Hilbert 90 for K_3 for degree-two extensions

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In this paper* we consider Milnor K-theory of fields [Mi]. Let F be a field of characteristic different from 2 and let L be an extension of degree two with generator σ of $\mathrm{Gal}(L|F)$. The purpose of this paper is to prove

Theorem A

The sequence

$$K_3L \xrightarrow{1-\sigma} K_3L \xrightarrow{N_{L|F}} K_3F$$

is exact.

It is a consequence of this theorem that the Galois symbol $K_3F/2 \to H^3(F, \mathbb{Z}/2)$ is an isomorphism. This will be considered elsewhere.

As in the proof of Hilbert 90 for K_2 ([Me], [MS]), Theorem A follows from the exactness of a certain part of the localization sequence of a Severi-Brauer variety with respect to Milnor K-Theory. Let D be a quaternion algebra over F and let X be the (one-dimensional) Severi-Brauer variety associated to D. The basic results needed in the proof of Theorem A are the injectivity of the reduced norm $\mathrm{Nrd} : K_2D \to K_2F$ ([R1]) and

Theorem B

The sequence

$$K_3F(X) \to \bigoplus_{v \in X^{(1)}} K_2\kappa(v) \xrightarrow{\theta} K_2D \to 0$$

is exact.

(Here v runs through the closed points of X. The homomorphism θ is induced by the natural map $\theta_K : K_2K \to K_2D$ for a splitting field K of D, finite over F; see [MS; § 1]).

Note that the corresponding statement for the K-Theory of Quillen follows from the computations $K_2(X) = K_2D \oplus K_2F$ and $H^0(X, \mathcal{K}_2) = K_2F$ ([MS]).

The philosophy of our proof is that Theorem A together with the injectivity of the Galois symbol is equivalent to Theorem B together with the injectivity of $\mathrm{Nrd} : K_2D \to K_2F$. Using Hilbert 90 for K_2 it is not difficult to see that Theorem A holds for the universal Kummer extension of degree two of a pure transcendental extension F of a prime field (§ 1). We use this to show that Theorem B holds for a generic quaternion algebra D over F (§ 3). To prove Theorem B in general we make use of Rehmann's description of K_2D in

* This is a TeXed version (Sept. 1996) of the original preprint.

terms of generators and relations and do some specialization arguments using the results of § 3 (Since I hope that it is possible to shorten some arguments, the proof of Theorem B given here is rather sketchy). Finally, in § 5 we show that Theorem B and the injectivity of $K_2D \to K_2F$ imply Theorem A.

§ 1 Hilbert 90 in a special case

The reader is assumed to be familiar with Milnor K-Theory of fields as defined in [Mi]. For the product $K_1K \otimes K_nK \to K_{n+1}K$ we use the notation $(x, u) \mapsto l(x)u = \{x, u\}$. For the rational function field $K(t)$ in one variable one has the exact sequence

$$0 \to K_nK \to K_nK(t) \xrightarrow{d} \bigoplus_{P \in \mathcal{P}_K} K_{n-1}K_P \to 0 \tag{1.0.1}$$

Here P runs over the set $\mathcal{P}_K$ of normed irreducible polynomials in t and $K_P = K[t]/(P)$ (see [Mi; Theorem 2.3]).
If $H|K$ is finite extension (H may be a field or a direct sum of field extensions $H_i|K$; in the latter case $K_nH = \oplus_i K_nH_i$ by definition), there is a restriction $\mathrm{res}_{H|K} : K_nK \to K_nH$ and a corestriction or norm homomorphism $\mathrm{cor}_{H|K} = N_{H|K} : K_nH \to K_nK$ (See [BT] for definition and [K] for uniqueness of the norm). One has the formulas

$$\begin{array}{lll} \mathrm{cor}_{H|K} \circ \mathrm{res}_{H|K} = [H : K] & & \\ \mathrm{cor}_{H|K}(\{u, v\}) = \{\mathrm{cor}_{H|K}(u), v\} & \text{for} & u \in K_nH, v \in K_mK \\ \mathrm{res}_{H|K} \circ \mathrm{cor}_{H|K} = \sum_{\sigma \in Gal(H|K)} & \text{if} & H|K \text{ is normal.} \end{array}$$

If P is irreducible over K and if $Q_1, \ldots, Q_1$ are the irreducible factors of P over H, one also has homomorphisms

$$\bigoplus_i K_nH_{Q_i} \underset{\mathrm{res}_{H|K}}{\overset{\mathrm{cor}_{H|K}}{\rightleftarrows}} K_nK_P$$

satisfying the above formulas. They fit into a commutative diagram

$$\begin{tikzcd} K_nH(t) \arrow[r, "d"] \arrow[d, shift left, "N_{H(t)|K(t)}"] & \bigoplus_{Q \in \mathcal{P}_H} K_{n-1}H_Q \arrow[d, shift left, "\mathrm{cor}_{H|K}"] \\ K_nK(t) \arrow[r] \arrow[u, shift left, "\mathrm{res}_{H(t)|K(t)}"] & \bigoplus_{P \in \mathcal{P}_K} K_{n-1}K_P \arrow[u, shift left, "\mathrm{res}_{H|K}"] \end{tikzcd}$$

(see [K]). Using this we construct an explicit section to d.

Lemma 1.1.
Let $u_P \in K_{n-1}K_P$ and let t_P be the residue class of t in K_P. Then

$$u_P = dN_{K_P(t)|K(t)}(t - t_P, u_P)$$

Proof
In order not to confuse the roles of K_P as residue class field and as base extension, let $\varphi : K_P \to H$ be an isomorphism over K. Then the statement reads as

$$u_P = dN_{H(t)|K(t)}(\{t - \varphi(t_P), \varphi(u_P)\})$$

Note that the composition $K_{n-1}H \to K_{n-1}H_{(t-\varphi(t_P))} \overset{\mathrm{cor}_{H|K}}{\longrightarrow} K_{n-1}K_P$ is the isomorphism induced by the inverse of φ. Hence

$$\begin{aligned} dN_{H(t)|K(t)}(\{t - \varphi(t_P), \varphi(u_P)\}) =& \mathrm{cor}_{H|K} \circ d(\{t - \varphi(t_P), \varphi(u_P)\}) \\ &= \mathrm{cor}_{H|K}(\varphi(u_P) \bmod (t - \varphi(t_P))) = u_P. \end{aligned}$$

qed.

Now let $F_0 \neq \mathbb{Z}/2$ be a prime field, let $F = F_0(a_1, \ldots, a_n, a)$ be pure transcendental over F_0 and let $L = F(\sqrt{a})$. The generator of $\mathrm{Gal}(L|F)$ is denoted by σ.

Proposition 1.2.
The following sequences are exact

(1.2.1) $$K_3L \overset{1-\sigma}{\longrightarrow} K_3L \overset{N_{L|F}}{\longrightarrow} K_3F$$

(1.2.2) $$K_2F/2 \overset{l(a)}{\longrightarrow} K_3F/2 \overset{\mathrm{res}_{L|F}}{\longrightarrow} K_3L/2$$

(1.2.3) $$K_2F \overset{l(-1)}{\longrightarrow} K_3F \overset{2}{\longrightarrow} K_3F$$

(1.2.4) $$K_3F \oplus U_F \overset{(\mathrm{res}_{L|F}, l(\sqrt{a}))}{\longrightarrow} K_3L \overset{1-\sigma}{\longrightarrow} K_3L$$

$$\text{where } U_F = \mathrm{Ker}(K_2F \overset{l(-1)}{\longrightarrow} K_3F)$$

Proof (Sketch)
(1.2.1): One uses Hilbert 90 for K_2 and (1.0.1) with respect to the variables a_i to reduce to the case $n = 0$. Then, if $\alpha \in \mathrm{Ker}N_{F_0(\sqrt{a})|F_0(a)}$, one uses again (1.0.1) for $F_0(\sqrt{a})|F_0$ and $F_0(a)|F_0$ to show that there exist $\beta \in K_3F_0(\sqrt{a})$ such that $\alpha - (1 - \sigma)(\beta) \in K_3F_0$. However $K_3F_0 = 0$ if F_0 is finite and $K_3\mathbb{Q} = \mathbb{Z}/2$ generated by $\{-1, -1, -1\}$, see [Mi]. In the latter case one has $\{-1, -1, -1\} = (1 - \sigma)(\{\sqrt{a}, -1, -1\})$.

(1.2.2) follows from the fact that the Galois symbol $K_3K/2 \to H^3(K, \mathbb{Z}/2)$ is an isomorphism for $K = F, L$ ([Mi; Lemma 6.2; Theorem 6.3]) and the corresponding exact sequence for Galois cohomology.

(1.2.3) can be derived from (1.2.1) in the same way as the corresponding result for K_2 (see [MS, Lemma 10.4] or [S; Lemma 3]); one also uses Lemma 1.1.

(1.2.4) will be proved in detail.
Let $\alpha \in K_3L$ such that $\sigma(\alpha) = \alpha$. Since $\mathrm{res}_{L|F} \circ N_{L|F}(\alpha) = (1+\sigma)(\alpha) \in 2K_3L$, (1.2.2) implies that

$$N_{L|F}(\alpha) = \{a, \beta\} + 2\gamma$$

for some $\beta \in K_2F$, $\gamma \in K_3F$. Replacing α by $\alpha - \mathrm{res}_{L|F}(\gamma)$ we may assume $\gamma = 0$. Put $\alpha' = \alpha - \{\sqrt{a}, \beta\}$. Then

$$(1-\sigma)(\alpha') = -\{\sqrt{a}, \beta\} + \{-\sqrt{a}, \beta\} = \{-1, \beta\}$$

On the other hand

$$(1+\sigma)(\alpha') = \mathrm{res}_{L|F} \circ N_{L|F}(\alpha') = \mathrm{res}_{L|F}(\{a, \beta\} - \{-a, \beta\}) = \{-1, \beta\}$$

Hence $2\alpha' = 0$ and (1.2.3) implies $\alpha' = \{-1, \delta\}$ for some $\delta \in K_2L$. Since

$$\{a, \beta\} = N_{L|F}(\alpha) = N_{L|F}(\alpha' + \{\sqrt{a}, \beta\}) = \{-1, N_{L|F}(\delta)\} + \{-a, \beta\}$$

we have $\beta' = \beta + N_{L|F}(\delta) \in U_F$. These facts yield

$$\begin{aligned} \alpha = \{\sqrt{a}, \beta\} + \{-1, \delta\} &= \{\sqrt{a}, \beta'\} - \{-\sqrt{a}, \delta\} - \{\sqrt{a}, \sigma(\delta)\} \\ &= \{\sqrt{a}, \beta'\} - \mathrm{res}_{L|F} \circ N_{L|F}(\{-\sqrt{a}, \delta\}). \end{aligned}$$

qed.

§ 2 Severi-Brauer Varieties

Let F be a field, $\mathrm{Char} F \neq 2$. For $a, b \in F^*$ let

$$D = D(a,b) =< A, B \mid A^2 = a, B^2 = b, AB = -AB > .$$

The Severi-Brauer variety to the quaternion algebra D is isomorphic to the quadric hypersurface X in $\mathbb{P}^3$ defined by $X_1^2 - aX_2^2 - bX_3^2 = 0$. It is well known that

$$D \simeq M_2(F) \iff X \simeq \mathbb{P}^1 \iff b \in N_{F(\sqrt{a})|F}(F(\sqrt{a})^* \iff \{a, b\} \in 2K_2F.$$

Now suppose $a \notin (F^*)^2$ and let $L = F(\sqrt{a})$. An explicit isomorphism $X_L \to \mathbb{P}^1_L$ is given by

$$\begin{aligned} &[X_1 : X_2 : X_3] \longrightarrow [(X_1 + \sqrt{a}X_2) : X_3] = [bX_3 : (X_1 - \sqrt{a}X_2)] \text{ with inverse} \\ &[S_1 : S_2] \qquad \longrightarrow [\sqrt{a}(S_1^2 + bS_2^2) : (S_1^2 - bS_2) : 2\sqrt{a}S_1S_2] \end{aligned}$$

The function $t = S_1/S_2$ is a generator of the function field of $\mathbb{P}^1_L$. In this paper we identify the function field $L(X)$ of X_L with $L(t)$ by means of the above isomorphism. Note that the action of $\mathrm{Gal}(L|F)$ is given by $t \to b/t$; in particular $N_{L(X)|F(X)}(t) = b$.

We have to use the following result of Merkur'ev and Suslin.

Proposition 2.1.

i) The sequence

$$0 \longrightarrow K_2F \xrightarrow{d} K_2F(X) \longrightarrow \bigoplus_{v\in X^{(1)}} K_1\kappa(v) \xrightarrow{\theta} K_1D \longrightarrow 0$$

is exact.

ii) For every $\alpha \in \bigoplus_{v\in X^{(1)}} K_1\kappa(v)$ there exist $v_0 \in X^{(1)}$ of degree two and $\alpha_0 \in K_1\kappa(v_0)$ such that $\alpha - \alpha_0 \in \operatorname{Im} d$.

For i) see [MS] or [S; Proposition 3]. To prove ii) represent $\theta(\alpha)$ by $x \in D^*$. Let F_x be a maximal commutative subfield of D containing x. Now choose v_0 such that $\kappa(v_0) \simeq F_x$ and take for α_0 the element corresponding to x. qed.

§ 3 Theorem B in a special case

Let $F_0 \neq \mathbb{Z}/2$ be a prime field and let $F/F_0(a_1, \ldots, a_n, a, b)$ be pure transcendental over F_0. Put $D = D(a,b)$ and let X be the Severi-Brauer variety corresponding to D. $L = F(\sqrt{a})$ is a splitting field of D.

Theorem 3.1. * The sequence

$$K_3F(X) \xrightarrow{d} \bigoplus_{v\in X^{(1)}} K_2\kappa(v) \xrightarrow{\mathcal{N}} K_2F$$

is exact.

Here $\mathcal{N}$ is induced by the norm for finite extensions. Since $\mathcal{N} = \operatorname{Nrd}\circ\theta$, Theorem 3.1 implies Theorem B in this case.

Note that, over L, the sequence of Theorem 3.1 reads as

$$\text{(3.1.1)} \qquad K_3L(t) \xrightarrow{d} \bigoplus_{p\in\mathcal{P}_L} K_2L_P \oplus K_2\kappa(w_\infty) \xrightarrow{\mathcal{N}} K_2L$$

under the identification $L(X) = L(t)$ of § 2 (w_∞ denotes the point of X_L defined by $t = \infty$). Since $\kappa(w_\infty) = L$, the exactness of (3.1.1) is clear by the exactness of (1.0.1).

* There is a simpler proof of Theorem 3.1 than the one given here: One has to compare the sequence of Theorem 3.1 via Galois symbol (which is an isomorphism for $F(X)$, see the proof of 3.1) with the spectral sequence for Galois cohomology associated to the field extension $F(X)|F$.

We have to consider the following commutative diagram

$$\begin{tikzcd}
K_3F \arrow[r] & K_3F(X) & & \\
K_3L \arrow[r] \arrow[u, "N_{L|F}"'] & K_3L(X) \arrow[r, "d_L"] \arrow[u] & \bigoplus_v K_2L_v & \\
K_3L \arrow[r] \arrow[u, "1-\sigma"'] & K_3L(X) \arrow[r, "d_L"] \arrow[u, "1-\sigma"'] & \bigoplus_v K_2L_v \arrow[r, "\mathcal{N}"] \arrow[u, "1-\sigma"'] & K_2L \\
 & K_3F(X) \arrow[r, "d_F"] \arrow[u] & \bigoplus_v K_2F_v \arrow[r, "\mathcal{N}"] \arrow[u] & K_2F \arrow[u]
\end{tikzcd}$$

Here I have changed notation a little bit. v runs everywhere (!) over the closed points of X, $F_v = \kappa(v)$ is the residue field of v and $L_v = F_v \otimes_F L$.

Lemma 3.2
Let $\alpha \in \bigoplus_v K_2F_v$ such that $\mathcal{N}(\alpha) = 0$. Then there exist $\beta \in K_3L(X)$ and $\gamma \in K_3L$ such that

i) $d_L(\beta) = \mathrm{res}_{L|F}(\alpha)$.
ii) $\mathrm{res}_{L(X)|L}(\gamma) = (1-\sigma)(\beta)$.
iii) $N_{L|F}(\gamma) = 0$.

Proof
The exactness of (3.1.1) implies the existence of β such that i) holds. Since $d_L \circ (1-\sigma)(\beta) = (1-\sigma) \circ d_L(\beta) = (1-\sigma) \circ \mathrm{res}_{L|F}(\sigma) = 0$, there exist a (unique) $\gamma \in K_3L$ such that ii) holds. Note that $N_{L|F}(\gamma)$ depends only on α. However, to prove that indeed $N_{L|F}(\gamma) = 0$ one constructs β more explicitly (I don't know a direct argument, because $K_3F \to K_3F(X)$ is not injective; e.g. $\{-1, a, b\} = 0$ in $K_3F(X)$). I use the identification $X_L \simeq \mathbb{P}^1_L$ of § 2. Let v_∞ be the closed point of X which splits over L into the points w_0, w_∞ given by $t = 0$, $t = \infty$ respectively. Let us first assume that $\alpha = (\alpha_v)_v \in \bigoplus_{v \neq v_\infty} K_2F_v$. Denote by t_v the residue class of t in L_v, $v \neq v_\infty$. Put

$$\beta = \sum_{v \neq v_\infty} N_{L_v(t)|L(t)}(\{t - t_v, \mathrm{res}_{L_v|F_v}(\alpha_v)\})$$

It is clear from Lemma 1.1 that i) holds. Since $\sigma(t) = b/t$ we also have $\sigma(t_v) = b/t_v$.

Hence

$$\begin{aligned}(1-\sigma)(\beta) &= \textstyle\sum_{v\neq v_\infty} N_{L_v(t)|L(t)}(\{\frac{t-t_v}{\sigma(t)-\sigma(t_v)}, \mathrm{res}_{L_v|F_v}(\alpha_v)\}) \\ &= \textstyle\sum_{v\neq v_\infty} N_{L_v(t)|L(t)}(\{\frac{tt_v}{-b}, \mathrm{res}_{L_v|F_v}(\alpha_v)\}) \\ &= \textstyle\sum_{v\neq v_\infty}\{\frac{-t}{b}, N_{L_v|F_v}(\alpha_v)\} + \mathrm{res}_{L(t)|L}(\gamma) \\ &= \{\tfrac{t}{b}, \mathrm{res}_{L|F}\circ\mathcal{N}(\alpha)\} + \mathrm{res}_{L(t)|L}(\gamma)\end{aligned}$$

where $\gamma = \sum_{v\neq v_\infty} N_{L_v|L}(\{t_v, \mathrm{res}_{L_v|F_v}(\alpha_v)\})$. With this choice of γ ii) and iii) hold, since $\mathcal{N}(\alpha) = 0$ and $N_{L|F}(\gamma) = \sum_v N_{F_v|F}\{b, \alpha_v\} = \{b, \mathcal{N}(\alpha)\}$.
For the general case it suffices to show $K_2F_{v_\infty} \subset \mathrm{Im}\, d_F \oplus \bigoplus_{v\neq v_\infty} K_2F_v$. Note that $\mathrm{cor}_{L|F} : K_2L_{v_\infty} \to K_2F_{v_\infty}$ induces an isomorphism $K_2\kappa(w_\infty) \to K_2F_{v_\infty}$. For $\alpha \in K_2F_{v_\infty}$ let $\alpha' \in K_2\kappa(w_\infty) = K_2L$ such that $\mathrm{cor}_{L|F}(\alpha') = \alpha$. Now, if $f \in L(X)$ is any function having a zero at w_∞ and no zero or pole at w_0, then
$\alpha - d_F\mathrm{cor}_{L(X)|F(X)}\{f, \alpha\} \in \bigoplus_{v\neq v_\infty} K_2F_v$. qed.

Proof of Theorem 3.1.
For $\alpha \in \mathrm{Ker}\,\mathcal{N}$ we have to show $\alpha \in \mathrm{Im}\, d_F$. Let β and γ be as in Lemma 3.2. By (1.2.1) there exist $\beta' \in K_3L$ such that $(1-\sigma)(\beta') = \gamma$. Replacing β by $\beta - \beta'$ we may assume $(1-\sigma)(\beta) = 0$. Hence, by (1.2.4), $\beta = \mathrm{res}_{L(X)|F(X)}(\beta'') + \{\sqrt{a}, \delta\}$ for some $\beta'' \in K_3F(X)$, $\delta \in U_{F(X)}$. (We can apply (1.2.4), since
$F(X) = \mathrm{qf}\ F_0[a_1, \ldots, a_n, a, b, X_1, X_2]/(X_1^2 - aX_2^2 - b) = F_0(a_1, \ldots, a_n, a, X_1, X_2))$. After replacing α by $\alpha - d_F(\beta'')$ and β by $\beta - \mathrm{res}_{L(X)|F(X)}(\beta'')$, we have the following situation

i) $d_L(\beta) = \mathrm{res}_{L|F}(\alpha)$.
ii) $\beta = \{\sqrt{a}, \delta\}$, $\delta \in K_2F(Y)$.
iii) $\{-1, \delta\} = 0$ in $K_3F(Y)$.

Claim
There exist $\rho \in K_2L(X)$ such that $\mathrm{cor}_{L|F}\circ d_L(\rho) = d_F(\delta)$.

Proof
Since

$$\begin{aligned}\{a, d_F(\delta)\} &= d_F(\{a, \delta\}) = d_F(\{-a, \delta\}) = d_F\circ N_{L(X)|F(X)}(\beta) \\ &= \mathrm{cor}_{L|F}\circ d_L(\beta) = \mathrm{cor}_{L|F}\circ\mathrm{res}_{L|F}(\alpha) = 2\alpha\end{aligned}$$

there exist $\mu \in \bigoplus_v K_1L_v$ such that $\mathrm{cor}_{L|F}(\mu) = d_F(\delta)$.
(use the general fact: $\{a, b\} \in 2K_2K \iff b \in N_{K(\sqrt{a})|K}(K(\sqrt{a})^*)$).
We now alter μ such that $\mathcal{N}(\mu) = 0$, i.e., $\mu \in \mathrm{Im}\, d_L$. Since $N_{L|F}\circ\mathcal{N}(\mu) = \mathcal{N}\circ d_F(\delta) = 0$, there exist by Hilbert 90 $\lambda \in K_1L$ such that $(1-\sigma)(\lambda) = \mathcal{N}(\mu)$. Let w be a rational point of X_L; the residue class field $\kappa(w)$ is a direct factor of L_v for some closed point v of X. Let $\varphi : L \to \kappa(w)$ be the natural isomorphism and put $\mu' = \mu - (1-\sigma)(\varphi(\lambda)) \in \bigoplus_v K_1L_v$. Then $\mathcal{N}(\mu') = \mathcal{N}(\mu) - (1-\sigma)(\lambda) = 0$, so there exist $\rho \in K_2L(X)$ such that $d_L(\rho) = \mu'$. The claim follows by

$$\mathrm{cor}_{L|F}\circ d_L(\rho) = \mathrm{cor}_{L|F}(\mu - (1-\sigma)(\varphi(\lambda))) = \mathrm{cor}_{L|F}(\mu) = d_F(\delta).$$

We continue the proof of Theorem 3.1.
Let $\alpha' = \alpha - d_F \circ N_{L(X)|F(X)}(\sqrt{a}, \rho)$. Since

$$\begin{aligned} 2\alpha = \{a, d_F(\delta)\} = \{a, \mathrm{cor}_{L|F} \circ d_L(\rho)\} &= d_F \circ N_{L(X)|F(X)}\{a, \rho\} \\ &= 2d_F \circ N_{L(X)|F(X)}(\{\sqrt{a}, \rho\}) \end{aligned}$$

we have $2\alpha' = 0$. The analogue to (1.2.3) for K_2 ([MS, Lemma 10.4]) implies $\alpha' = \{-1, \xi\}$ for some $\xi \oplus_v K_1F_v$. By Proposition 2.1 ii) there exist a closed point $v \in X$ of degree two, $\xi_0 \in K_1F_v$ and $\eta \in K_2F(X)$ such that $d_F\eta = \xi - \xi_0$. Then

$$\alpha = \{-1, \xi_0\} + d_F(N_{L(X)|F(X)}(\{\sqrt{a}, \rho\}) + \{-1, \eta\}).$$

Hence we may assume that α is concentrated in some point v of X of degree two, i.e., $\alpha \in K_2F_v$. Let ε be the generator of $\mathrm{Gal}(F_v|F)$. Since $N_{F_v|F}(\alpha) = \mathcal{N}(\alpha) = 0$, Hilbert 90 for K_2 implies $\alpha = (1-\varepsilon)(\lambda)$ for some $\lambda \in K_2F_v$. We consider the base extension $F \to F'$, where $F'|F$ is isomorphic to $F_v|F$. Let ε' be the generator of $\mathrm{Gal}(F'|F)$ and let v_0 and $v_1 = \varepsilon'(v_0)$ be the points over v. Moreover let $\varphi_i : F_v = F' \to \kappa(v_i)$ be the natural identification. If we put $x = \varphi_0(\lambda) - \varphi_1(\lambda) \in \oplus_v K_2F'_v$. then $\mathrm{cor}_{F'|F}(x) = (1-\varepsilon)(\lambda) = \alpha$. Now take $y \in K_3F'(X)$ such that $d_{F'}(y) = x$; this is possible since D is split over F' and $\mathcal{N}(x) = 0$. Then $\alpha = \mathrm{cor}_{F'|F}(x) = d_F \circ N_{F'(X)|F(X)}(y) \in \mathrm{Im}\, d_F$ qed.

§ 4 Proof of Theorem B (Sketch)

The hard point in the proof of Theorem B is

Theorem 4.1.
If $\alpha \in \bigoplus_{\substack{v \in X^{(1)} \\ \deg\, v=2}} K_2\kappa(v)$ and $\theta(\alpha) = 0$, then $\alpha \in \mathrm{Im}\, d$.

The general case is covered by the following two lemmas

Lemma 4.2
Every element of $\mathrm{Ker}\,\theta/\mathrm{Im}\, d$ is of order 2.

This follows by adjoining a splitting field of D of degree two and the usual transfer arguments. So we may assume that F has no extension of odd degree.

Lemma 4.3
If F has no extension of odd degree, then

$$\bigoplus_{\substack{v \in X^{(1)} \\ \deg v > 2}} K_2\kappa(v) \subset \mathrm{Im}\, d + \bigoplus_{\substack{v \in X^{(1)} \\ \deg v = 2}} K_2\kappa(v).$$

The proof is similar to that of the K_1-case in [R2].

In the following we use the notation of [Re]. One has an exact sequence

$$1 \to K_2 D \to U_D \xrightarrow{\pi} [D^*, D^*] \to 1$$

where U_D is generated by elements $c(x, y)$, $x, y \in D^*$ and $\pi(c(x, y)) = [x, y]$.
Now choose maps $\psi_0, \psi_1 : [D^*, D^*] \to D^*$ such that $[\psi_0(x), \psi_1(x)] = x$ and $\psi_i(1) = 1$. The defining relations for the $c(u, v)$ in [Re] and the Reidemeister-Schreier method [MKS] yield the following representation of $K_2 D$.

Lemma 4.4
$K_2 D$ is generated by the elements

$$d(u; x, y) = c(\psi_0(u), \psi_1(u)) \cdot c(x, y) \cdot c(\psi_0(u[x, y]), \psi_1(u[x, y]))^{-1}$$
$$u \in [D^*, D^*], x, y \in D^*$$

with the following set of defining relations:

$$\begin{array}{ll}
R_0(u, x) & d(u; x, 1 - x) = 1 \\
R_1(u, x, y, z) & d(u; xy, z) = d(u; xyx^{-1}, xzx^{-1}) \cdot d(ux[y, z]x^{-1}; x, z) \\
R_2(u, x, y, z) & d(u; x, yz) \cdot d(u[x, yz]; y, zx) \cdot d(u[xy, z]; z, xy) = 1 \\
R_3(u) & d(1; \psi_0(u), \psi_1(u)) = 1
\end{array}$$

Let

$$H_D = \langle\, h(x, y); x, y \in D^*, [x, y] = 1 \mid h(x, 1 - x) = 1;$$
$$h(x, y)h(x, z) = h(x, yz); [h(x, y), h(x', y')] = 1 \,\rangle$$

There is a natural map $\mu : H_D \to K_2 D$, sending $h(x, y)$ to $c(x, y) = d(1; x, y)$. By [RS; § 4] μ is surjective. Note that $[x, y] = 1$ implies that x and y are contained in a maximal commutative subfield of D which is unique if $x \notin F^*$ or $y \notin F^*$.
There is a bijection

$$v : \{\text{maximal commutative subfields of } D\} \xrightarrow{\approx} \{\text{closed points of } X \text{ of degree } 2\},$$

such that $\kappa(v(L)) \simeq L$.
Let $\Omega_D = \oplus_{\deg v = 2} K_2 \kappa(v) / \mathrm{Im}\, d$. One defines an homomorphism $\phi : H_D \to \Omega_D$ by $\phi(h(x, y)) = \{x, y\} \in K_2 \kappa(v(L)) \bmod \mathrm{Im}\, d$, where $L \subset D$ is a maximal commutative subfield containing x and y. It turns out that ϕ is well defined, surjective and that $\theta \circ \phi = \mu$. (I can show that ϕ is also injective, at least if F has no extension of odd degree. Theorem B then implies $H_D \cong K_2 D$). So we have a commutative diagram

$$\begin{tikzcd}
H_D \arrow[rr, "\phi"] \arrow[dr, "\mu"'] & & \Omega_D \arrow[dl, "\theta"] \\
& K_2 D \arrow[ur, dashed, bend right, "s"'] &
\end{tikzcd}$$

To prove Theorem 4.1 we construct a surjective section s as follows. For every generator $d(u;x,y)$ choose a preimage $g(u;x,y) \in H_D$. Now put $s(d(u;x,y)) = \phi(g(u;x,y))$. The problem is of course to show that s is well defined. (To guarantee surjectivity of s one takes $g(1;x,y) = h(x,y)$ if $[x,y]=1$). In any case one gets an homomorphism $s' : G_D \to \Omega_D$, where G_D is the free group generated by the $d(u;x,y)$. To show that s' vanishes on the relations of Lemma 4.4. one has to be very careful in the choice of ψ_0, ψ_1 and the $g(u,x,y)$. For a certain specific choice of s' (I don't see another way than to give explicit formulas using the method of proof of [RS; Proposition 4.1]) one shows:

Lemma

For every $u \in [D^*, D^*]; x, y, z \in D^*$ there exist

i) a rational function field $F_0(\bar{a}_1, \ldots \bar{a}_n, \bar{a}, \bar{b})$ over the prime field F_0 of F.

ii) elements $\bar{u} \in [\bar{D}^*, \bar{D}^*]; \bar{x}, \bar{y}, \bar{z} \in \bar{D}^*$, where $\bar{D} = D(\bar{a}, \bar{b})$

iii) maps $\bar{\psi}_0, \bar{\psi}_1 : [\bar{D}^*, \bar{D}^*] \to \bar{D}^*$ and an homomorphism $\bar{s} : G_{\bar{D}} \to \Omega_{\bar{D}}$ with the corresponding properties as ψ_0, ψ_1 and s'.

iv) a specialization $\rho : F[\bar{a}_1, \ldots, \bar{a}_n, \bar{a}, \bar{b}] \to F$ such that $\rho(\bar{a}) = a$, $\rho(\bar{b}) = b$, $\rho(\bar{u}) = u$, $\rho(\bar{x}) = x$, $\rho(\bar{y}) = y$, $\rho(\bar{z}) = z$.

v) a diagram of homomorphisms

$$\begin{array}{ccc} G_{\bar{D}} & \xrightarrow{\bar{s}'} & \Omega_{\bar{D}} \\ \downarrow \rho & & \downarrow \rho \\ G_D & \xrightarrow{s'} & \Omega_D \end{array}$$

which is commutative at least on the elements $d(.;.,.)$ which occur in the relations $R_0(\bar{u}, \bar{x})$, $R_1(\bar{u}, \bar{x}, \bar{y}, \bar{z})$ etc., that is $d(\bar{u}, \bar{x}, 1-\bar{x})$, $d(\bar{u}, \bar{x}\bar{y}, \bar{z})$ etc.

Using this lemma one argues as follows:

By Theorem 3.1 $\bar{s}'$ vanishes on the relations for $K_2\bar{D}$; in particular $\bar{s}'(R_0(\bar{u}; \bar{x})) = 0$, $\bar{s}'(R_1(\bar{u}; \bar{x}, \bar{y}, \bar{z}) = 0$, etc. Then v) shows $s'(R_0(u,x)) = 0$, $s'(R_1(u,x,y,z)) = 0$ etc., which is the desired conclusion.

§ 5 Proof of Hilbert 90 (Theorem A)

Theorem B and the injectivity of the reduced norm $\mathrm{Nrd} : K_2D \to K_2F$ (see [R2]) yield:

Theorem 5.1
The sequence

$$K_3F(X) \xrightarrow{d} \bigoplus_{v\in X^{(1)}} K_2\kappa(v) \xrightarrow{\mathcal{N}} K_2F$$

is exact.

We have to generalize this theorem to a product of Severi-Brauer varieties. Let $X_1, \ldots, X_n$ be a family of Severi-Brauer varieties over F of dimension 1. Put $X = X_1 \times \ldots \times X_n$ and $\hat{X}_i = X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$. Let $\bar{X}_i$ be the fiber over the generic point of $\hat{X}_i$ with respect to the natural projection. $\bar{X}_i$ is a Severi-Brauer variety over the function field $F(\hat{X}_i)$; thus we have an exact sequence as in Theorem 5.1:

$$K_3F(\bar{X}_i) \xrightarrow{d} \bigoplus_{v\in \bar{X}_i^{(1)}} K_2\kappa(v) \xrightarrow{\mathcal{N}_i} K_2F(\hat{X}_i).$$

We have $\bar{X}_i^{(1)} \subset X^{(1)}$ and a bijection $X^{(1)} \setminus \bar{X}_i^{(1)} \to \hat{X}_i^{(1)}$ induced by projection. Therefore

$$\bigoplus_{v\in X^{(1)}} K_2\kappa(v) = \bigoplus_{v\in \bar{X}_i^{(1)}} K_2\kappa(v) \oplus \bigoplus_{v\hat{X}_i^{(1)}} K_2(\kappa(X_i \times_F \kappa(v))).$$

Let $\pi_i : \bigoplus_{v\in X^{(1)}} K_2\kappa(v) \to \bigoplus_{v\in \bar{X}_i^{(1)}} K_2\kappa(v)$ be the corresponding projection. Put $N_i = \mathcal{N}_i \circ \pi_i$.

Corollary 5.2.
The sequence

$$K_3F(X) \xrightarrow{d} \bigoplus_{v\in X^{(1)}} K_2\kappa(v) \xrightarrow{(d', \oplus N_i)} \bigoplus_{v\in X^{(2)}} K_1\kappa(v) \oplus \bigoplus_{i=1}^{n} K_2F(\hat{X}_i)$$

is exact.

Proof
If $n = 1$ this is Theorem 5.1. For an induction proof let us denote $X = X^n$, $\bar{X}_i = \bar{X}_i^n$ and $N_i = N_i^n$ to make clear the dependency on n $(i \leq n)$. Consider the commutative diagram

$$\begin{tikzcd}
K_3F(X^{n-1}) \arrow[r, "d"] \arrow[d] & \bigoplus\limits_{v \in (X^{n-1})^{(1)}} K_2\kappa(v) \arrow[r, "(d', \oplus N_i^{n-1})"] \arrow[d, hook, "f"] & \bigoplus\limits_{v \in (X^{n-1})^{(2)}} K_1\kappa(v) \oplus \bigoplus\limits_{i=1}^{n-1} K_2F(\hat{X}_i^{n-1}) \arrow[d, hook, "g"] \\
K_3F(X^n) \arrow[r, "d"] \arrow[d, "\wr"] & \bigoplus\limits_{v \in (X^n)^{(1)}} K_2\kappa(v) \arrow[r, "(d', \oplus N_i^n)"] \arrow[d, "\pi_n"] & \bigoplus\limits_{v \in (X^n)^{(2)}} K_1\kappa(v) \oplus \bigoplus\limits_{i=1}^{n} K_2F(\hat{X}_i^n) \\
K_3F(\bar{X}_n^n) \arrow[r, "d"] & \bigoplus\limits_{v \in (\bar{X}_n^n)^{(1)}} K_2\kappa(v) \arrow[r, "\mathcal{N}_n"] & K_2F(\hat{X}_n^n) \arrow[u, hook]
\end{tikzcd}$$

Note that $X^{n-1} = \hat{X}_n$. The homomorphisms denoted by f and g are injective by Proposition 2.1. i).
Now let $\alpha \in \bigoplus_{v \in (X^n)^{(1)}} K_2\kappa(v)$ such that $d'(\alpha) = 0$ and $N_i^n(\alpha) = 0$; we have to show $\alpha \in \operatorname{Im} d + \operatorname{Im} f$. Since $\mathcal{N}_n \circ \pi_n(\alpha) = 0$ and the lower sequence is exact, there is a $\beta \in K_3F(X^n)$ such that $\pi_n(\alpha - d(\beta)) = 0$. So we may assume $\pi_n(\alpha) = 0$, that is

$$\alpha \in \bigoplus_{v \in (X^{n-1})^{(1)}} K_2\kappa(X_n \times_F \kappa(v)).$$

The homomorphism d' in the middle row can be written as

$$? \oplus \bigoplus_{v \in (X^{n-1})^{(1)}} K_2\kappa(X_n \times_F \kappa(v)) \xrightarrow{d'} \bigoplus_{v' \in X_n^{(1)}, v \in (X^{n-1})^{(1)}} K_1\kappa(v' \times v) \oplus ?$$

Hence $d'(\alpha) = 0$ and Proposition 2.1. i) imply

$$\alpha \in \bigoplus_{v \in (X^{n-1})^{(1)}} K_2\kappa(v) = \operatorname{Im} f.$$

qed.

Now we are ready to start the proof of Hilbert 90.

Lemma 5.3
Let $\alpha \in K_3L$ such that $N_{L|F}(\alpha) = 0$. Then there exist $r, n, m, p_{ij} \in \mathbb{N}$, $b_i \in F^*$, $\alpha_i \in K_2L$, $c_j \in F^*$ $\quad (0 \leq i \leq n, 1 \leq j \leq m)$ and $\rho \in K_2F$ such that

i) $\alpha = \sum_i \{b_i, \alpha_i\}$
ii) $b_0^r = 1$
iii) $N_{L|F}(\alpha_0) = \sum_j p_{0j}\{1 - d_j, c_j\} + r\rho$,
$N_{L|F}(\alpha_i) = \sum_j p_{ij}\{1 - d_j, c_j\}, i \geq 1$ where $d_j = \pi_i b_i^{p_{ij}}$.

The proof is completely analogous to that of [MS; Lemma 13.3]. $\square$

Let X_i be the Severi-Brauer variety associated to $D(a, b_i)$ and let $X = X_1 \times \ldots \times X_n$. $L(X)$ denotes the function field of X_L.

Lemma 5.4

There exist $\beta \in K_3L(X)$ and $\gamma \in \bigoplus_{v\in X^{(1)}} K_2\kappa(v)$ such that

i) $\mathrm{res}_{L(X)|L}(\alpha) = (1-\sigma)(\beta)$.

ii) $d\beta = \mathrm{res}_{L|F}(\gamma)$

iii) $(d', \oplus N_i)(\gamma) = 0$.

Suppose the lemma holds. Then, by iii) and Corollary 5.2, we have $\gamma = d(\delta)$ for some $\delta \in K_3F(X)$. Put $\beta' = \beta - \mathrm{res}_{L(X)|F(X)}(\delta)$. Then ii) implies $d\beta' = 0$, i.e., $\beta' \in K_3L$ and i) yields $\alpha = (1-\sigma)(\beta') \in (1-\sigma)(K_3L)$, which was to be shown.

Proof of Lemma 5.4

We identify $L(X_i)$ with $L(t_i)$ as in § 2; then $L(X) = L(t_1, \ldots, t_n)$. Moreover $\sigma(t_i) = b_i/t_i$, hence $N_{L(X)|F(X)}(t_i) = b_i$. Put $s_j = \prod_i t^{p_{ij}}$; then $N_{L(X)|F(X)}(s_j) = d_j$. Let $F_j = F[x_j]/(x_j^2 - d_j)$ and $L_j = F_j \otimes_F L$.

We have
$$\begin{aligned}\alpha &= \textstyle\sum_i\{b_i, \alpha_i\} = \sum\limits_i\{t_i, N_{L|F}(\alpha_i)\} - (1-\sigma)\sum\limits_i\{t_i, \sigma(\alpha_i)\}\\ &= \sum\limits_j\{s_j, 1-d_j, c_j\} + \{t_0^r, \rho\} - (1-\sigma)\sum\limits_i\{t_i, \sigma(\alpha_i)\}\end{aligned}$$

by Lemma 5.3. Put
$$\beta = \sum_j N_{L_j(X)|L(X)}\{x_j + s_j, 1-x_j, c_j\} + \{1+t_0^r, \rho\} - \sum_i\{t_i, \sigma(\alpha_i)\}.$$

Then $\alpha = (1-\sigma)(\beta)$, since
$$N_{L_j(X)|L(X)} \circ (1-\sigma)(\{x_j+s_j, 1-x_j\}) = N_{L_j(X)|L(X)}(\{\frac{s_j}{x_j}, 1-x_j\}) =$$
$$= \{s_j, N_{F_j|F}(1-x_j)\} = \{s_j, 1-d_j\} \quad \text{and} \quad N_{L(X)|F(X)}(t_0^r) = b_0^r = 1.$$

Denote by $P_i, P_i' \in \bigoplus_{w\in X_L^{(1)}} K_0\kappa(w)$ the canonical generators of $K_0\kappa(\{t_i = 0\})$, $K_0\kappa(\{t_i = \infty\})$, respectively; in particular $d(t_i) = P_i - P_i'$. Define $R_0 \in \bigoplus_{w\in X_L^{(1)}} K_0\kappa(w)$ and $Q_j \in \bigoplus_{w\in X_{L_j}^{(1)}} K_0\kappa(w)$ by
$$\begin{aligned}d(1+t_0^r) &= R_0 - rP_0'\\ d(x_j+s_j) &= Q_j - \sum_i p_{ij}\mathrm{res}_{L_j|L}(P_j')\end{aligned}$$

A little calculation shows
$$d\beta = \sum_j \mathrm{cor}_{L_j|L}(\{1-x_j, c_j\}\cdot Q_j) + \rho R_0 - \sum_i(\sigma(\alpha_i)P_i + \alpha_i P_i')$$

Note that $\sigma(P_i) = P_i'$, $\sigma(R_0) = R_0$ and $\sigma(Q_j) = Q_j$. In particular $R_0 \in \bigoplus_{v\in X^{(1)}} K_0\kappa(w)$

and $Q_j \in \bigoplus_{v \in X^{(1)}_{F_j}} K_0\kappa(w)$. Therefore $d\beta = \mathrm{res}_{L|F}(\gamma)$, where

$$\gamma = \sum_j \mathrm{cor}_{F_j|F}(\{1 - x_j, c_j\} \cdot Q_j) + \rho R_0 - \mathrm{cor}_{L|F}(\sum_i \alpha_i P'_i) \in \bigoplus_{v \in X^{(1)}} K_2\kappa(v).$$

It is straight forward to verify iii) for this choice of γ. qed.

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