Some new results on the Chowgroups of quadrics

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§ 1 Introduction

The computation of some K-cohomology groups of certain normvarieties plays an essential role in the investigation of the bijectivity of the Galoissymbol

$$K_n^M F/p \to H^n(F;\mu_p^{\otimes n}).$$

In the case $p=2$ these normvarieties are quadrics associated with Pfisterforms.

In this note we describe some new results and conjectures about such quadrics. In short we have the following results:

It turns out that for a n-fold Pfisterform φ the Chow-motive of its associated quadric X_φ can be described as

$$X_\varphi \simeq M_\varphi \times \mathbb{P}^{d_\varphi}, \quad d_\varphi = 2^{n-1}-1,$$

where M_φ is a certain Chow-motive associated with φ. We give a complete description of the Chowgroups of M_φ and of the generators of $H^p(M_\varphi; K_{p+1})$; moreover we have a precise conjecture about (the Milnor-K-theory version of) $H^p(M_\varphi; K_{p+r})$ for $r \le 2$.

This note contains no proofs. Detailed proofs will be prepared as soon as possible.

§ 2 Motific decomposition of certain quadrics

We work in the category of Chow-motives over a field $F(\mathrm{Char}\ F \neq 2)$ (see e.g. [Fulton; Intersection Theory, § 16]). For a motive $M=(X,p)$ let $CH_k(M)=p_*(CH_k(X))$, where $CH_k(X)$ is the group of k-dimensional cycles modulo rational equivalence. We denote by $L=(\mathbb{P}^1,p)$ the Tate motive and by L^i its i-th power. For a quadratic form φ over F we denote by X_φ the corresponding projective quadric $(\dim X_\varphi = \dim\varphi - 2)$ and by $\overline{X}_\varphi$ its associated Chow-motive.

One first observation is

Proposition 1
Let $\varphi = \psi \perp h$, where h is a hyperbolic plane and $\dim \psi \geq 0$. Then

$$\overline{X}_\varphi = \overline{X}_\psi \otimes L \oplus L^0 \oplus L^d;\ d = \dim \psi.$$

Hence the motive of the quadric X_φ depends essentially only on the class of φ in the Wittring of F. Moreover the proposition shows that the motive of a trivial quadric X_φ (i.e. $\varphi = \Sigma(-1)^i x_i^2$) is just a sum of certain powers of the Tate motive.

Proposition 2
Let $K|F$ be any field extension. Then $\alpha \in \mathrm{End}(\overline{X}_\varphi)$ is invertible if and only if $\alpha_K \in \mathrm{End}((\overline{X}_\varphi)_K)$ is invertible.

This is basic to all what follows. To proof proposition 2 one considers the filtration on $\mathrm{End}(\overline{X}_\varphi) = CH_d(X_\varphi \times X_\varphi)$ $(d = \dim X_\varphi)$ given by the dimension of cycles after projection to one factor.

For a n-fold Pfisterform φ we call $n_\varphi = n$ the degree of φ; moreover we put $d_\varphi = 2^{n-1} - 1$. Note that $\dim X_\varphi = 2d_\varphi$.

Theorem 3
For a Pfisterform φ there exists a unique motive $M = M_\varphi$ such that for extensions $K|E|F$ there exists

i) an isomorphism

$$M_E = L_E^0 \oplus L_E^{d_\varphi}$$

if φ_E is isotropic.

ii) a commutative diagram of isomorphisms

$$\begin{array}{ccc} & \mathbb{Z} & \\ i_E \swarrow & & \searrow i_K \\ CH_{d_\varphi}(M_E) & \xrightarrow{\mathrm{res}_{K|E}} & CH_{d_\varphi}(M_K) \end{array}$$

iii) a commutative diagram of injective homomorphisms

$$\begin{array}{ccc} CH_0(M_E) & \xrightarrow{\mathrm{res}_{K|E}} & CH_0(M_K) \\ \searrow N_E & & \swarrow N_K \\ & \mathbb{Z} & \end{array}$$

with

$$\mathrm{Im}\ N_E = \begin{cases} 2\,\mathbb{Z} & \text{if } \varphi_E \text{ is anisotropic} \\ \mathbb{Z} & \text{if } \varphi_E \text{ is isotropic.} \end{cases}$$

The motive M_φ can be described more precisely as follows: Let $\varphi = \mu \otimes \langle\langle a \rangle\rangle$ where μ is a Pfisterform of degree $n_\mu = n_\varphi - 1$ and put $\rho = \mu \perp \langle -a \rangle$. Then $M_\varphi = (X_\rho, p)$ for a certain projector $p \in \operatorname{End}(\overline{X}_\rho)$ such that p_* is the identity on $CH_{d_\varphi}(X_\rho)$ and $CH_0(X_\rho)$ (note that $d_\varphi = \dim X_\rho$). Hence $CH_{d_\varphi}(M_\varphi) = \mathbb{Z}$ with the cycle X_ρ as canonical generator; we denote this generator by $[M_\varphi]$. Moreover $CH_0(M_\varphi) = CH_0(X_\rho)$; the homomorphisms N in iii) are given by the degree of a zero-cycle.
The motive of X_ρ decomposes as

$$\overline{X}_\rho = M_\varphi \oplus \overline{X}_{\mu'} \otimes L,$$

where μ' is the pure subform of μ, i.e. $\mu = \mu' \perp \langle 1 \rangle$. More generally, one has

Proposition 4
Let φ be a Pfisterform and let ρ be a subform of φ with $\dim \rho = \frac{1}{2} \dim \varphi + k$, $k > 0$. Suppose that

$$\rho_{F(X_\varphi)} \simeq \eta_{F(X_\varphi)} \perp h,$$

where η is a form over F and h is hyperbolic of dimension $2k$. Then

$$\overline{X}_\rho = \bigoplus_{i=0}^{k-1} M_\varphi \otimes L^i \oplus \overline{X}_\eta \otimes L^k.$$

Hence, if ρ is an excellent form, i.e. the class of φ in the Wittring of F is the alternating sum of a sequence $\varphi_0, \dots, \varphi_n$ of Pfisterforms φ_i with φ_i a subform of φ_{i+1}, one has a decomposition of the motive X_ρ in terms of the motives M_{φ_i}.

§ 3 On the K-cohomology of M_φ

For a variety X over F we denote by $A_p(X, K_n^M)$ the homology of the complex

$$\bigoplus_{v \in X_{(p+1)}} K^M_{n+p+1}\kappa(v) \xrightarrow{d} \bigoplus_{v \in X_{(p)}} K^M_{n+p}\kappa(v) \xrightarrow{d} \bigoplus_{v \in X_{(p-1)}} K^M_{n+p-1}\kappa(v),$$

where K_n^M denotes the n-th Milnor K-group, $X_{(p)}$ denotes the set of all points of X of dimension p and d is given by the tame symbol.

Using the deformation to the normal cone (see Fulton) one can define intersection theory for the groups $A_p(X; K_n^M)$. Hence the functors $A_p(\quad, K_n^M)$ are defined on the category of Chow-motives.

Note that $A_p(X; K^M_{-p}) = CH_p(X)$ and that for a smooth variety X of dimension d there is a natural homomorphism

$$A_p(X; K_n^M) \to H^{d-p}(X; K_{n+d})$$

induced by the canonical map from Milnor-K-theory to Quillen's K-groups (which is an isomorphism for $n+p \leq 2$).

For a nonsingular quadratic form φ let $D_0(\varphi) \subset K_0F = \mathbb{Z}$ be the subgroup

$$D_0(\varphi) = \begin{cases} K_0F & \text{if } \varphi \text{ is isotropic} \\ 2K_0F & \text{if } \varphi \text{ is nonisotropic} \end{cases}$$

and for $n \geq 1$ let $D_n(\varphi) \subset K_n^M F$ be the subgroup generated by symbols in which one entry is represented by φ.

One can show that $D_n(\varphi)$ is exactly the image of the normmap

$$N : A_0(X_\varphi K_n^M) \to K_n^M F.$$

Theorem 5
For a Pfisterform φ of degree $n_\varphi \geq 2$ one has

$$A_p(M_\varphi; K_{-p}) = CH_p(M_\varphi) = \begin{cases} K_0F & \text{for } p = d_\varphi = 2^{n_\varphi - 1} - 1 \\ K_0F/D_0(\varphi) & \text{for } p = 2^k - 1;\ k = 1, \ldots, n_\varphi - 2 \\ D_0(\varphi) & \text{for } p = 0 \\ 0 & \text{else} \end{cases}$$

The generators of $CH_p(M_\varphi)$ can be described as follows.

For a Pfistersubform ψ of φ there is a natural morphism

$$i_{\psi,\varphi} : M_\psi \to M_\varphi\,,$$

compatible with the norm maps $CH_0 \to \mathbb{Z}_0$. It is induced by inclusion $X_{\tilde{\rho}} \to X_\rho$ for appropriate choices of representations $M_\varphi = (X_\rho, p)$, $M_\psi = (X_{\tilde{\rho}}, \tilde{p})$ as described above.

The generator of $CH_p(M_\varphi)$, $p = 2^k - 1$ for some $k \in \{1, \ldots, n_\varphi - 1\}$, is then given by the image $(i_{\psi,\varphi})_*([M_\psi])$ of the fundamental cycle $[M_\psi] \in CH_{d\psi}(M_\psi)$, where ψ is any Pfistersubform of φ of degree $n_\psi = k+1$.

Theorem 6
Let φ be a Pfisterform of degree $n_\varphi \geq 2$.

i) If $p = 2^k + 2^\ell - 1$, $0 < \ell < k < n_\varphi - 1$, then $A_p(M_\varphi; K_{1-p})$ is cyclic of order at most 2.

ii) If $p = 0$, then the normmap $A_0(M_\varphi; K_1) \to K_1F$ is injective with image $D_1(\varphi)$.

iii) For all other values of p the multiplication map

$$CH_p(M_\varphi) \otimes K_1F \to A_p(M_\varphi; K_{1-p})$$

is surjective.

To give some information about the generators for the groups in i) we describe now a general conjecture about certain elements in the groups $A_p(M_\varphi; K_n)$.

Let $\ell : \mathbb{N} \to \mathbb{N}$ be the function with the property

$$p+1 = \sum_{i=0}^{\ell(p)} 2^{k_i} \quad \text{for some} \quad 0 \le k_0 < k_1 < \cdots < k_{\ell(p)}$$

Conjecture 7
For Pfisterforms φ of degree $n_\varphi \ge 2$ there exist unique classes

$$\gamma_p(\varphi) \in A_p(M_\varphi; K^M_{\ell(p)-p}) \quad \text{for } 0 < p \le d_\varphi\,,\ p \text{ odd,}$$

such that

I) $\gamma_{d_\varphi}(\varphi) \in CH_{d_\varphi}(M_\varphi)$ is the generator $[M_\varphi]$.

II) For a Pfister-subform $\psi < \varphi$ one has

$$(i_{\psi,\varphi})_*(\gamma_p(\psi)) = \gamma_p(\varphi) \quad \text{for } 0 < p \le d_\psi\,,\ p \text{ odd.}$$

III) For a Pfister-subform $\psi < \varphi$ with $\varphi = \psi \otimes \langle\langle a \rangle\rangle$ one has

$$(i_{\psi,\varphi})^*(\gamma_p(\varphi)) = \{a\} \cdot \gamma_{p-d_\varphi+d_\psi}(\psi) \quad \text{for } d_\psi < p < d_\varphi\,,\ p \text{ odd.}$$

For III) note that $\ell(p - d_\varphi + d_\psi) = \ell(p) - 1$.
For $\ell(p) = 0$ the classes $\gamma_p(\varphi)$ are exactly the generators of $CH_p(M_\varphi)$ as described above (this follows from I) and II)). I have constructed the classes $\gamma_p(\varphi)$ for $\ell(p) \le 1$ (which are the generators for $A_p(M_\varphi; K_{1-p})$) and it is probable that I can do this for $\ell(p) \le 2$. Moreover our methods should lead to a proof of

Conjecture 8
For a Pfisterform φ of degree $n_\varphi \ge 2$ one has for $n \le 2$:

$$A_p(M_\varphi; K^M_{n-p}) = \begin{cases} K^M_n F & \text{for } p = d_\varphi \\ [\,K^M_{n-\ell(p)}F / D_{n-\ell(p)}(\varphi)\,] & \text{for } 0 < p < d_\varphi,\ p \text{ odd,} \\ 0 & \text{for } 0 < p < d_\varphi,\ p \text{ even} \\ D_n(\varphi) & p = 0 \end{cases}$$

Here we understand $K^M_r F = 0$ for $r < 0$. For $0 < p \le d_\varphi$, p odd, the isomorphism is given by multiplication with $\gamma_p(\varphi)$ and for $p = 0$ by the normmap to $K^M_n F$.

Conjecture 8 is definitely false for $n \ge 3$ (e.g. for $n_\varphi = 2$ the motive M_φ is the conic corresponding to φ and $K^M_3 F \xrightarrow{[M_\varphi]} A_1(M_\varphi; K^M_2)$ is neither surjective nor injective in general). Nevertheless the classes $\gamma_p(\varphi)$ should form a (part of a) fundamental set of generators of the groups $A_p(M_\varphi; K^M_n)$.