A DESCENT PROPERTY FOR PFISTER FORMS

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Summary

The Rosenberg-Ware theorem states that for a Galois extension K/F of odd degree the natural map of Witt rings of quadratic forms

$$W(F) \to W(K)^{\mathrm{Gal}(K/F)}$$

is an isomorphism. We extend this result to arbitrary field extensions K/F of odd degree. Basically we show that (Proposition 1)

$$0 \to W(F) \xrightarrow{r_{K/F}} W(K) \xrightarrow{i_1 - i_2} W(K \otimes K)$$

is exact, where i_1, i_2 are induced from the two natural maps $K \to K \otimes K$. Further it is shown that an element of the graded Witt ring is represented by a Pfister form if this is true after an extension of odd degree (Proposition 2). We apply this to trace forms of exceptional Jordan algebras (Proposition 3). In the last section similar questions for symbols in Milnor's K-theory and Galois cohomology are considered.

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1. The transfer map

For generalities on quadratic forms we refer to [7, 9].

Let F be a field of characteristic different from 2 and let $W(F)$ be the Witt group of quadratic forms over F. Due to the Witt cancelation theorem one may identify $W(F)$ with the set of isomorphism classes of anisotropic quadratic forms over F.

For a field extension K/F we denote by

$$r_{K/F} \colon W(F) \to W(K),$$
$$r_{K/F}([\varphi]) = [\varphi_K]$$

the homomorphism given by extension of scalars.

Let $s\colon K \to F$ be a nontrivial F-linear map. According to Scharlau [9, Chap. 2, § 5] there is an associated transfer map

$$s_* \colon W(K) \to W(F),$$
$$s_*([\psi]) = [s \circ \psi]$$

such that

$$s_* \circ r_{K/F}(x) = s_*(1)x.$$

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If K/F has odd degree, then s can be chosen so that

$$s_*(1) = 1,$$

cf. [9, Chap. 2, Lemma 5.8]. In this case s_* is a left inverse to $r_{K/F}$. It follows that the restriction map $r_{K/F}$ is injective for extensions K/F of odd degree.

If K/F is purely inseparable (necessarily of odd degree since $\operatorname{char} F \neq 2$), then K and F have the same square class groups. Hence every quadratic form over K is extended from a form over F, as can be seen via diagonalization. Therefore $W(F) = W(K)$ in the purely inseparable case.

Let $I(F) \subset W(F)$ be the fundamental ideal consisting of the classes of even dimensional quadratic forms and let $I^n(F)$ denote its n-th power. The transfer maps respect the filtration of the Witt ring by the powers of its fundamental ideal, i. e.,

$$s_*\big(I^n(K)\big) \subset I^n(F),$$

cf. [1, Lemma 3.2]. It follows that

$$I^n(F) = r_{K/F}^{-1}\big(I^n(K)\big)$$

for an extension K/F of odd degree.

2. Descent for extensions of odd degree

Let K/F be a finite field extension. For each prime ideal $\mathfrak{m}$ of $K \otimes_F K$ let

$$H_{\mathfrak{m}} = (K \otimes_F K)/\mathfrak{m},$$

and let

$$i^1_{\mathfrak{m}},\, i^2_{\mathfrak{m}} \colon W(K) \to W(H_{\mathfrak{m}})$$

be the restriction maps induced from the homomorphisms

$$K \to H_{\mathfrak{m}}, \quad a \mapsto a \otimes 1 \bmod \mathfrak{m}, \quad a \mapsto 1 \otimes a \bmod \mathfrak{m},$$

respectively. We put

$$\delta \colon W(K) \to \prod_{\mathfrak{m}} W(H_{\mathfrak{m}}),$$
$$\delta(x) = \big((i^1_{\mathfrak{m}} - i^2_{\mathfrak{m}})(x)\big)_{\mathfrak{m}}.$$

Proposition 1. *Let K/F be an extension of odd degree. Then the sequence*

$$0 \to W(F) \xrightarrow{r_{K/F}} W(K) \xrightarrow{\delta} \prod_{\mathfrak{m}} W(H_{\mathfrak{m}})$$

is exact.

Proof. The injectivity of $r_{K/F}$ has been discussed already. Further, using the bijectivity of $r_{K/F}$ in the purely inseparable case, one easily reduces to separable extensions K/F. In this case one has

$$K \otimes_F K = \bigoplus_{\mathfrak{m}} H_{\mathfrak{m}}.$$

Let $s \colon K \to F$ be F-linear with $s_*(1) = 1$ and let $s_{\mathfrak{m}} \colon H_{\mathfrak{m}} \to K$ be the components of $s \otimes \mathrm{id}_K$, i. e.,

$$s(a)b = \sum_{\mathfrak{m}} s_{\mathfrak{m}}(a \otimes b \bmod \mathfrak{m}).$$

With these settings one has for $x \in W(K)$

$$\sum_{\mathfrak{m}} (s_{\mathfrak{m}})_* \big(i^1_{\mathfrak{m}}(x)\big) = r_{K/F} \circ s_*(x),$$
$$\sum_{\mathfrak{m}} (s_{\mathfrak{m}})_* \big(i^2_{\mathfrak{m}}(x)\big) = s_*(1)x = x,$$

as may be verified on the level of forms.

If $x \in \ker \delta$, then $i^1_{\mathfrak{m}}(x) = i^2_{\mathfrak{m}}(x)$ for all $\mathfrak{m}$, whence $x = r_{K/F} \circ s_*(x)$. □

The Rosenberg-Ware theorem appears as a special case of Proposition 1. Namely if K/F is a Galois extension with Galois group G, the map δ can be identified with the homomorphism

$$W(K) \to \prod_{\sigma \in G} W(K),$$
$$x \mapsto \big((1-\sigma)(x)\big)_\sigma.$$

Therefore $\ker \delta$ equals the subgroup of Galois invariants.

3. Descent of Pfister forms

An n-fold Pfister form is a quadratic form of type

$$\langle\langle a_1, \dots, a_n \rangle\rangle = \bigotimes_{i=1}^{n} \langle 1, -a_i \rangle$$

with $a_i \in F^*$.

The classes of n-fold Pfister forms generate $I^n(F)$. If a Pfister form is isotropic, then it is hyperbolic [7, Chap. Ten].

A basic theorem of Pfister [9, Chap. 4, Theorem 4.4] asserts that an anisotropic quadratic form φ in indeterminates $X = (x_1, \dots, x_m)$ is isomorphic to a Pfister form if and only if

$$\varphi(X)\varphi_{F(X)} \simeq \varphi_{F(X)}. \tag{1}$$

If α and β are n-fold Pfister forms such that

$$\alpha = \beta \bmod I^{n+1}(F)$$

in $W(F)/I^{n+1}(F)$, then α and β are isomorphic [7, Chap. Ten, Corollary 3.4].

Lemma. *Let φ be a quadratic form over F, let K/F be an extension of odd degree, and suppose that φ_K is isomorphic to a Pfister form. Then φ is isomorphic to a Pfister form.*

Proof. Since φ_K is a Pfister form, the dimension of φ is a 2-power. If φ_K is isotropic, it is hyperbolic and therefore φ is hyperbolic. It follows that φ is a Pfister form.

Assume that φ_K is anisotropic. Then φ is anisotropic and the claim follows from the criterion (1) and the injectivity of $W\big(F(X)\big) \to W\big(K(X)\big)$. □

Proposition 2. *Let φ be a quadratic form over F, let K/F be an extension of odd degree, and suppose that there exists an n-fold Pfister form β over K such that*

$$\varphi_K = \beta \bmod I^{n+1}(K).$$

Then there exists an n-fold Pfister form α over F such that

$$\varphi = \alpha \bmod I^{n+1}(F).$$

Proof. Let H/F be a field extension and let f, $g\colon K \to H$ be two homomorphisms over F. We denote the by r_f resp. r_g the extension of scalars via f resp. g. Obviously

$$r_f(\beta) = r_g(\beta) \bmod I^{n+1}(H).$$

Hence $r_f(\beta) = r_g(\beta)$ in $W(H)$.

Using the last equality for the fields $H_{\mathfrak{m}}$, it follows from Proposition 1 that $\beta \in r_{K/F}\big(W(F)\big)$. The Lemma shows $\beta = \alpha_K$ for some Pfister form α. Then $(\varphi - \alpha)_K \in I^{n+1}(K)$ and therefore $\varphi - \alpha \in I^{n+1}(F)$. □

Corollary 1. *Let $0 \le n_1 < n_2 < \cdots < n_r$ be integers and let $c_1, \ldots, c_r \in F^*$. Let further φ be a quadratic form over F, let K/F be an extension of odd degree, and suppose that there exist n_i-fold Pfister forms β_i over K $(i = 1, \ldots, r)$ such that*

$$\varphi_K = c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r$$

in $W(K)$. Then there exist n_i-fold Pfister forms α_i over F such that $\beta_i = (\alpha_i)_K$ $(i = 1, \ldots, r)$ and

$$\varphi = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$$

in $W(F)$. Moreover, if for some $j > i$ one has $\beta_j = \langle\langle b_1, \ldots, b_s\rangle\rangle\beta_i$ for some $b_k \in K^*$, then $\alpha_j = \langle\langle a_1, \ldots, a_s\rangle\rangle\alpha_i$ for some $a_k \in F^*$.

Proof. For the first statement we use induction on $r \ge 0$. One has $\varphi_K = \beta_1 \bmod I^{n_1+1}(K)$. By Proposition 2 there exists an n_1-fold Pfister form α over F such that $\varphi = \alpha \bmod I^{n_1+1}(F)$. Then necessarily $\beta_1 = \alpha_K$. The claim follows by applying the induction hypothesis to the form $\varphi \perp -c_1\alpha$.

For the second statement first note that α_i is a subform of α_j by Springer's theorem [9, Chap. 2, Theorem 5.3]. The claim follows from [3, Theorem 2.7]. □

Quadratic forms φ of the type as in Corollary 1 appear when studying trace forms of various algebras. An example is considered in the next section.

4. Serre's (mod 2) invariants for F_4

It has been noticed by Serre that there are cohomological invariants

$$f_3\colon H^1(F, F_4) \to H^3(F, \mathbf{Z}/2),$$
$$f_5\colon H^1(F, F_4) \to H^5(F, \mathbf{Z}/2),$$

cf. [10, III. Annexe, § 3.4] or [11, III. Appendix 2, 3.4] and [6, § 40]. The construction of these invariants is based on the interpretation of $H^1(F, F_4)$ as the set of isomorphism classes of exceptional Jordan algebras (cf. [5, 6, 12]) and the following description of their trace forms:

Proposition 3. *Let J be an exceptional Jordan algebra over F and let*

$$q_J\colon J \to F,$$
$$q_J(x) = T_J(x^2),$$

where T_J denotes the trace map of J. Then there exist elements $a_1, \ldots, a_5 \in F^*$ such that

$$q_J \simeq \langle 1, 1, 1\rangle \perp 2\langle\langle a_1, a_2, a_3\rangle\rangle\langle -a_4, -a_5, a_4a_5\rangle. \tag{2}$$

In terms of this description, Serre's invariants are given by

$$f_3([J]) = (a_1)(a_2)(a_3), \quad f_5([J]) = (a_1)(a_2)(a_3)(a_4)(a_5).$$

For a proof of Proposition 3 using an analysis of the Tits constructions of exceptional Jordan algebras see [6, Lemma 40.1].

Alternatively one can prove Proposition 3 using Corollary 1 as follows. Let $\varphi = 2\big(q_J \perp -\langle 1,1,1\rangle\big)$. After passing to an appropriate cube extension K/F, the Jordan algebra J has zero divisors. In this case (the so called "reduced" case), the description (2) of the trace form can be read off a presentation of J in terms of 3×3 matrices over an octonion algebra [5, 6]. Hence $\varphi_K = \langle\langle b_1, b_2, b_3, b_4, b_5\rangle\rangle - \langle\langle b_1, b_2, b_3\rangle\rangle$ in $W(K)$ for some $b_i \in K^*$. By Corollary 1 one has $\varphi = \langle\langle a_1, a_2, a_3, a_4, a_5\rangle\rangle - \langle\langle a_1, a_2, a_3\rangle\rangle$ in $W(F)$ for some $a_i \in F^*$, whence (2).

5. Descent for (mod 2) symbols in Milnor's K-ring

Let $K_n^M F$ be Milnor's K-group of F [2, 8]. For $a_1, \ldots, a_n \in F^*$ one denotes by $\{a_1, \ldots, a_n\}$ the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M F$.

Let $p \neq \operatorname{char} F$ be a prime. By a (mod p)-*symbol* we understand an element in $K_n^M F/p$ of the form $\{a_1, \ldots, a_n\}$ mod $pK_n^M F$. Let us call an element $x \in K_n^M F/p$ a *weak* (mod p)-*symbol*, if there exists a finite field extension K/F of degree prime to p such that x_K is a (mod p)-symbol.

Is a weak (mod p)-symbol always a (mod p)-symbol?

I don't know any counterexample. For $p = 2$ one has:

Corollary 2. *Every weak* (mod 2)*-symbol is a* (mod 2)*-symbol.*

Proof. Milnor [8] defined a homomorphism

$$s_n \colon K_n^M F/2 \to I^n(F)/I^{n+1}(F),$$
$$s_n(\{a_1, \ldots, a_n\}) = \langle\langle a_1, \ldots, a_n\rangle\rangle \bmod I^{n+1}(F).$$

The map s_n is surjective and it is injective on symbols [3, Prop. 2.1].

Let $x \in K_n^M F/2$, let K/F be of odd degree, and suppose that

$$x_K = \{b_1, \ldots, b_n\}$$

with $b_i \in K^*$. Let φ be a quadratic form such that

$$s_n(x) = \varphi \bmod I^{n+1}(F).$$

Applying Milnor's homomorphism over K yields

$$\langle\langle b_1, \ldots, b_n\rangle\rangle = \varphi_K \bmod I^{n+1}(K).$$

By Proposition 2 there exist $a_i \in F^*$ with

$$s_n(x) = s_n(\{a_1, \ldots, a_n\}).$$

Since s_n is injective on symbols one has

$$x_K = \{a_1, \ldots, a_n\}_K \bmod 2K_n^M K.$$

Applying the transfer map in Milnor's K-theory yields

$$x = \{a_1, \ldots, a_n\} \bmod 2K_n^M F$$

since K/F is of odd degree. $\square$

If $n = 2$ and $\mu_p \subset F$, the question whether a weak symbol is a symbol is equivalent to the question whether an algebra of prime degree p is cyclic. This is known for $p \leq 3$ and unsettled otherwise.

Using the results of [4] one can show that every weak symbol in $K_2^M F/3$ is a symbol for any field F of characteristic different from 2 and 3.

Beyond the cases $p = 2$ and $n = 2$, $p = 3$ not much seems to be known, even for instance for the following question: Let K/F be a quadratic extension, let $x \in K_3^M F/3$ and suppose that x_K is a symbol. Is then x itself a symbol? Similar for $x \in K_2^M F/5$.

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