminus y$ fixed). As a special case, we write

$$\tau_i = \tau_{x_i} \in \Sigma_n \qquad (1 \leq i \leq n-1)$$

Thus $\tau_i = (i\ i{+}1)$ in a common notation for the symmetric group.

For $\alpha \in \Sigma_n$ and $y \in Y$ define the integer

$$[\alpha, y] = \begin{cases} +1 & \alpha|y \text{ is order preserving} \\ -1 & \alpha|y \text{ is order reversing} \end{cases}$$

Hence $[\alpha, y] = -1$ if and only if y is an inversion of α. Obviously $[\alpha, y]$ is invariant under order reversing of the underlying set $[n]$.

Let

$$\rho \in \Sigma_n, \qquad \rho(k) = n + 1 - k$$

be the unique order reversing bijection.

The following formulas are immediate:

$$\textbf{(1.1)} \qquad [\rho, y] = -1$$

$$\textbf{(1.2)} \qquad [\alpha\beta, y] = [\alpha, \beta(y)][\beta, y]$$

$$\textbf{(1.3)} \qquad [\alpha\tau_y, y] = -[\alpha, y]$$

$$\textbf{(1.4)} \qquad [\tau_i, y] = 1 \qquad (y \neq x_i)$$

Let $1 \leq i, j \leq n-1$ with $|i-j| = 1$ which means $|x_i \cap x_j| = 1$. Then $M = x_i \cup x_j$ consists of 3 consecutive elements. The symmetric group $\Sigma_M \simeq \Sigma_3$ has exactly 3 transpositions: τ_i, τ_j and τ_y where $y = M \setminus (x_i \cap x_j)$. Thus $\tau_i\tau_j = \tau_y\tau_i$. Since τ_y is also the order reversing permutation of M it follows that

$$\textbf{(1.5)} \qquad [\tau_i\tau_j, x_i] = 1 \qquad (|i-j| = 1)$$

1.2. **The groups B_n, S_n, F_n.** Define the following groups by generators and relations:

$$B_n = \langle s_i \mid s_i s_{i+1} s_i (s_{i+1} s_i s_{i+1})^{-1},\ s_i s_j (s_j s_i)^{-1} \rangle$$
$$S_n = \langle t_i \mid t_i t_{i+1} t_i (t_{i+1} t_i t_{i+1})^{-1},\ t_i t_j (t_j t_i)^{-1},\ t_i^2 \rangle$$
$$F_n = \langle u_i \mid u_i^2 \rangle$$

Here the generators are indexed by

$$i \in \{1, \dots, n-1\}$$

and the three types of relations by

$$\begin{aligned} i &\in \{1,\dots,n-2\} \\ i,j &\in \{1,\dots,n-1\},\ |i-j|\geq 2 \\ i &\in \{1,\dots,n-1\} \end{aligned}$$

respectively.

Thus B_n is the braid group on n strings and S_n is the permutation group Σ_n in their usual presentations. The group F_n is the free product of $n-1$ copies of the group of order 2.

There are the evident surjective group homomorphisms

$$\begin{aligned} p\colon B_n &\to S_n, \qquad p(s_i)= t_i \\ q\colon F_n &\to S_n, \qquad q(u_i)= t_i \end{aligned}$$

The letter π denotes the obvious homomorphisms from the three groups to Σ_n:

$$\begin{aligned} \pi\colon B_n, S_n, F_n &\to \Sigma_n \\ \pi(s_i) = \pi(t_i) &= \pi(u_i) = \tau_i \end{aligned}$$

(In the following we do need not the fact that $\pi\colon S_n \to \Sigma_n$ is an isomorphism.)

Further let

$$J\colon B_n \to B_n, \qquad J(s_i) = s_i^{-1}$$

be the standard involutive automorphism of B_n.

1.3. **Reduced words in F_n.** Every element $w \in F_n$ is represented uniquely by a reduced word in the u_i: $w = u_{i_1}\cdots u_{i_r}$ $(i_{k+1} \neq i_k)$. The integer $\mathrm{len}(w) = r$ is called the length of w. Moreover one says that w starts in u_{i_1} and ends in u_{i_r} $(w \neq 1)$.

Let $F_n^{(i)} \subset F_n$ be the subset of elements $w \in F_n$ which end in u_i. Then

$$F_n = \{1\} \amalg \coprod_i F_n^{(i)} \qquad \text{(disjoint union)} \tag{1.6}$$

1.4. **The section $S_n \to B_n$.**

(1.7) Theorem. *There exists a map*

$$\mathcal{S}\colon S_n \to B_n$$

with $\mathcal{S}(1) = 1$ *and*

$$\mathcal{S}(tt_i) = S(t)s_i^{[\pi(t),x_i]} \qquad (t \in S_n,\ 1 \leq i \leq n-1) \tag{1.8}$$

Since the t_i generate S_n, it is clear that $\mathcal{S}$ is unique if it exists. By the same reason and $t_i^2 = 1$ it follows that the map $\mathcal{S}$ is a set-theoretic section (not a group homomorphism) to p: $p(\mathcal{S}(t)) = t$ for $t \in S_n$.

The proof of Theorem (1.7) takes up the rest of this section.

(1.9) Lemma. *There exists a unique map*

$$\Psi\colon \Sigma_n \times F_n \to B_n$$

with $\Psi(\alpha, 1) = 1$ *and*

$$\Psi(\alpha, wu_i) = \Psi(\alpha, w)s_i^{[\alpha\pi(w),x_i]} \tag{1.10}$$

for $\alpha \in \Sigma_n$, $w \in F_n$ *and* $1 \leq i \leq n-1$.

Proof: Uniqueness is obvious since the u_i generate F_n.

Define Ψ using the decomposition (1.6) by induction on the length of w by $\Psi(\alpha, 1) = 1$ and for $w \in F_n^{(i)}$ by

$$\Psi(\alpha, w) = \Psi(\alpha, wu_i) s_i^{-[\alpha\pi(w), x_i]} \tag{1.10′}$$

(as $u_i^2 = 1$ one has $\mathrm{len}(wu_i) = \mathrm{len}(w) - 1$).

Formula (1.10′) is a reformulation of (1.10), since $u_i^2 = 1$ and by (1.2):

$$[\alpha\pi(wu_i), x_i] = [\alpha\pi(w)\tau_i, x_i] = -[\alpha\pi(w), x_i]$$

It remains to check that for a given i formula (1.10′) holds for $w \notin F_n^{(i)}$. But then $wu_i \in F_n^{(i)}$ and (1.10′) applied to wu_i yields indeed

$$\Psi(\alpha, wu_i) = \Psi(\alpha, wu_iu_i) s_i^{-[\alpha\pi(wu_i), x_i]} = \Psi(\alpha, w) s_i^{[\alpha\pi(w), x_i]} \qquad \square$$

(1.11) Lemma. *One has*

$$\Psi(\alpha, vw) = \Psi(\alpha, v)\Psi(\alpha\pi(v), w) \tag{1.12}$$

for $\alpha \in \Sigma_n$, v, $w \in F_n$. *Moreover,*

$$\Psi(\rho\alpha, w) = J(\Psi(\alpha, w)) \tag{1.13}$$

where $\rho \in \Sigma_n$ *is the order reversing bijection and* J *the standard involution of* B_n.

Proof: Use (1.10) and induction on the length of w. $\square$

(1.14) Lemma.

$$\Psi(1, u_i) = s_i \tag{1.15}$$

$$\Psi(1, u_iu_j) = s_is_j \qquad (i \neq j) \tag{1.16}$$

$$\Psi(1, u_iu_ju_i) = s_is_js_i \qquad (|i - j| = 1) \tag{1.17}$$

Proof: (1.15) is obvious from the definition. For the other claims one uses (1.4) resp. (1.5):

$$\Psi(1, u_iu_j) = \Psi(1, u_i)\Psi(\tau_i, u_j) = s_i s_j^{[\tau_i, x_j]}$$
$$\Psi(1, u_iu_ju_i) = \Psi(1, u_iu_j)\Psi(\tau_i\tau_j, u_i) = s_is_js_i^{[\tau_i\tau_j, x_i]} \qquad \square$$

(1.18) Corollary.

$$\Psi(1, (u_iu_ju_i)^2) = 1 \qquad (|i - j| = 1)$$
$$\Psi(1, (u_iu_j)^2) = 1 \qquad (|i - j| \leq 2)$$

Proof: For the first claim one may restrict to the case $n = 3$. Then $\rho = \tau_i\tau_j\tau_i$ is order reversing and we can apply (1.13):

$$\begin{aligned}\Psi(1, (u_iu_ju_i)^2) &= \Psi(1, u_iu_ju_i)\Psi(\rho, u_iu_ju_i)\\ &= s_is_js_is_i^{-1}s_j^{-1}s_i^{-1} = 1\end{aligned}$$

For the second claim one can argue similarly (or compute directly using $\tau_i\tau_j = \tau_j\tau_i$):

$$\begin{aligned}\Psi(1, (u_iu_j)^2) &= \Psi(1, u_iu_j)\Psi(\tau_i\tau_j, u_iu_j)\\ &= s_is_js_i^{-1}s_j^{-1} = 1\end{aligned}$$

$\square$

Let

$$K \subset \{1, \dots, n-1\}$$

be a subset and put

$$\overline{K} = \bigcup_{k \in K} x_k = K \cup (K+1) \subset [n]$$

Consider the graph Γ_K with $[n]$ as set of vertices and x_k $(k \in K)$ as edges. The graph Γ_K is a disjoint of graphs

$$\Gamma_K = \coprod_t \Gamma_t$$

which gives rise to a decomposition

$$[n] = \coprod_t K_t$$

This is the finest decomposition such that each x_k $(k \in K)$ is contained in some K_t. The elements of $[n] \setminus \overline{K}$ form exactly the K_t with $|K_t| = 1$ (the isolated points of the graph Γ_K).

(1.19) Lemma. *Let*

$$K \subset \{1, \dots, n-1\}$$

be a subset and let $w \in F_n$ be in the subgroup generated by u_k $(k \in K)$.

If $\alpha \in \Sigma_n$ is order preserving on each K_t, then

$$\Psi(\alpha, w) = \Psi(1, w) \tag{1.20}$$

Proof: Again use (1.10) and induction on the length of w. □

(1.21) Corollary. *There exists a unique map*

$$\Phi \colon \Sigma_n \times S_n \to B_n$$

with

$$\Phi(\alpha, q(w)) = \Psi(\alpha, w)$$

Proof: The claim amounts to

$$\Psi(\alpha, vw) = \Psi(\alpha, w)$$

for $v \in \ker q$. By (1.12) and $\pi(\ker q) = 1$ one has

$$\Psi(\alpha, vw) = \Psi(\alpha, v)\Psi(\alpha, w)$$

Thus it suffices to show

$$\Psi(\alpha, v) = 1 \tag{1.22}$$

with v from a set of group generators of $\ker q$.

Again by (1.12) one has

$$\Psi(\alpha, wvw^{-1}) = \Psi(\alpha, w)\Psi(\alpha\pi(w), v)\Psi(\alpha, w^{-1}) \tag{1.23}$$

for $v \in \ker q$. Hence it suffices to show (1.22) with v from a set of normal generators of $\ker q$ for which we take the elements $r = (u_i u_j)^3$ $(|i-j| = 1)$, $r = (u_i u_j)^2$ $(|i-j| \geq 2)$.

Given such a pair i, j and $r = (u_i u_j)^m$, write $\alpha = \alpha_0 \pi(w)$ with $w \in F_n$ in the subgroup U_{ij} generated by the two elements u_i, u_j and with α_0 order preserving on $x_i \cup x_j$ (in the case $m = 3$) resp. order preserving on x_i and on x_j (in the case $m = 2$).

Again by (1.23) it remains to show

$$\Psi(\alpha_0, wrw^{-1}) = 1$$

By Lemma (1.19) one may assume $\alpha_0 = 1$.

And again by (1.23) it suffices to show

$$\Psi(1, w(u_iu_j)^m w^{-1}) = 1 \qquad (w \in U_{ij})$$

However the conjugates of $(u_iu_j)^m$ in U_{ij} are just $(u_iu_j)^m$, $(u_ju_i)^m$. The claim follows from Corollary (1.18). □

Proof of Theorem (1.7): Take $\mathcal{S}(t) = \Phi(1, t)$. □

§2. Abelian version

Here is the way I stumbled on the map $\mathcal{S}$.

Let

$$P_n = \ker(B_n \xrightarrow{p} S_n)$$

be the subgroup of pure braids. The abelianization

$$\overline{P}_n = P_n/[P_n, P_n]$$

can be identified with the S_n-module

$$T = \mathbf{Z}[\binom{[n]}{2}] = \mathbf{Z}[Y]$$

having the 2-element subsets of $\{1, \dots, n\}$ as $\mathbf{Z}$-basis (see Kassel and Turaev 2008 [3, p. 19, Corollary 1.20, p. 21] describing the generators A_{ij} of P_n). Writing

$$\overline{B}_n = B_n/[P_n, P_n]$$

one gets an extension

$$0 \mapsto T \to \overline{B}_n \to S_n \to 1 \tag{2.1}$$

We are now in the world of cohomology: The extension (2.1) determines an element

$$\eta \in H^2(S_n, T)$$

The section $\mathcal{S}$ taken mod $[P_n, P_n]$ yields the 2-cocycle

$$f\colon S_n \times S_n \to P_n/[P_n, P_n] = T = \mathbf{Z}[Y]$$
$$f(g_1, g_2) = \mathcal{S}(g_1 g_2)^{-1}\mathcal{S}(g_1)\mathcal{S}(g_2) \mod [P_n, P_n]$$

(The standard 2-cocycle would be given by $\mathcal{S}(g_1)\mathcal{S}(g_2)\mathcal{S}(g_1 g_2)^{-1}$, but our form yields nicer formulas.)

Let

$$\ell\colon S_n \times Y \to \mathbf{Z}$$
$$\ell(g, y) = \begin{cases} 0 & g|y \text{ is order preserving} \\ 1 & g|y \text{ is order reversing} \end{cases}$$

Thus

$$[g, y] = (-1)^{\ell(g,y)}$$

in the earlier notation. Note that ℓ takes integral values (not in $\mathbf{Z}/2\mathbf{Z}$).

(2.2) Proposition. *One has*

$$f(g_1, g_2) = \sum_{y \in Y} \ell(g_1, g_2(y))\ell(g_2, y)y \tag{2.3}$$

Proof: No proof yet, but see below. □

(2.4) Example. Take $g_2 = \tau_x$ for some $x = x_i \in X$. Since

$$\ell(\tau_x, y) = \begin{cases} 0 & y \neq x \\ 1 & y = x \end{cases}$$

one gets

$$f(g, \tau_x) = \ell(g, \tau_x(x))x = \ell(g, x)x$$

in accordance with

$$\mathcal{S}(g\tau_x)^{-1}\mathcal{S}(g)\mathcal{S}(\tau_x) = (\mathcal{S}(g)s_i^{[g,x_i]})^{-1}\mathcal{S}(g)s_i = s_i^{-[g,x_i]+1} = (s_i^2)^{\ell(g,x_i)}$$

and the fact that x_i is in $P_n/[P_n,P_n]$ represented by $A_{ii} = s_i^2$.

2.1. **More on the class η.** One has

$$Y = (S_n/(S_2 \times S_{n-2})) \cdot x_1$$

and therefore

$$H^2(S_n, T) = H^2(S_2 \times S_{n-2}, \mathbf{Z})$$

by Shapiros Lemma. Thus η is given by an element

$$\eta' \in H^2(S_2 \times S_{n-2}, \mathbf{Z})$$

Shapiros Lemma is given by restriction and projection:

$$H^2(G, \mathbf{Z}[G/H]) \to H^2(H, \mathbf{Z}[G/H]) \to H^2(H, \mathbf{Z})$$

It easy to see that restriction of η' in $H^2(S_{n-2}, \mathbf{Z})$ is trivial. Therefore η' is the pullback of its restriction $\eta'' \in H^2(S_2, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ to S_2. This is the class of the non-trivial extension

$$0 \to \mathbf{Z} \to B_2 \to S_2 \to 1$$

Now one can undergo the exercise and construct an explicit 2-cocycle representing η starting from an explicit 2-cocycle representing η''. For the latter one takes the 2-cocycle relative to the section $S_2 \to B_2$, $t_1 \mapsto s_1$. Along the way one has to choose a set of coset representatives for $Y = S_n/(S_2 \times S_{n-2})$ for which one takes

$$1, \quad \tau_{1i}, \tau_{2i}\,(2 < i), \quad \tau_{1i}\tau_{2j}\,(2 < i < j)$$

The resulting 2-cocycle on S_n is a bit complicated when restricted to S_2 (one gets the trace of η'') but takes a simpler form after restriction to S_{n-2}. After replacing n with $n+2$ one gets the 2-cocycle f described in Proposition (2.2).

Thus the section $\mathcal{S}$ mod $[P_n,P_n]$ can be deduced from the case $n = 2$. It came as a surprise to me that this section actually lifts to a nice section $S_n \to B_n$.

References

[1] J. Birman, K. H. Ko, and S. J. Lee, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. **139** (1998), no. 2, 322–353. MR 1654165 2

[2] E. A. El-Rifai and H. R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 479–497. MR 1315459 2

[3] C. Kassel and V. Turaev, *Braid groups*, Graduate Texts in Mathematics, vol. 247, Springer, New York, 2008, With the graphical assistance of Olivier Dodane. MR 2435235 9

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

Email address: rost *at* math.uni-bielefeld.de

URL: www.math.uni-bielefeld.de/~rost