BASIC NOTES ON ELLIPSES AND KEPLER/NEWTON

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Introduction

The core of the solution to the Newton differential equation for planet orbits can be formulated as follows:

Theorem. *Let a, b, $c \in \mathbf{R}$ be real numbers with*

$$a^2 = b^2 + c^2, \qquad a > 0,\, b \neq 0$$

and let

$$z\colon \mathbf{R} \to \mathbf{C} \setminus \{0\}$$
$$z(s) = c + a\cos s + ib\sin s$$

The map

$$t\colon \mathbf{R} \to \mathbf{R}$$
$$t(s) = as + c\sin s$$

is a diffeomorphism and z satisfies the differential equation

$$\frac{d}{dt}\frac{dz}{dt} = -a\frac{z}{|z|^3}$$

A common form of the Newton equation is to choose a fixed positive real number $k > 0$ and then ask for solutions to

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{k}{|z|^2}\frac{z}{|z|}$$

with initial position at the shortest (or longest) distance from the sun. For negative energy and nonzero angular momentum (i.e., for closed planet orbits) the solutions are easily obtained from the Theorem by setting

$$\tau = \sqrt{\frac{a}{k}}t$$

See Section 4 for details.

The Theorem is proved in Section 1 in an ad hoc fashion, without references to geometry or physics. The idea is to display the computations in the most straightforward and simpleminded way. They are easy enough and meant to serve as a basis for forthcoming considerations.

A remarkable fact is that the absolute value $|z|$ of z has the simple form

$$|z(s)| = r(s) = a + c\cos s$$

In Section 2 we consider the angle function $\theta(s)$ given by

$$z(s) = r(s)e^{i\theta(s)}$$

and derive the relevant basic properties.

Section 3 contains excursional remarks not needed later.

The application to planet orbits is discussed in Section 4. Here we assume basic knowledge of conic sections and celestial mechanics. Given the standard constants k, L, E, the corresponding ellipse parameters are

$$a = -\frac{k}{2E}, \qquad b = \frac{L}{\sqrt{-2E}}, \qquad c = P + \frac{k}{2E}$$

where P is the longest distance from the sun.

The text is directly inspired by Milnor 1983 [2, Appendix 1. Conic sections, p. 359].

1. The main statement

We denote by $\mathcal{F}$ the ring of real-analytic functions

$$f\colon \mathbf{R} \to \mathbf{R}, \qquad s \mapsto f(s)$$

Let a, b, $c \in \mathbf{R}$ be real numbers with

$$a^2 = b^2 + c^2, \qquad a, b \neq 0$$

and consider the following elements of $\mathcal{F}$:

$$\begin{aligned} x &= c + a\cos s \\ y &= b\sin s \\ r &= a + c\cos s \\ t &= as + c\sin s \end{aligned} \tag{1.1}$$

Obviously

$$\frac{dt}{ds} = r$$

Lemma.

$$r^2 = x^2 + y^2 \tag{1.2}$$

Proof.
$$\begin{aligned} x^2 + y^2 &= (c + a\cos s)^2 + (b\sin s)^2 \\ &= c^2 + 2ac\cos s + a^2\cos^2 s + b^2\sin^2 s \\ &= a^2 + 2ac\cos s - a^2\sin^2 s + c^2 + (a^2 - c^2)\sin^2 s \\ &= a^2 + 2ac\cos s + c^2\cos^2 s = r^2 \end{aligned}$$ □

Let us stress that we do not assume $a > 0$. This assumption is unnecessary and would be rather disruptive.

Note that $r = a + c\cos s$ vanishes nowhere since $|a| > |c|$ because of $b \neq 0$. The function

$$r\colon \mathbf{R} \to \mathbf{R} \setminus \{0\}$$

has the same sign as a.

Since $dt/ds = r$ and $\lim_{s\to\pm\infty} t = \pm\infty$ (with a sign switch if $a < 0$), it follows that t is an analytic diffeomorphism $\mathbf{R} \to \mathbf{R}$ and one has

$$\frac{d}{dt} = \frac{1}{r}\frac{d}{ds}$$

Put

$$z = x + iy\colon \mathbf{R} \to \mathbf{C} \setminus \{0\}$$

Then

$$z(0) = a + c = r(0) = \frac{dt}{ds}(0), \qquad \frac{dz}{ds}(0) = ib \tag{1.3}$$

Clearly, $z(s) = Z(e^{is})$ and $r(s) = R(e^{is})$ where

$$\begin{aligned} &Z\colon \mathbf{S}^1 \to \mathbf{C} \setminus \{0\}, && Z(u + iv) = c + au + ibv \\ &R\colon \mathbf{S}^1 \to \mathbf{R} \setminus \{0\}, && R(u + iv) = a + cv \end{aligned}$$

with $\mathbf{S}^1 \subset \mathbf{C}$ the unit circle ($u^2 + v^2 = 1$). Moreover, (1.2) shows $|z/r| = 1$. Hence z/r and Z/R map to $\mathbf{S}^1$. For a related diagram see Section 2.

The $\times$-product of two complex numbers is defined here as

$$\times\colon \mathbf{C}\times\mathbf{C}\to\mathbf{R}$$
$$z\times w = \Im(\overline{z}w)$$
$$(x+iy)\times(u+iv) = xv - yu$$

Theorem 1.

$$r^3\frac{d}{dt}\frac{dz}{dt} = -az \tag{1.4}$$
$$z\times\frac{dz}{dt} = b \tag{1.5}$$
$$\frac{dz}{dt}\frac{d\overline{z}}{dt} - \frac{2a}{r} = -1 \tag{1.6}$$

Proof. One has
$$\frac{dz}{ds} = -a\sin s + ib\cos s$$
Equation (1.4) follows from
$$\begin{aligned}\frac{d}{dt}\frac{dz}{dt} &= \frac{1}{r}\frac{d}{ds}\left(\frac{1}{r}\frac{dz}{ds}\right) = \frac{1}{r}\left(\frac{1}{r}\frac{d}{ds}\frac{dz}{ds} - \frac{1}{r^2}\frac{dr}{ds}\frac{dz}{ds}\right)\\ &= \frac{1}{r^3}\left[(a+c\cos s)(-a\cos s - ib\sin s) - (-c\sin s)(-a\sin s + ib\cos s)\right]\\ &= \frac{1}{r^3}\left[-a^2\cos s - ac(\cos^2 s+\sin^2 s) - ib\sin s(a+c\cos s - c\cos s)\right]\\ &= -\frac{1}{r^3}\left[a(a\cos s + c) + iab\sin s\right] = -\frac{a}{r^3}z\end{aligned}$$
Moreover, (1.5) follows from
$$\begin{aligned}rz\times\frac{dz}{dt} = z\times\frac{dz}{ds} &= (c + a\cos s + ib\sin s)\times(-a\sin s + ib\cos s)\\ &= cb\cos s + ab\cos^2 s + ab\sin^2 s = cb\cos s + ab = rb\end{aligned}$$
and (1.6) from
$$\begin{aligned}r^2\frac{dz}{dt}\frac{d\overline{z}}{dt} = \frac{dz}{ds}\frac{d\overline{z}}{ds} &= a^2\sin^2 s + b^2\cos^2 s\\ &= a^2(1-\cos^2 s) + (a^2-c^2)\cos^2 s\\ &= a^2 - c^2\cos^2 s\\ r^2\left(\frac{2a}{r} - 1\right) &= r\big(2a - (a + c\cos s)\big)\\ &= (a+c\cos s)(a - c\cos s) = a^2 - c^2\cos^2 s\end{aligned}$$
□

Remark. Standard arguments use (1.4) to show that the left hand sides of (1.5), (1.6) have vanishing derivatives, i.e., they are constant. In brief (with $a>0$),
$$\begin{aligned}(z\times\dot z)^{\cdot} &= \dot z\times\dot z + z\times\ddot z = 0+0\\ (\dot z\overline{\dot z})^{\cdot} &= \ddot z\overline{\dot z} + \dot z\overline{\ddot z} = \Re(\ddot z\overline{\dot z}) = -a\Re(z\overline{\dot z})|z|^{-3}\\ \left(\frac{1}{(z\overline{z})^{1/2}}\right)^{\cdot} &= -\frac{1}{2}\frac{\dot z\overline{z} + z\overline{\dot z}}{(z\overline{z})^{3/2}} = -\frac{1}{2}\Re(\dot z\overline{z})|z|^{-3}\end{aligned}$$
Then (1.5), (1.6) follow by evaluating at $s=0$.

2. The angle

Define $\theta \in \mathcal{F}$ as the lift of $z/r\colon \mathbf{R} \to \mathbf{S}^1$ to the standard universal cover $\mathbf{R} \to \mathbf{S}^1$,

$$\begin{tikzcd}
\mathbf{R} \arrow[r, "\theta"] \arrow[d, "s \mapsto e^{is}"'] \arrow[dr, "\frac{z}{r}"] & \mathbf{R} \arrow[d, "\varphi \mapsto e^{i\varphi}"] \\
\mathbf{S}^1 \arrow[r, "\frac{Z}{R}"] & \mathbf{S}^1
\end{tikzcd}$$

with $\theta(0) = 0$ (note that $(z/r)(0) = (c+a)/(a+c) = 1$).

Thus

$$z = re^{i\theta}, \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{2.1}$$

Lemma.

$$r(a - c\cos\theta) = b^2 \tag{2.2}$$

Proof. $r(a - c\cos\theta) = ra - cx = (a + c\cos s)a - c(c + a\cos s) = a^2 - c^2$ $\square$

For a system (2.1) one has generally

$$rdr = xdx + ydy, \qquad r^2 d\theta = xdy - ydx$$

as can be seen from

$$\begin{aligned} \overline{z}dz &= (x - iy)(dx + idy) = (xdx + ydy) + i(xdy - ydx) \\ \overline{z}dz &= \overline{z}d(re^{i\theta}) = \overline{z}(dr + ird\theta)e^{i\theta} = r(dr + ird\theta) \end{aligned}$$

Proposition 1.

$$r^2 \frac{d\theta}{dt} = b \tag{1.5'}$$

Proof. (1.5′) is a reformulation of (1.5). This follows from

$$z \times dz = \Im(\overline{z}dz) = r^2 d\theta$$ $\square$

3. Remarks

The following comments will not be used in the next section.

3.1. Equations (1.5′) or (1.5) show that

$$Z/R\colon \mathbf{S}^1 \to \mathbf{S}^1$$

and its lift to the universal cover

$$\theta\colon \mathbf{R} \to \mathbf{R}$$

are analytic diffeomorphisms.

3.2. To prove Theorem 1 and Proposition 1 it would suffice to look at the Taylor expansions at 0, i.e., one may pass from $\mathcal{F}$ to the formal power series ring $\mathbf{R}((s))$.

Hence one may formulate and verify the statements at first in a purely algebraic way, without referring to convergence etc.

Obviously

$$\theta = \arcsin\left(\frac{y}{r}\right) = \arcsin\left(\frac{b\sin s}{a + c\cos s}\right) \qquad \text{(at } s = 0\text{)}$$

3.3. The function

$$Z\colon \mathbf{S}^1 \to \mathbf{C} \setminus \{0\}$$

(with $Z(e^{is}) = z(s)$) extends to a complex analytic function as follows:

$$Z\colon \mathbf{C} \setminus \{0\} \to \mathbf{C}$$

$$\begin{aligned} 2Z(w) &= 2c + a\left(w + \frac{1}{w}\right) + b\left(w - \frac{1}{w}\right) \\ &= 2c + (a+b)w + (a-b)\frac{1}{w} \end{aligned}$$

One finds

$$2Z(\lambda^2) = (a+b)\left(\lambda + \frac{c}{a+b}\frac{1}{\lambda}\right)^2 = (a-b)\left(\frac{1}{\lambda} + \frac{c}{a-b}\lambda\right)^2$$

(note that $a \pm b \neq 0$ in at least one case).

It follows that Z has for $c \neq 0$ the only zero

$$w = \frac{-c}{a+b} = \frac{a-b}{-c}$$

For $c = 0$ one has $Z(w) = aw^{\pm 1}$ and there are no zeros in the domain $\mathbf{C} \setminus \{0\}$.

The fact that

$$\frac{2Z(\lambda^2)}{a+b}, \qquad \frac{2Z(\lambda^2)}{a-b}$$

are the squares of

$$\lambda + \frac{c}{a+b}\frac{1}{\lambda}, \qquad \frac{1}{\lambda} + \frac{c}{a-b}\lambda$$

respectively, is employed in Arnold 1990 (1989) [1, Appendix 1. Proof that orbits are elliptic, p. 95].

4. Newton and Kepler

We assume $a > 0$ (now this assumption is convenient). Then $r > 0$ and (1.2) shows

$$r = |z|$$

The function $z\colon \mathbf{R} \to \mathbf{C} \setminus \{0\}$ is a solution to the Newton equation (cf. (1.4))

$$\frac{d}{dt}\frac{dz}{dt} = -\frac{az}{|z|^3}$$

with initial conditions (cf. (1.3))

$$z(0) = a + c$$

$$\frac{dz}{dt}(0) = i\frac{b}{a+c} = i\frac{a-c}{b} = \pm i\sqrt{\frac{a-c}{a+c}}$$

The constants (cf. (1.5) and (1.6))

$$L = z \times \frac{dz}{dt} = b, \qquad E = \frac{1}{2}\left|\frac{dz}{dt}\right|^2 - \frac{a}{|z|} = -\frac{1}{2}$$

are the angular momentum and the energy of the system.

The first Kepler law says that a planet orbit is an ellipse with a focus point at the location of the sun. This is clear for z from $c^2 = a^2 - b^2$ and the definitions of x, y in (1.1) or from (2.2) (and standard definitions/descriptions for ellipses).

The second Kepler law holds by (1.5′), since the area A has differential

$$dA = \frac{1}{2}r^2 d\theta$$

Choose some positive constant $k \in \mathbf{R}$ and put

$$\tau = \sqrt{\frac{a}{k}}t = \sqrt{\frac{a}{k}}(as + c\sin s)$$

Then z is a solution to

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{kz}{|z|^3}$$

with initial conditions

$$z(0) = P = a + c$$

$$\frac{dz}{d\tau}(0) = iQ = i\sqrt{\frac{k}{a}}\frac{b}{a+c}$$

The angular momentum and the energy of the system (with respect to τ) are

$$L = PQ = b\sqrt{\frac{k}{a}}, \qquad E = \frac{1}{2}Q^2 - \frac{k}{P} = -\frac{k}{2a}$$

The period time T with respect to τ is $T = \tau|_{s=2\pi}$ since the period time with respect to s is 2π. Hence

$$T = 2\pi\sqrt{\frac{a^3}{k}} = \frac{2\pi k}{(-2E)^{3/2}}$$

in accordance with the third Kepler law (cf. Milnor [2, (6), p. 356]).

The orthogonality of position $z(0) = P$ and velocity $dz/d\tau(0) = iQ$ means that P is an orbit point with extremal distance to the sun (since $\langle z, z\rangle^{\cdot} = 2\langle z, \dot{z}\rangle$).

The following description of the solutions is as in Milnor [2, p. 360] but with the scale factor $\alpha = \sqrt{k/a}$ for the parameter s eliminated. More precisely, Milnor's function $t(s)$ equals $\tau(\alpha s)$ with c replaced by $-c$.

Given $k > 0$ and P, Q with

$$L = PQ \neq 0, \qquad E = \frac{1}{2}Q^2 - \frac{k}{P} < 0$$

the solution to the Newton equation

$$\frac{d}{d\tau}\frac{dz}{d\tau} = -\frac{k}{|z|^2}\frac{z}{|z|}, \qquad z(0) = P, \qquad \frac{dz}{d\tau}(0) = iQ$$

is

$$z(s) = c + a\cos s + ib\sin s, \qquad \tau(s) = \sqrt{\frac{a}{k}}(as + c\sin s)$$

where

$$a = -\frac{k}{2E}, \qquad b = \frac{L}{\sqrt{-2E}}, \qquad c = P + \frac{k}{2E}$$

Here P is the longest or shortest distance to the sun, resulting in $c \geq 0$ resp. $c \leq 0$ ($E < 0$ implies $P > 0$ and $a > 0$). One has

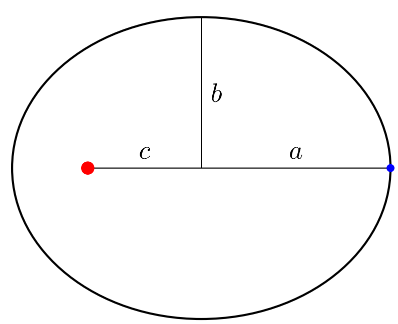
$$|z(s)| = a + c\cos s$$

and the polar ellipse equation of the orbit is (cf. (2.2))

$$|z| = \frac{b^2}{a - c\cos\theta} = \frac{L^2/k}{1 + (1 + 2PE/k)\cos\theta} \qquad (z = |z|e^{i\theta})$$

The eccentricity and the semi-latus rectum of the ellipse are

$$\varepsilon = |c|/a = |1 + 2PE/k|, \qquad \Lambda = b^2/a = L^2/k$$



$$\frac{c}{a} = \frac{3}{5}$$

References

[1] V. I. Arnol'd, *Huygens and Barrow, Newton and Hooke*, Birkhäuser Verlag, Basel, 1990, Pioneers in mathematical analysis and catastrophe theory from evolvents to quasicrystals, Translated from the Russian by Eric J. F. Primrose. MR 1078625 zbMATH 0721.01001 7

[2] J. Milnor, *On the geometry of the Kepler problem*, Amer. Math. Monthly **90** (1983), no. 6, 353–365. MR 707149 zbMATH 0518.70008 3, 8, 9

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