F-MAGMAS

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`preliminary version`

Contents

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Introduction

A (classical) magma is a multiplication in the simplest sense, a set M together with a map $\mu\colon M \times M \to M$.

In Bourbaki (Algebra) such a pair (M, μ) is called a magma with μ the composition law on M [2, Chapter I, §1, 1. Definition 1, p. 1]. The term magma appears also in Bourbaki (Groupes et algèbres de Lie) [1, Chap. II, §2 Algèbres de Lie libres, p. 17] and in Serre (Lie algebras and Lie groups) [5, Chap. IV. Free Lie Algebras, 1. Free magmas, Definition 1.1, p. 18].

If M_X is the free magma on a set X, the map

$$(*) \qquad X \amalg (M_X \times M_X) \to M_X$$

given by inclusion and multiplication is bijective [5, Properties 2), p. 18]. A similar fact holds for multi-magmas as described in [4], see [4, (1.3), p. 6].

The decomposition $M_X = X \amalg M_X^2$ is immediate from the explicit construction of M_X in [5], but can be also directly deduced from the universality of $X \to M_X$. Namely one may define right away on $X \amalg M_X^2$ the structure of a magma (that is, a multiplication) and the universality of M_X gives a map $M_X \to X \amalg M_X^2$ yielding the inverse of $(*)$.

The starting point of this text was to formalize this argument. We ended up with a very simple generalization of magmas, F-magmas. Here F is an endofunctor on a category $\mathcal{C}$ and an F-magma is an object M together with a morphism $\mu_M\colon F(M) \to M$.

The basic idea to construct the free F-magma on an object X of $\mathcal{C}$ is to take the limit of a straightforward iteration, see Summary (4.4). The rest of the paper arose from that.

If $\mathcal{C}$ has colimits and F preserves filtered colimits there are universal F-magmas and the free F-magma on an object. Further, the bijectivity of $(*)$ generalizes to the F-decomposition (4.2).

Interestingly, in the case of classical magmas the construction of free magmas is different from that in [5]. The result is the same of course, but the constructions yield different filtrations. See Example (5.2) and also Example (5.3).

The dual notion of an F-comagma appears naturally when constructing universal F-magmas. I haven't looked much into F-comagmas themselves and further possible interplays with F-magmas.

There is an apparent formal similarity of convolution-stable morphisms between comagmas and magmas (see §2 and Proposition (3.3)) to twisting morphisms for differential graded associative (co)algebras [3, Chapter 2, Twisting Morphisms, p. 37]. Again, I haven't looked into this further.

General provisions

The general framework is a category $\mathcal{C}$ and an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

Beginning in §3 we assume that colimits (aka direct limits) of the form

$$L = \operatorname*{Lim}_{k\to\infty} X_k$$

exist in $\mathcal{C}$ and that F preserves such limits:

$$F(L) = \operatorname*{Lim}_{k\to\infty} F(X_k)$$

From §4 on we assume that $\mathcal{C}$ has coproducts $X \amalg Y$ and an initial object 0 (that is, $\operatorname{Hom}_{\mathcal{C}}(0, X)$ consists of single element). The latter is not really necessary, see Remark (4.3).

The basic example is the category **Sets** of sets and $F(Z) = Z^2$. Here $X \amalg Y$ is disjoint union and $0 = \emptyset$.

Another example is the category of R-modules for some ring R and $F(Z) = Z^{\otimes 2}$. Here $X \amalg Y = X \oplus Y$ is the direct sum and 0 is, well, 0.

§1. Magmas

(1.1) Definition.

An F-*magma* is a pair (M, μ) consisting of an object M of $\mathcal{C}$ and a $\mathcal{C}$-morphism

$$\mu\colon F(M) \to M$$

An F-*comagma* is a pair (A, δ) consisting of an object A of $\mathcal{C}$ and a $\mathcal{C}$-morphism

$$\delta\colon A \to F(A)$$

(1.2) Examples. In **Sets** let $F(Z) = Z^2$. Then an F-magma is a magma in the classical sense, consisting of a set M and a map $M^2 \to M$, see [5, p. 18].

In **Sets** let

$$F(Z) = \coprod_{n\geq 2} Z^n$$

Then an F-magma is a multi-magma in the sense of [4].

Let R be a ring and let $F(V) = V^{\otimes 2}$ in the category of R-modules. Then an F-magma is an R-algebra (non-unital, non-associative, non-commutative).[1] Similarly, an F-comagma is an R-coalgebra.

In the following (until §4) the functor F is fixed and we call an F-magma simply a magma. Similarly for comagmas.

A magma is mostly written in the form $M = (M, \mu_M^F) = (M, \mu_M)$ and μ_M^F is called the F-*multiplication* of M. Similarly for comagmas, which appear as $A = (A, \delta_A^F) = (A, \delta_A)$ with δ_A^F called the F-*diagonal* of A.

A homomorphism of magmas is a $\mathcal{C}$-morphism f such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & N \\ {\scriptstyle \mu_M}\uparrow & & \uparrow{\scriptstyle \mu_N} \\ F(M) & \xrightarrow{F(f)} & F(N) \end{array}$$

[1] The prefix "non-" stands for "not required".

is commutative. We denote by

$$\mathrm{Hom}_F(M,N) = \{\, f \in \mathrm{Hom}_{\mathcal{C}}(M,N) \mid f\mu_M = \mu_N F(f)\,\}$$

the set of magma homomorphisms $M \to N$.

If $f \in \mathrm{Hom}_F(M,N)$ is invertible in $\mathcal{C}$, then $f^{-1} \in \mathrm{Hom}_F(N,M)$.

If M is a magma, then $F(M)$ is a magma with

$$\mu_{F(M)} = F(\mu_M)$$

Obviously $\mu_M \in \mathrm{Hom}_F(F(M),M)$.

Similarly, if (A,δ_A) is a comagma, so is $(F(A),F(\delta_A))$. (We don't elaborate much on comagma homomorphisms, as there is no real need for this.)

A magma M is called *stable* if μ_M is an isomorphism. For a stable magma M the magma $F(M)$ is stable as well.

A magma M is called *universal* if for any magma N the set $\mathrm{Hom}_F(M,N)$ has exactly one element. In other words, M is an initial object in the category of magmas.

A key fact is that universal magmas are stable:

(1.3) Lemma. *If M is a universal magma, then μ_M is an isomorphism.*

Proof: Let $s\colon M \to F(M)$ be the unique magma homomorphism. Then $\mu_M s = \mathrm{id}_M$ by uniqueness. Moreover

$$s\mu_M = \mu_{F(M)}F(s) = F(\mu_M)F(s) = F(\mu_M s) = F(\mathrm{id}_M) = \mathrm{id}_{F(M)} \qquad \square$$

(It was this computation which started this paper.)

§2. **Convolutions**

Let A be a comagma and let M be a magma.

(2.1) Definition. The self-map

$$c_F\colon \mathrm{Hom}_{\mathcal{C}}(A,M) \to \mathrm{Hom}_{\mathcal{C}}(A,M)$$
$$c_F(f) = \mu_M F(f)\delta_A$$

is called *convolution*.

A $\mathcal{C}$-morphism $f\colon A \to M$ is called *c-stable* (convolution-stable) if $c_F(f) = f$. We denote by

$$S_F(A,M) = \{\, f \in \mathrm{Hom}_{\mathcal{C}}(A,M) \mid c_F(f) = f\,\}$$

the set of c-stable $\mathcal{C}$-morphisms $A \to M$.

(2.2) Example. Let $F(V) = V^{\otimes 2}$ in the category of R-modules. Then $c_F(f) = f * f$ is the convolution square of an R-module homomorphism from an R-coalgebra to an R-algebra (see for instance [3, 1.6 Convolution, p. 32]).

We use the notations $(k,h \geq 0)$

$$\mu\colon \mathrm{Hom}_{\mathcal{C}}(F^k(A),F^{h+1}(M)) \to \mathrm{Hom}_{\mathcal{C}}(F^k(A),F^h(M))$$
$$\mu(f) = \mu_{F^h(M)}f = F^h(\mu_M)f$$
$$\delta\colon \mathrm{Hom}_{\mathcal{C}}(F^{k+1}(A),F^h(M)) \to \mathrm{Hom}_{\mathcal{C}}(F^k(A),F^h(M))$$
$$\delta(f) = f\delta_{F^k(A)} = fF^k(\delta_A)$$

These maps and

$$\begin{gathered} F\colon \operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M)) \to \operatorname{Hom}_{\mathcal{C}}(F^{k+1}(A), F^{h+1}(M)) \\ f \mapsto F(f) \end{gathered}$$

commute whenever the composites are defined. More precisely, on

$$\operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M))$$

one has

$$\begin{aligned} \mu\delta &= \delta\mu && (k, h \geq 1) \\ \mu F &= F\mu && (k \geq 0, h \geq 1) \\ \delta F &= F\delta && (k \geq 1, h \geq 0) \end{aligned}$$

For instance,

$$(\mu F)(f) = \mu_{F^h(M)} F(f) = F(\mu_{F^{h-1}(M)}) F(f) = F(\mu_{F^{h-1}(M)} f) = (F\mu)(f)$$

Note that F, μF, δF and the convolution

$$c = \mu\delta F = \delta\mu F$$

are defined on $\operatorname{Hom}_{\mathcal{C}}(F^k(A), F^h(M))$ for $k, h \geq 0$.

In particular, the diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(A, M) \arrow[r, shift left, "\mu F"] \arrow[d, shift left, "\delta F"] \arrow[dr, "Fc"] & \operatorname{Hom}_{\mathcal{C}}(F(A), M) \arrow[l, shift left, "\delta"] \arrow[d, shift left, "\delta F"] \\ \operatorname{Hom}_{\mathcal{C}}(A, F(M)) \arrow[r, shift left, "\mu F"] \arrow[u, shift left, "\mu"] & \operatorname{Hom}_{\mathcal{C}}(F(A), F(M)) \arrow[l, shift left, "\delta"] \arrow[u, shift left, "\mu"] \end{tikzcd}$$

yields 4 commutative square diagrams (one for each corner) and 2 commutative triangles. On the c-stable subsets (defined by $\mu\delta F = \mathrm{id}$) these induce bijections

$$\begin{tikzcd} S_F(A, M) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, leftrightarrow, "\simeq"'] \arrow[dr, "F"] & S_F(F(A), M) \arrow[d, leftrightarrow, "\simeq"] \\ S_F(A, F(M)) \arrow[r, leftrightarrow, "\simeq"] & S_F(F(A), F(M)) \end{tikzcd}$$

§3. Limits

For a comagma A let

$$L(F, A) = \operatorname*{Lim}_{k \to \infty} (F^k(A), F^k(\delta_A))$$

and let

$$\begin{gathered} j_k\colon F^k(A) \to L(F, A) \\ j_k = j_{k+1} F^k(\delta_A) \end{gathered} \qquad (k \geq 0)$$

be the corresponding morphisms. In particular, j_0 is a morphism $A \to L(F, A)$.

Thus a sequence of $\mathcal{C}$-morphisms

$$\varphi_k \colon F^k(A) \to N$$

with

$$\varphi_k = \varphi_{k+1} F^k(\delta_A)$$

defines a $\mathcal{C}$-morphism

$$\varphi = \operatorname*{Lim}_{k\to\infty} \varphi_k \colon L(F,A) \to N$$

and any $\mathcal{C}$-morphism $\varphi\colon L(F,A) \to N$ is of this form by taking $\varphi_k = \varphi j_k$.

We consider $L(F,A)$ as magma with

$$\mu_{L(F,A)} \colon F(L(F,A)) = \operatorname*{Lim}_{k\to\infty} (F^{k+1}(A), F^{k+1}(\delta_A)) \to L(F,A)$$

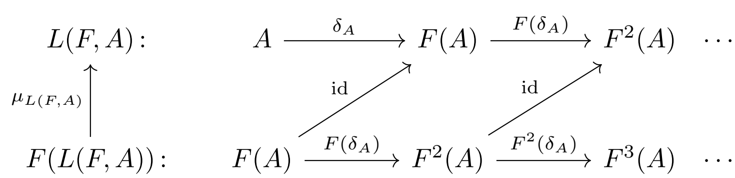
the colimit of the sequence

$$j_{k+1} \colon F^{k+1}(A) \to L(F,A)$$

so that

$$\mu_{L(F,A)} F(j_k) = j_{k+1}$$

This means that $\mu_{L(F,A)}$ is induced by the identity maps on $F^{k+1}(A)$:

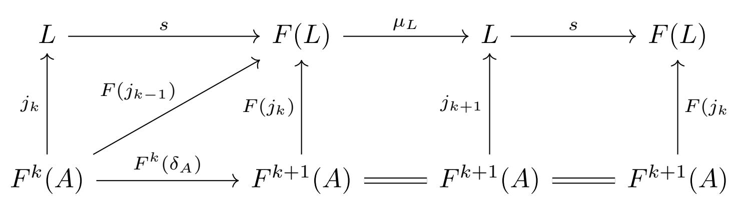


(3.1) Lemma. *The magma $L(F,A)$ is stable.*

Proof: The inverse s of $\mu_{L(F,A)}$ is the colimit of the sequence

$$s j_k = F(j_{k-1}) \colon F^k(A) \to F(L(F,A)) \qquad (k \geq 1)$$

as can be seen from the commutative diagram



with $L = L(F,A)$. □

On the other hand, if (M, μ_M) is stable, then

$$(M, \mu_M) = L(F, (M, \mu_M^{-1}))$$

since all $F^k(\mu_M^{-1})$ are isomorphisms.

(3.2) Remark. If $F = \mathrm{id}_{\mathcal{C}}$ is the identity functor, then a comagma is just an endomorphism $\delta \in \mathrm{End}_{\mathcal{C}}(A)$. In this case $L(F, A)$ is the standard construction to invert δ. For example, in the category of abelian groups consider $L = L(\mathrm{id}, (\mathbf{Z}, \delta))$ with δ the multiplication by 2. Then

$$L = \varinjlim(\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{2} \cdots) = \mathbf{Z}[\tfrac{1}{2}]$$
$$j_k(x) = \tfrac{1}{2^k}x$$
$$\mu_L(x) = \tfrac{1}{2}x$$

(3.3) Proposition. *For any magma M, the map*

$$\mathrm{Hom}_{\mathcal{C}}(L(F, A), M) \to \mathrm{Hom}_{\mathcal{C}}(A, M)$$
$$\varphi \mapsto \varphi j_0$$

induces a bijection of subsets

$$\mathrm{Hom}_F(L(F, A), M) \to S_F(A, M)$$

Proof: For a $\mathcal{C}$-morphism $\varphi\colon L(F, A) \to M$ the corresponding sequence

$$\varphi_k = \varphi j_k \colon F^k(A) \to M$$

satisfies

(3.4) $$\varphi_k = \varphi_{k+1} F^k(\delta_A)$$

If φ is a magma homomorphism, the commutative diagrams

$$\begin{array}{rccccc}
\varphi_{k+1}\colon & F^{k+1}(A) & \xrightarrow{j_{k+1}} & L(F, A) & \xrightarrow{\varphi} & M \\
& \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mu_{L(F,A)}} & & \uparrow{\scriptstyle \mu_M} \\
F(\varphi_k)\colon & F^{k+1}(A) & \xrightarrow{F(j_k)} & F(L(F, A)) & \xrightarrow{F(\varphi)} & F(M)
\end{array}$$

yield

(3.5) $$\varphi_{k+1} = \mu_M F(\varphi_k)$$

Together with (3.4) this implies

$$\varphi_k = \mu_M F(\varphi_k) F^k(\delta_A) = c_F(\varphi_k)$$

so that $\varphi_k \in S_F(F^k(A), M)$.

In particular $\varphi j_0 = \varphi_0 \in S_F(A, M)$. On the other hand, (3.5) shows that $\varphi = \varinjlim \varphi_k$ is determined by φ_0. (One has $\varphi_k = (\mu F)^k(\varphi_0)$ in the notation of §2.) $\square$

It follows that $L(F, A)$ is universal if and only if $S_F(A, N)$ consists of a single element.

(3.6) Example. A constant functor is a functor with constant value on objects and sending a morphism to the identity.

Let $F(Z) = Y$ be a constant functor. Then an F-magma is a pair $(M, Y \to M)$ and (Y, id_Y) is universal.

In this case $L(F, A) = (Y, \mathrm{id}_Y)$ for any A. Indeed, $F^k(A) = Y$, $F^k(\delta_A) = \mathrm{id}_Y$ for $k \geq 1$. Moreover, $S_F(A, N)$ consists of $\mu_N \delta_A$.

An initial object 0 of $\mathcal{C}$ is a comagma with δ_0 the unique morphism $0 \to F(0)$.

(3.7) Corollary. *Let* 0 *be an initial object of* $\mathcal{C}$. *Then* $L(F, 0)$ *is a universal* F*-magma.*

Proof: The unique element of $\mathrm{Hom}_{\mathcal{C}}(0, M)$ is clearly c-stable and therefore the only element of $S_F(0, M)$. The claim follows from Proposition (3.3). □

§4. Free magmas

Given the endofunctor F and an object X of $\mathcal{C}$, define the endofunctor F_X of $\mathcal{C}$ by

$$F_X(M) = X \amalg F(M)$$
$$F_X(f) = \mathrm{id}_X \amalg F(f)$$

In other words, F_X is the coproduct of the constant functor with value X and F. Or, if Φ_X is the endofunctor

$$\Phi_X(Y) = X \amalg Y$$
$$\Phi_X(f) = \mathrm{id}_X \amalg f$$

then F_X is the composite

$$F_X = \Phi_X \circ F$$

It follows that F_X commutes with colimits $\operatorname*{Lim}_{k\to\infty}$ since F and Φ_X do.

An F_X-magma M consists of an F-magma M and a $\mathcal{C}$-morphism $\lambda_M\colon X \to M$:

$$\mu_M^{F_X} = (\lambda_M, \mu_M^F)\colon X \amalg F(M) \to M$$

In the following definition we assume the existence of an initial object 0, but see Remark (4.3).

(4.1) Definition. The *free* F*-magma on* X is the universal F_X-magma

$$M(F, X) = L(F_X, 0)$$

Hence (abbreviating $M_X = M(F, X)$)

$$(M_X, \mu_{M_X}^F, \lambda_{M_X})$$

is universal among triples

$$(M, \mu\colon F(M) \to M, \lambda\colon X \to M)$$

Since universal magmas are stable (Lemma (1.3)) it follows that

$$\textbf{(4.2)} \qquad X \amalg F(M_X) \xrightarrow{(\lambda_{M_X}, \mu_{M_X}^F)} M_X$$

is an isomorphism. We call (4.2) the F*-decomposition* of the free F-magma on X.

(4.3) Remark. One has

$$L(F_X, 0) = L(F_X, F_X^h(0)) \qquad (h \geq 0)$$

The F_X-comagma

$$F_X(0) = X \amalg 0 = X$$

can be defined without reference to 0 as follows.

One considers X as F_X-comagma with

$$\delta_X: X \to F_X(X) = X \amalg F(X)$$

the inclusion of the first term.

For an F_X-magma M and a $\mathcal{C}$-morphism $f\colon X \to M$ one finds

$$c_{F_X}(f) = \mu_M^{F_X} F_X(f)\delta_X = \lambda_M$$

as can be seen by following the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad c_{F_X}(f) \quad} & M \\ {\scriptstyle \delta_X}\big\downarrow & & \big\uparrow {\scriptstyle \mu_M^{F_X} = (\lambda_M, \mu_M^F)} \\ X \amalg F(X) & \xrightarrow[= \mathrm{id}_X \amalg F(f)]{F_X(f)} & X \amalg F(M) \end{array}$$

Hence $S_{F_X}(X, M) = \{\lambda_M\}$ and $L(F_X, X)$ is a universal F_X-magma.

Therefore one may generally define the free F-magma on X as

$$M(F, X) = L(F_X, X)$$

(4.4) Summary. The free F-magma on X is the limit of terms

$$\begin{aligned} M_1 &= X \\ M_2 &= X \amalg F(X) \\ M_3 &= X \amalg F(M_2) \\ M_4 &= X \amalg F(M_3) \\ &\cdots \end{aligned}$$

with the transitions given by the identity on X and the F-transform of the preceding transition morphism.

§5. Examples

(5.1) Example. If $F = \mathrm{id}_{\mathcal{C}}$ is the identity functor, then

$$F_X^k(0) = \coprod_{h=1}^{k} X = X \times \{1, \dots, k\}$$

and the free F-magma $M_X = M(\mathrm{id}_{\mathcal{C}}, X)$ on X is

$$M_X = X \times \mathbf{N} = X \amalg X \amalg X \amalg \cdots$$

with $\lambda_{M_X} = \mathrm{id}_X \times \{1\}$ the inclusion of the first term and $\mu_{M_X} = \mathrm{id}_X \times \{+1\}$ the shift to the right.

For a triple $(M, \mu\colon M \to M, \lambda\colon X \to M)$ the corresponding morphism $\varphi\colon M_X \to M$ is given by

$$\varphi | X \times \{k\} = \mu^{k-1}\lambda$$

The F-decomposition (4.2) is the isomorphism

$$X \amalg (X \times \mathbf{N}) \xrightarrow{\simeq} X \times \mathbf{N}$$

induced from the bijection

$$\mathbf{N}_0 \xrightarrow{+1} \mathbf{N}$$

(5.2) Example. We consider the case of classical magmas. So let $\mathcal{C} = \textbf{Sets}$ and $F(Z) = Z^2$ (this stands of course for $F(Z) = Z \times Z$, $F(f) = f \times f$).

Then the free magma $M(F, X)$ on X is the limit of

$$\begin{aligned} M_1 &= X \\ M_2 &= X \amalg X^2 \\ M_3 &= X \amalg (X \amalg X^2)^2 \\ &= X \amalg X^2 \amalg (X \times X^2) \amalg (X^2 \times X) \amalg (X^2 \times X^2) \\ M_4 &= X \amalg (X \amalg (X \amalg X^2)^2)^2 \\ &\dots \end{aligned}$$

This limit is actually a union with $M_{k+1} \setminus M_k$ consisting of the parenthesized expressions with maximal depth of nested paren pairs equal to k (here $X^2 = (X \times X)$ counts for 1 pair).

In contrast, in [5, p. 18] (also in [2, Chapter I, §7.1, p. 81], [1, p. 17]) the free magma on X (denoted as M_X) is defined as follows:

$$\begin{aligned} X_1 &= X \\ X_n &= \coprod_{p+q=n} X_p \times X_q \qquad (n \geq 2) \\ M_X &= \coprod_{n=1}^{\infty} X_n \end{aligned}$$

This description corresponds to the filtration by length with first terms

$$\begin{aligned} X_1 &= X \\ X_2 &= X^2 \\ X_3 &= (X \times X^2) \amalg (X^2 \times X) \\ X_4 &= (X \times (X \times X^2)) \amalg (X \times (X^2 \times X)) \\ &\quad \amalg (X^2 \times X^2) \\ &\quad \amalg ((X \times X^2) \times X) \amalg ((X^2 \times X) \times X) \end{aligned}$$

The filtration by length is more natural and convenient for classical magmas. However for general F there is no notion similar to length.

The F-decomposition (4.2) is

$$M_X = X \amalg M_X^2$$

as noted in [5, Properties 2), p. 18] and in [2, p. 81], [1, p. 17].

(5.3) Example. Similar remarks apply to multi-magmas (see (1.2)). Here the F-decomposition (4.2) is

$$M_X = X \amalg \coprod_{n \geq 2} M_X^n$$

In [4] it is called arity-decomposition ([4, (1.3), p. 6]) and an indispensable tool for inductive definitions and proofs.

(5.4) Example. More generally, let P be a set of ordered finite nonempty sets and consider in **Sets** the endofunctor

$$F(Z) = \coprod_{I \in P} Z^I$$

This set up includes Examples (5.1) (for $\mathcal{C} = \mathbf{Sets}$), (5.2), (5.3).

The general construction of $M(F, X)$ (Definition (4.1)) establishes the existence of free F-magmas right away without much ado about the details of F.

One way to construct the free F-magma M_X directly is to consider parenthetical expressions with nested "I-paren pairs" looking like this

$$(\alpha_1 \cdots \alpha_{|I|})_I$$

Alternatively, M_X can be identified with the set of isomorphism classes of finite rooted planar trees with labels as follows: Each leaf (a vertex of valency 1, excluding the root) is marked with an element of X. Further, for each inner node (a vertex with valency ≥ 2) the ordered set of incoming edges (coming from a leaf) is identified with some $I \in P$ (so the valency of the node is $|I| + 1$).

If $|I| = 1$ for some $I \in P$, then the subsets of M_X of a given number of leaves (the length in the preceding examples) are not finite already for $|X| = 1$ since any number of nodes of valency 2 is possible.

In the particular case

$$F(Z) = Z$$

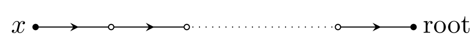
(the case $P = \{\{*\}\}$) the element

$$(x, k) \in M_X = X \times \mathbf{N}$$

(see Example (5.1)) is represented in terms of parenthetical expressions by

$$(\cdots((x))\cdots)$$

with $k - 1$ paren pairs and in terms of trees by



with $k - 1$ inner nodes.

References

[1] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie*, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1349, Hermann, Paris, 1972. MR 573068 2, 10

[2] N. Bourbaki, *Algebra I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)]. MR 1727844 2, 10

[3] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. MR 2954392 2, 4

[4] M. Rost, *The associahedral chain complex and the cubical associahedron*, Notes, 2026, `www.math.uni-bielefeld.de/~rost/assoc.html#assoc2` [pdf]. 2, 3, 10

[5] J.-P. Serre, *Lie algebras and Lie groups*, Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 2006, 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. MR 2179691 2, 3, 10

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