NOTES ON CUBIC EQUATIONS

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For a cubic element (or a triangle) we define its “basic line”. It leads to the normalization (4) of cubic equations. The method works in all characteristics for generic cubic elements. We describe for this 1-parameter family the discriminant extension and the corresponding variant of Cardano’s formula.

1. **Basic invariants.** Let $F$ be a field and let $K$ be a cubic extension of $F$. We consider the functions

$$T, Q, N, A, B, D, M, f, g: K \to F, \quad \delta, \varphi: K \to K$$

defined as follows: For $x \in K$ the polynomial

$$P_x(r) = r^3 - T(x)r^2 + Q(x)r - N(x)$$

is the characteristic polynomial of $x$. In other words, $T(x)$ is the trace of $x$, $N(x)$ is the norm of $x$, and, for invertible $x$,

$$Q(x) = T(x^{-1})N(x)$$

Moreover

$$\delta(x) = \frac{dP_x(r)}{dr}_{r=x} = 3x^2 - 2Tx + Q(x)$$

$$\varphi(x) = 3x - T(x)$$

$$\Delta(x) = N(\varphi(x)^2 - 4\delta(x))$$

$$A(x) = N(\varphi(x))$$

$$B = \frac{\Delta - A^2}{4}$$

$$D(x) = T(\delta(x))$$

$$M = QT - 9N$$

$$f = \frac{TD - M}{D}$$

$$g = -\frac{A}{D}$$

One finds

$$D = T^2 - 3Q$$

$$A = -(3M - 2TD)$$

$$3f + g = T$$

$$2f + g = \frac{M}{D}$$

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The polynomial $\Delta(x)$ is the discriminant of $x$.

Remark 1. Suppose $K = F \times F \times F$. Then for $x = (x_0, x_1, x_2)$ one has

$$D(x) = x_0^2 + x_1^2 + x_2^2 - x_0x_1 - x_1x_2 - x_2x_0$$

$$= (x_0 + \zeta x_1 + \zeta^2 x_2)(x_0 + \zeta^2 x_1 + \zeta x_2)$$

where $\zeta$ is subject to

$$1 + \zeta + \zeta^2 = 0$$

Remark 2. Note that $1 + \zeta + \zeta^2 = 0$ means that $\zeta$ is a cube root of unity (and is primitive as long as $\text{char } F \neq 3$). Thus, if $F = \mathbb{C}$ (complex numbers) in Remark 1, then $D(x) = 0$ if and only if the Euclidean triangle $x_0, x_1, x_2$ is equilateral.

Remark 3. Suppose $K = F \times F \times F$. Then for $x = (x_0, x_1, x_2)$ one has

$$A(x) = (2x_0 - x_1 - x_2)(2x_1 - x_2 - x_0)(2x_2 - x_0 - x_1)$$

Lemma. For $x \in K$ with $D(x) \neq 0$ and $a, b \in F$ one has

(1) $D(ax + b) = a^2 D(x)$

(2) $g(ax + b) = ag(x)$

(3) $f(ax + b) = af(x) + b$

Proof. Claim (1) is easy to check, for instance using Remark 1, or by using a similar property for $\delta(x)$.

As for (2), note that $T(x) - 3x$ is invariant under translations $x \mapsto x + b$. The same is true for $A(x)$ and also for $D$ by (1). The claim is now clear from $\deg g = 1$.

Claim (3) follows easily from (2) and $3f + g = T$. $\square$

2. The basic line.

Definition. For $x \in K$ with $D(x) \neq 0$ the function

$$\ell_x(s) = f(x) + sg(x), \quad s \in F$$

is called the basic line of $x$.

Clearly one has

$$\ell_{ax+b}(s) = a\ell_x(s) + b$$

Remark 4. Let $x_0, x_1, x_2$ be an Euclidean triangle which is not equilateral. Then $\ell_{(x_0, x_1, x_2)}(s)$ (understanding $K = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$) determines a natural line associated to the triangle.

Apart from the degenerate cases $D(x) = 0$ and $A(x) = 0$, the basic line is defined and non-degenerate. We thus can normalize it by means of an affine transformation $x \mapsto ax + b$, so that $f(x) = 0$ and $g(x) = 1$. This yields the following normalization for the basic parameters:

$$T = 1, \quad 9N - 4Q + 1 = 0$$

This gives

$$(T, Q, N) = (1, 9t - 2, 4t - 1)$$

with $t \in F$ and we get the following normalized form of a cubic equation

(4) $x^3 - x^2 + (9t - 2)x - (4t - 1) = 0$
Remark 5. One finds
\[ t = \frac{B}{A^2} \]
The invariant \( t \) is the basic modulus for triangles up to affine transformations.
I think it can serve as a toy model for the \( j \)-invariant for elliptic curves.

Remark 6. For \( t = 0 \) equation (4) becomes \( x^3 - x^2 - 2x + 1 = 0 \) which has the complex roots \( x = 2 \cos \frac{\pi}{7}, 2 \cos \frac{3\pi}{7}, 2 \cos \frac{5\pi}{7} \).

Remark 7. After the change of variables
\[ y = 3x - 1 \]
one gets the family
\[ y^3 - 3Dy + D = 0 \]
with \( D = 7 - 27t \).

3. The discriminant. The discriminant of the cubic equation (4) is
\[ (1 - 4t)(27t - 7)^2 \]
The discriminant algebra is after a normalization simply given by the quadratic equation
\[ u^2 - u + t = 0 \]
In fact, one may check that the linear fractional transformation
\[ \Phi(x) = \frac{ux + 3t - 1}{x + u - 1} \]
defines an \( F[u] \)-automorphism of \( F[u][x] \) of order 3. In particular, \( \Phi(x) \) and \( \Phi^2(x) \) are the conjugates of \( x \).

4. Cardano’s formula. Explicit solutions of (4) in terms of radicals are given by:
\[ D = 7 - 27t \]
\[ w^2 + w + D = 0 \]
\[ w + \overline{w} = -1 \]
\[ w\overline{w} = D \]
\[ \alpha^3 = w^2\overline{w} \]
\[ \overline{\alpha}^3 = \overline{w}^2w \]
\[ \alpha\overline{\alpha} = D \]
\[ x = \frac{1 + \alpha + \overline{\alpha}}{3} \]
This gives
\[ x = \frac{1}{6} \left( 2 - \sqrt[3]{(1 - E)(1 + \sqrt{E})} - \sqrt[3]{(1 - E)(1 - \sqrt{E})} \right) \]
with
\[ E = 1 - 4D = 27(4t - 1) \]
Here are further related formulas:

\[ X := 3x - 1 \]

\[ D \left( \frac{1}{X + w} \right) = D \left( \frac{1}{X + \overline{w}} \right) = 0 \]

\[ N(X + w) = 27(4t - 1)w \]

\[ Y := \frac{X + w}{X + \overline{w}} \]

\[ Y^3 = \frac{w}{\overline{w}} \]

5. **Appendix.** (Added April 2004)

Here are more remarks:

Let

\[ \phi_t: K \to K \]

\[ \phi_t(x) = (1 - 3t)x + tT(x) \]

Then

\[ D(x) = Q(x) - Q(\phi_1(x)) \]

\[ D(x) = \frac{-1}{2} \frac{d}{dt} Q(\phi_t(x))|_{t=0} \]

\[ f(x) = \frac{\frac{4}{3} N(\phi_t(x))|_{t=0}}{\frac{d}{dt} Q(\phi_t(x))|_{t=0}} \]

\[ f(x) = \frac{N(x) - N(\phi_1(x))}{Q(x) - Q(\phi_1(x))} \]

This is perhaps helpful to give a more geometric definition of \( f \).

Moreover one has

\[ f(x_1, x_2, x_3) = x_1 \Rightarrow x_2 = x_3 \]

This is perhaps helpful to characterize \( f \).

Can one draw any analogies between \( x \mapsto f(x) \) and the orthocenter of a Euclidean triangle?