# THE MOTIVE OF A PFISTER FORM

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`preliminary version`

## Contents



## Introduction

This is a draft which deserves some reorganization and supplements. However it contains all details concerning the construction and properties of the motive.

Unfortunately only very late I noticed that Proposition 1 and Proposition 9 with Corollaries should have been formulated for motives instead only for varieties. But this may be complemented without much difficulty.

## 1. Correspondences

In this section we recall some basic facts about correspondences and Grothendieck motives. The basic reference is [1], in particular [1, Example 16.1.12].

Let $k$ be a fixed ground field.

1.1. **Algebra of correspondences.** By a variety $X$ we understand a separated scheme of finite type over $k$. We denote by $\mathrm{CH}_p(X)$ the Chow group of $p$-dimensional cycles on $X$. If $X$ is smooth, we denote by $\mathrm{CH}^p(X)$ the Chow group of $p$-codimensional cycles on $X$. In case $X$ is irreducible, one has $\mathrm{CH}^p(X) = \mathrm{CH}_{d-p}(X)$ where $d = \dim X$.

*Date*: November 12, 1998.

Let $\overline{\mathcal{V}}$ be the following additive category: The objects of $\overline{\mathcal{V}}$ are the smooth proper varieties over $k$. The morphism groups in $\overline{\mathcal{V}}$ are

$$\operatorname{Hom}_{\overline{\mathcal{V}}}(X,Y)=\operatorname{CH}_*(X\times Y)=\oplus_p\operatorname{CH}_p(X\times Y).$$

The composition law in $\overline{\mathcal{V}}$ is given by the composition of correspondences:

$$\operatorname{CH}_*(X\times Y)\times\operatorname{CH}_*(Y\times Z)\to\operatorname{CH}_*(X\times Z),$$
$$(f,g)\mapsto g\circ f=\pi_*\big(\Delta^*(f\times g)\big),$$

with

$$\Delta\colon X\times Y\times Z\to X\times Y\times Y\times Z,\qquad \Delta(x,y,z)=(x,y,y,z),$$
$$\pi\colon X\times Y\times Z\to X\times Z,\qquad \pi(x,y,z)=(x,z).$$

The *transpose* of $f\in\operatorname{Hom}_{\overline{\mathcal{V}}}(X,Y)$ is $f^t=\tau_*(f)\in\operatorname{Hom}_{\overline{\mathcal{V}}}(Y,X)$ where $\tau\colon X\times Y\to Y\times X$, $\tau(x,y)=(y,x)$. The category $\overline{\mathcal{V}}$ is endowed with the duality ${}^t\colon\overline{\mathcal{V}}\to\overline{\mathcal{V}}^{\mathrm{op}}$ given by $X^t=X$ on objects and by the transpose on morphisms.

One has $\operatorname{Hom}_{\overline{\mathcal{V}}}(\operatorname{Spec}k,X)=\operatorname{CH}_*(X)$. This way $X\mapsto\operatorname{CH}_*(X)$ defines a functor on $\overline{\mathcal{V}}$ (by composition of morphisms).

Let $\mathcal{V}=\mathcal{V}(k)$ be the following subcategory of $\overline{\mathcal{V}}$: The objects of $\mathcal{V}$ are the objects of $\overline{\mathcal{V}}$. The morphism groups in $\mathcal{V}$ are

$$\operatorname{Hom}(X,Y)=\oplus_i\operatorname{CH}_{\dim X_i}(X_i\times Y)=\oplus_j\operatorname{CH}^{\dim Y_j}(X\times Y_j)\subset\operatorname{Hom}_{\overline{\mathcal{V}}}(X,Y).$$

Here $X_i$ resp. $Y_j$ are the connected components of $X$ resp. $Y$. The composition law in $\mathcal{V}$ is induced from the composition law in $\overline{\mathcal{V}}$.

We write

$$\operatorname{End}(X)=\operatorname{Hom}(X,X)$$

for the endomorphism ring of $X$. If $X$ is irreducible, then

$$\operatorname{End}(X)=\operatorname{CH}_d(X\times X)=\operatorname{CH}^d(X\times X),\quad d=\dim X.$$

The assignments $X\mapsto\operatorname{CH}_p(X)$ resp. $X\mapsto\operatorname{CH}^p(X)$ are covariant resp. contravariant functors on $\mathcal{V}$. For $f\in\operatorname{Hom}(X,Y)$ we denote the associated maps by $f_*\colon\operatorname{CH}_p(X)\to\operatorname{CH}_p(Y)$ resp. $f^*\colon\operatorname{CH}^p(Y)\to\operatorname{CH}^p(X)$.

For a morphism $f\colon X\to Y$ of varietes over $k$ we denote by the same letter the class of its graph:

$$f=[\operatorname{Graph}(f)]\in\operatorname{Hom}(X,Y).$$

In this case the maps $f_*$ resp. $f^*$ are the standard push forward resp. pull back maps.

1.2. **Motives.** By a *motive* we understand a pair $(X,p)$ with $X$ an object in $\mathcal{V}$ and $p\in\operatorname{End}(X)$ a projector: $p\circ p=p$. The Chow groups of $(X,p)$ are defined by

$$\operatorname{CH}_r\big((X,p)\big)=p_*\big(\operatorname{CH}_r(X)\big),$$
$$\operatorname{CH}^r\big((X,p)\big)=p^*\big(\operatorname{CH}^r(X)\big).$$

The category $\mathcal{M}$ is defined as follows: Its objects are the pairs $(X,p)$. Its morphism groups are

$$\operatorname{Hom}\big((X,p),(Y,q)\big)=q\circ\operatorname{Hom}(X,Y)\circ p\subset\operatorname{Hom}(X,Y).$$

The composition of morphisms in $\mathcal{M}$ is induced from the composition law in $\mathcal{V}$.

Associating to $X$ the motive $(X, \mathrm{id}_X)$ identifies $\mathcal{V}$ as a full subcategory of $\mathcal{M}$. We simply write $X = (X, \mathrm{id}_X)$. The functors $M \mapsto \mathrm{CH}_p(M)$ resp. $M \mapsto \mathrm{CH}^p(M)$ extend the corresponding functors on $\mathcal{V}$.

The sum and the product in $\mathcal{M}$ (and in $\mathcal{V}$ and $\overline{\mathcal{V}}$) are defined by disjoint union and product:

$$(X, p) \oplus (Y, q) = (X \cup Y, p + q),$$
$$(X, p) \otimes (Y, q) = (X \times Y, p \times q).$$

1.3. **The Tate motive.** The endomorphism ring of $\mathbb{P}^1$ is

$$\mathrm{End}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z},$$

generated by the cycle classes $p_0 = [\mathbb{P}^1 \times P]$ and $p_1 = [P \times \mathbb{P}^1]$ with $P$ a $k$-point on $\mathbb{P}^1$. The endomorphisms $p_0$, $p_1$ are orthogonal projectors. One has $(\mathbb{P}^1, p_0) = \operatorname{Spec} k$.

The *Tate motive* is the motive $L = (\mathbb{P}^1, p_1)$. For $i \geq 0$ we denote by $L^{\otimes i}$ the $i$-fold product $L \otimes \cdots \otimes L$ with $L^{\otimes 0} = \operatorname{Spec} k$. One has

$$\mathrm{CH}_p\big(M \otimes L^{\otimes i}\big) = \mathrm{CH}_{p-i}(M)$$

If $X$ is irreducible, then

$$\mathrm{Hom}(X \otimes L^{\otimes i}, Y \otimes L^{\otimes j}) = \mathrm{CH}_{(\dim X)+i-j}(X \times Y)$$

For projective spaces there is the following decomposition in $\mathcal{M}$:

$$\mathbb{P}^n = \oplus_{i=0}^n L^{\otimes i}.$$

1.4. **Splitting off a point.** Let $X$ be equidimensional of dimension $d$. We denote by $[X] \in \mathrm{CH}_d(X)$ the fundamental class of $X$.

Let

$$P \in \coprod_{x \in X_{(0)}} \mathbb{Z}$$

be a zero cycle. The classes

$$[P] \in \mathrm{CH}_0(X) = \mathrm{Hom}(L^{\otimes 0}, X) = \mathrm{Hom}(X, L^{\otimes d}),$$
$$[X] \in \mathrm{CH}_d(X) = \mathrm{Hom}(L^{\otimes d}, X) = \mathrm{Hom}(X, L^{\otimes 0}),$$

define morphisms

$$L^{\otimes 0} \underset{g_X}{\overset{f_P}{\rightleftarrows}} X, \qquad L^{\otimes d} \underset{g_P}{\overset{f_X}{\rightleftarrows}} X.$$

If $P$ has degree 1, one has $g_X \circ f_P = \mathrm{id}_{L^{\otimes 0}}$ and $g_P \circ f_X = \mathrm{id}_{L^{\otimes d}}$. Then $f_P \circ g_X$, $f_X \circ g_P \in \mathrm{End}(X)$ are projectors. If additionally $d > 0$, these projectors are orthogonal. They identify $L^{\otimes 0} \oplus L^{\otimes d}$ as a direct summand of $X$ and we have a decomposition

$$X = L^{\otimes 0} \oplus (X, \pi_P) \oplus L^{\otimes d} \tag{1}$$

with $\pi_P = \mathrm{id}_X - f_P \circ g_X - f_X \circ g_P$.

Suppose that the degree map

$$\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$$

is bijective. Then the correspondences $f_P$, $g_P$, and $\pi_P$ do not depend on the choice of $P$. Moreover we have a canonical decomposition

$$\mathrm{End}(X) = \mathrm{End}(L^{\otimes 0}) \oplus \mathrm{End}\big((X,\pi_P)\big) \oplus \mathrm{End}(L^{\otimes d}).$$

1.5. **Chow groups of fibrations.** Let $X$, $B$ be smooth and proper varieties over $k$ and let $\pi\colon B\times X\to B$ be the projection. For $b\in B$ we write $X_b=\operatorname{Spec}\kappa(b)\times X$ for the fibre over $b$. Moreover, for $f\in\mathrm{End}(X)$ we denote by $f_b\in\mathrm{End}_{\mathcal{V}(\kappa(b))}(X_b)$ the element obtained by base change $k\to\kappa(b)$.

We will need the following observation:

**Proposition 1.** *Let $f\in\mathrm{End}(X)$ and suppose that*

$$(f_b)_*\big(\mathrm{CH}_p(X_b)\big)=0$$

*for all $b\in B$ and $0\le p\le\dim B$. Then*

$$f^{(1+\dim B)}\circ\mathrm{Hom}(B,X)=0.$$

(Here the power of $f$ is taken in the ring $\mathrm{End}(X)$.)

*Proof.* We use the setting of [3]. For simplicity we assume that $X$ and $B$ are irreducible of dimensions $d=\dim X$ and $e=\dim B$.

First note that the map $\mathrm{Hom}(B,X)\to\mathrm{Hom}(B,X)$, $g\mapsto f\circ g$ is given by the composite of the maps

$$\mathrm{CH}_p(B\times X)\xrightarrow{\times f}\mathrm{CH}_{p+d}(B\times X\times X\times X),\tag{2}$$

$$\mathrm{CH}_{p+d}(B\times X\times X\times X)\xrightarrow{\Delta^*}\mathrm{CH}_p(B\times X\times X),\tag{3}$$

$$\mathrm{CH}_p(B\times X\times X)\xrightarrow{g_*}\mathrm{CH}_p(B\times X),\tag{4}$$

with $\Delta(b,x,y)=(b,x,x,y)$ and $g(b,x,y)=(b,y)$.

Similarly, after replacing $B$ by $\operatorname{Spec}\kappa(b)$ and $f$ by $f_b$, the composition of these maps yield the action of $f_b$ on $\mathrm{CH}_p\big(X_b\big)$.

Let $Z=X$, $X\times X$, or $X\times X\times X$. The projections $\pi\colon B\times Z\to B$ induce spectral sequences

$$E^2_{p,q}=A_p\big(B,A_q[\pi,M]\big)\Longrightarrow A_{p+q}(B\times Z,M),\tag{5}$$

see [3, Sect. 8]. Here $M$ can be any cycle module in the sense of [3]; for our purpose we may restrict to the case $M=K_*^M$ given by Milnor's $K$-theory.

The maps (2), (3), and (4) extend to the Chow groups $A_p(B\times Z,M)$ and, similarly, to the groups $A_p(Z_b,M)$. Moreover they are compatibel with the spectral sequences (5). For the product map $\times f$ this is fairly obvious from the definitions, see [3, Sect. 14]. For the pull back map $\Delta^*$ see [3, Theorem 12.7]. For the push forward map $g_*$ see [3, Proposition 8.5.1].

The filtration on the Chow groups of $B\times X$ corresponding to the spectral sequence (5) for $Z=X$ is

$$\mathrm{CH}_{p,0}(\pi)\subset\mathrm{CH}_{p,1}(\pi)\subset\cdots\subset\mathrm{CH}_p(B\times X)$$

where $\mathrm{CH}_{p,r}(\pi)\subset\mathrm{CH}_p(B\times X)$ is the subgroup generated by the classes of cycles $V\subset Z$ with $\dim\pi(V)\le r$.

Consider the quotients

$$\overline{\mathrm{CH}}_{p,r}(\pi)=\mathrm{CH}_{p,r}(\pi)/\,\mathrm{CH}_{p,r-1}(\pi)$$

The direct summands $\widetilde{E}^2_{r,p-r}$ of the groups $E^2_{r,p-r}$ corresponding to the quotients $\overline{\mathrm{CH}}_{p,r}(\pi)$ are the cokernels of the divisor maps

$$\coprod_{b\in B_{(r+1)}} A_{p-r}(X_b, K^M_*, 1-p+r) \xrightarrow{d} \coprod_{b\in B_{(r)}} \mathrm{CH}_{p-r}(X_b).$$

The group $\overline{\mathrm{CH}}_{p,r}(\pi)$ is a quotient of $\widetilde{E}^2_{r,p-r}$. Therefore $f$ acts trivially on $\overline{\mathrm{CH}}_{p,r}(\pi)$ as long as $f_b$ acts trivially on $\mathrm{CH}_{p-r}(X_b)$ for all $b \in B$.

Since the filtration on $\mathrm{CH}_p(B \times X)$ has length $1+e$ and since $f$ acts trivially on the filtration quotients, it follows that the action of $f$ on $\mathrm{CH}_p(B\times X)$ is nilpotent of order $1+e$. □

## 2. Isotropic and split quadrics

For the remaining parts of this article, the characteristic of the ground field $k$ is different from 2. For generalities on quadratic forms and in particular Pfister forms, see [2, 4].

For a quadratic form $\varphi$ we denote by $X_\varphi$ the associated projective quadric. If not mentioned otherwise, we assume $\varphi$ to be regular so that $X_\varphi$ is smooth. One has $\dim X_\varphi = \dim \varphi - 2$.

2.1. **Motives of quadrics and Witt equivalence.** We call a quadric $X_\varphi$ *isotropic* if $\varphi$ is isotropic. A quadric $X$ is isotropic if and only if $X$ has a $k$-rational point.

The following proposition shows that the motive of a quadric $X_\varphi$ depends essentially (up to elementary operations involving the Tate motive) on the class of $\varphi$ in the Witt ring of $k$.

**Proposition 2.** *Let $\varphi = \mathbb{H} \perp \psi$ where $\mathbb{H}$ is a hyperbolic plane and let $X = X_\varphi$ and $Y = X_\psi$. Then*

$$X = L^{\otimes 0} \oplus Y \otimes L \oplus L^{\otimes d}$$

*where $d = \dim X$.*

The case $d = 0$ is easily verified and therefore we may assume $d > 0$. We choose coordinates $(u, v, y)$ such that $\varphi(u, v, y) = uv + \psi(y)$.

The proof of Proposition 2 is based on the stratification of $X$ given by the following data:

$$\begin{aligned} &Z = \{\, u = 0 \,\} \subset X, \\ &Z' = Z \setminus P, \quad P = [0, 1, 0], \\ &r\colon Z' \to Y, \quad r\big([0, v, y]\big) = [y]. \end{aligned}$$

Then $X \setminus Z \simeq \mathbb{A}^d$ with $d = \dim X$, the point $P$ is the singular point of $Z$, and $r$ is a 1-dimensional vector bundle.

Suppose for a moment that we work in a category of motives which include the motives of arbitrary (possibly non smooth and non proper) varieties. Then the data above immediately give the desired decomposition of the motive of $X$: The decomposition $X = Z \cup \mathbb{A}^d$ indicates that $X = Z \oplus L^{\otimes d}$. Similarly one would have $Z = L^{\otimes 0} \oplus Z'$. Furthermore, since $r$ is a 1-dimensional vector bundle, homotopy invariance gives $Z' = Y \otimes L$.

Constructions of such a category of motives have been announced by Morel and Voevodsky, and (in a different manner by extending the framework of [3] to a bivariant theory of cycles) by the author. However, for the sake of completeness, we continue here to work in the category $\mathcal{M}$. The arguments are of a completely formal nature starting from the described data.

Let

$$S \subset Y \times X$$

be the closure of the image of the morphism

$$Z' \xrightarrow{(r,i)} Y \times X$$

where $i\colon Z' \to X$ is the inclusion. The cycle class $f = [S] \in \mathrm{CH}_{d-1}(Y \times X)$ defines a morphism

$$Y \otimes L \xrightarrow{f} X.$$

Recall the decomposition (1). One has $g_X \circ f = 0$ and $g_P \circ f = 0$ by dimension reasons. Therefore $f$ defines actually a morphism

$$Y \otimes L \xrightarrow{\hat{f}} (X, \pi_P).$$

By Manin's identity principle [1, Example 16.1.12], $\hat{f}$ is an isomorphism, if for any $B$ in $\mathcal{V}$ and $p \geq 1$ the map

$$(\mathrm{id}_B \times \hat{f})_*\colon\ \mathrm{CH}_{p-1}(B \otimes Y) \to \mathrm{CH}_p\big(B \otimes (X, \pi_P)\big)$$

is bijective. This follows from the following two Lemmata.

**Lemma 3.** *Let $X' = X \setminus P$ and let $i\colon Z' \to X'$, $j\colon X' \to X$ be the inclusions. The following diagram is commutative*:

$$\begin{array}{ccc}
\mathrm{CH}_{p-1}(B \otimes Y) & \xrightarrow{f_*} & \mathrm{CH}_p(B \otimes X) \\
{\scriptstyle r^*}\big\downarrow & & \big\downarrow{\scriptstyle j^*} \\
\mathrm{CH}_p(B \otimes Z') & \xrightarrow{i_*} & \mathrm{CH}_p(B \otimes X')
\end{array}$$

*The pull back map $r^*$ is bijective.*

*Proof.* The bijectivity of $r^*$ follows from the fact that $r$ is a 1-dimensional vector bundle.

Let $S' = S \cap (Y \times X')$. The scheme $S'$ is just the image of the closed immersion $Z' \to Y \times X'$.

The map $j^* \circ f_*$ is given by taking cross product with $X'$, intersecting with $B \times S'$ and taking push forward along $Y$. Now $S' \subset Y \times Z'$ and $Z' \subset X'$ is a smooth hyperplane. Therefore we can also take first cross product with $Z'$, intersect with $B \times S'$ inside $B \times Y \times Z'$, take push forward along $Y$ (this yields the pull back map $r^*$, since $S'$ is the transpose of the graph of $r$) and finally take the direct image with $i_*$. ∎

**Lemma 4.** *There is an isomorphism* $\rho\colon \mathrm{CH}_p\big(B\otimes(X,\pi_P)\big) \to \mathrm{CH}_p(B\otimes Z')$ *making the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{CH}_p\big(B\otimes(X,\pi_P)\big) & \xrightarrow{\text{inclusion}} & \mathrm{CH}_p(B\otimes X) \\ {\scriptstyle\rho}\big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle j^*} \\ \mathrm{CH}_p(B\otimes Z') & \xrightarrow{i_*} & \mathrm{CH}_p(B\otimes X') \end{array}$$

*The map* $i_*$ *is injective.*

*Proof.* The map $j^*$ fits into an exact sequence

$$\mathrm{CH}_p(B\times P) \xrightarrow{(i_P)_*} \mathrm{CH}_p(B\times X) \xrightarrow{j^*} \mathrm{CH}_p(B\times X') \to 0$$

The push forward map $(i_P)_*$ is injective, since the projection $\lambda_X\colon X \to \operatorname{Spec} k$ gives a left inverse.

The map $i_*$ fits into a long exact sequence (see [3, Sect. 5])

$$\cdots \to A_{p+1}(B\times X', K_*^M, -p) \xrightarrow{\ell^*} A_{p+1}(B\times \mathbb{A}^d, K_*^M, -p) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \mathrm{CH}_p(B\times Z') \xrightarrow{i_*} \mathrm{CH}_p(B\times X') \xrightarrow{\ell^*} \mathrm{CH}_p(B\times \mathbb{A}^d) \to 0$$

Here the maps $\ell^*$ are induced from the inclusion $\ell\colon \mathbb{A}^d \to X'$.

Let $\lambda\colon \mathbb{A}^d \to \operatorname{Spec} k$ be the projection. The pullback maps $\lambda^*\colon A_{p-d}(B) \to A_p(B\times\mathbb{A}^d)$ are isomorphims. The maps $\lambda_X^*\circ(\lambda^*)^{-1}$ are left inverses to $\partial$. Hence the long exact sequence splits up into short exact sequences.

Putting things together yields a decomposition

$$\mathrm{CH}_p(B\times X) = \mathrm{CH}_p(B\times P) \oplus \mathrm{CH}_p(B\times Z') \oplus \mathrm{CH}_p(B\times\mathbb{A}^d)$$

It is easy to see that this decomposition coincides with the decomposition given by (1). □

The proof of Proposition 2 is now complete.

One may check that the inverse of the correspondence $\hat{f}$ is given by the cycle $T = \sigma(S) \subset X\times Y$ where

$$\sigma\colon Y\times X \to X\times Y,$$
$$\sigma\big([y'],[u,v,y]\big) = \big([v,u,y],[y']\big).$$

($T$ is the transpose of $S$ reflected by $u \leftrightarrow v$.)

2.2. **Zero cycles on a quadric.** We recall the computation of the group of zero cycles of a quadric (see [5]).

**Lemma 5.** *For a quadric* $X$ *the degree map* $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ *is injective. Its image is*

$$\deg\big(\mathrm{CH}_0(X)\to\mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \textit{if } X \textit{ is isotropic,} \\ 2\mathbb{Z} & \textit{if } X \textit{ is anisotropic.} \end{cases}$$

2.3. **The Chow ring of a split quadric.** The following material is well known.

For $n \geq 2$ let $\tau_n$ be the following $n$-dimensional quadratic form:

$$\tau_n = \begin{cases} \sum_{i=1}^m x_i y_i + z^2 & \text{if } n = 2m+1, \\ \sum_{i=1}^m x_i y_i & \text{if } n = 2m. \end{cases}$$

We call a quadratic form $\varphi$ *split* if $\varphi$ is similar to $\tau_n$ for some $n$.

**Lemma 6.** *A quadratic form $\varphi$ ($\dim \varphi \geq 2$) is split if and only if $X_\varphi$ is a direct sum of powers of the Tate motive.*

*In this case one has with $d = \dim X_\varphi$*

$$X_\varphi = \begin{cases} \oplus_{i=0}^d L^{\otimes i} & \text{if } d \text{ is odd,} \\ \oplus_{i=0}^d L^{\otimes i} \oplus L^{\otimes (d/2)} & \text{if } d \text{ is even.} \end{cases}$$

*Proof.* Using Proposition 2 one reduces to the case when $\varphi$ is anisotropic. But then $X_\varphi$ has no rational point and cannot be a sum of powers of the Tate motive (cf. Lemma 5). □

A quadric is called *split* if it satisfies the properties of Lemma 6.

**Lemma 7.** *Let $X$ be a split quadric. Then*

$$\operatorname{End}(X) = \oplus_p \operatorname{End}_{\mathbb{Z}}\bigl(\mathrm{CH}_p(X)\bigr)$$

*Proof.* This holds in fact for any sum of powers of the Tate motive. □

**Lemma 8.** *Let $X = X_{\tau_n}$ and $d = \dim X = n - 2$. Then*

(i) *For $p < d/2$ one has $\mathrm{CH}^p(X) = \mathbb{Z}$ generated by the class of the plane section*
$$\{\, x_1 = \cdots = x_p = 0 \,\}.$$

(ii) *If $d = 2m - 2$, then $\mathrm{CH}^{d/2}(X) = \mathbb{Z} \oplus \mathbb{Z}$ generated by the classes of the two maximal linear subspaces*
$$\{\, x_1 = \cdots = x_{m-1} = x_m = 0 \,\},$$
$$\{\, x_1 = \cdots = x_{m-1} = y_m = 0 \,\}.$$

(iii) *For $p > d/2$ one has $\mathrm{CH}^p(X) = \mathbb{Z}$ generated by the class of the linear subspace*
$$\{\, x_1 = \cdots = x_m = z = y_1 = \cdots = y_{p-m} = 0 \,\} \qquad \text{if } d = 2m-1,$$
$$\{\, x_1 = \cdots = x_m = y_1 = \cdots = y_{p+1-m} = 0 \,\} \qquad \text{if } d = 2m-2.$$

*Proof.* This well known fact follows e.g. by analyzing the proof of Proposition 2. In fact, it easy to check that the maps

$$f_* \colon \mathrm{CH}_{p-1}(Y) \to \mathrm{CH}_p(X)$$

sends plane sections to plane sections and linear subspaces to linear subspaces. □

Let $h \in \mathrm{CH}^1(X)$ and $u \in \mathrm{CH}^{(d+1)/2}(X)$, $v$, $w \in \mathrm{CH}^{d/2}(X)$ the generators described in Lemma 8. The Chow ring of $X_{\tau_n}$ has as a ring over $\mathbb{Z}$ the following presentation ($d > 0$):

$$\langle\, h, u \mid h^{(d+1)/2} = 2u,\ h^{d+1} = 0,\ u^2 = 0 \,\rangle \qquad \text{if } d \equiv 1 \bmod 2,$$
$$\langle\, h, u \mid h^{d/2} = v + w,\ hv = hw,\ vw = 0,\ v^2 = w^2 = h^d,\ h^{d+1} = 0 \,\rangle \qquad \text{if } d \equiv 0 \bmod 4,$$
$$\langle\, h, u \mid h^{d/2} = v + w,\ hv = hw,\ vw = h^d,\ v^2 = w^2 = 0,\ h^{d+1} = 0 \,\rangle \qquad \text{if } d \equiv 2 \bmod 4.$$

## 3. Endomorphisms of quadrics

We are now ready to prove the following proposition, which provides an important tool to construct correspondences between quadrics.

**Proposition 9.** *For $d \geq 0$ there exist a number $N(d)$ with the following property:*
*Let $X$ be a smooth projective quadric of dimension $d$, let $f \in \mathrm{End}(X)$ and let $F/k$ be some field extension. If $f_F = 0 \in \mathrm{End}_{\mathcal{V}(F)}(X_F)$, then $f^{N(d)} = 0$.*

*Proof.* We argue by induction on $d$. Let $\varphi$ be a quadratic form defining $X$.

Suppose first that $X$ has a $k$-rational point $P$. Then $\varphi = \mathbb{H} \perp \psi$ as in Proposition 2 and

$$\mathrm{End}(X) = [X \times P]\mathbb{Z} \oplus \mathrm{End}(X_\psi) \oplus [P \times X]\mathbb{Z}.$$

If $f_F = 0$, then necessarily $f \in \mathrm{End}(X_\psi)$, since the other summands do not change under field extensions. By induction we have $f^{N(d-2)} = 0$.

In the general case we apply Proposition 1 with $B = X$. Since all the quadrics $X_b$ have a rational point, we may apply the previous step and see that $f_b^{N(d-2)} = 0$ for all $b \in B$. It suffices to take $N(d) = (1 + d)N(d - 2)$. □

**Corollary 10.** *Let $X$ be a smooth projective quadric, let $f \in \mathrm{End}(X)$ and let $F/k$ be some field extension.*

(i) *If $f_F$ is nilpotent, then $f$ is nilpotent.*
(ii) *If $f_F$ is an isomorphism, then $f$ is an isomorphism.*

*Proof.* (i) is clear from Proposition 9.

Suppose that $f_F$ is an isomorphism. We may assume that $X_F$ is a split quadric. Then, by Lemma 7, $f_F$ is completely determined by its action on the groups $\mathrm{CH}_p(X_F) \simeq \mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$. Thus $f_F$ satisfies an equation $(t^2 - 1)(t^2 + mt \pm 1) = 0$ with $m \in \mathbb{Z}$. It follows that the inverse of $f_F$ is an integral polynomial in $f_F$ and therefore defined over $k$.

Hence we may assume that $f_F = (\mathrm{id}_X)_F$. But then (ii) is immediate from (i): If $f = 1 + g$ and $g$ is nilpotent, then $f$ is invertible. □

**Corollary 11.** *Let $X$, $Z$ be quadrics and $f \in \mathrm{Hom}(X, Z)$. If $F/k$ is a field extension such that $X_F$ and $Z_F$ are split and if*

$$(f_F)_*\colon\ \mathrm{CH}_p(X_F) \to \mathrm{CH}_p(Z_F)$$

*is an isomorphism for all $p$, then $f$ is an isomorphism.*

*Proof.* We must have $\dim X = \dim Z$. Let $g = f^t$. Then $(g_F)_*\colon\ \mathrm{CH}_p(Z_F) \to \mathrm{CH}_p(X_F)$ is also an isomorphism for all $p$. By Lemma 7, the endomorphism $(f \circ g)_F$ is an isomorphism. By Corollary 10 (ii) $f \circ g$ is an isomorphism. Similarly it follows that $g \circ f$ is an isomorphism. □

**Definition 12.** Let $X$ be a quadric of dimension $d$. We call a projector $p \in \mathrm{End}(X)$ *special*, if there exists a quadric $Z$ of dimension $d - 2$ such that

$$(X, \mathrm{id}_X - p) \simeq Z \otimes L.$$

*Example*: (cf. (1) and Proposition 2) If $X$ is isotropic, then $\pi_P$ is a special projector on $X$.

**Proposition 13.** *On a quadric $X$ there exists at most one special projector.*

*Proof.* First suppose that $X$ is isotropic. Then there is an isomorphism

$$(X,p)\oplus Z\otimes L \xrightarrow{\sim} (X,\pi_P)\oplus Y\otimes L$$

with $Y$ as in Proposition 2. Since

$$\operatorname{Hom}\bigl(Z\otimes L,(X,\pi_P)\bigr)=0$$

by dimension reasons, the induced morphism $Z\otimes L\to Y\otimes L$ has a left inverse. Hence $Z$ is a direct summand of $Y$. The morphism $Z\to Y$ must be an isomorphism by Corollary 11. It follows that $(X,\pi_P)\simeq(X,p)$. Hence

$$\operatorname{Hom}\bigl((X,p),Y\otimes L\bigr)=0$$

similar as above and the decomposition is unique.

In the general case let $p$, $p'$ be two special projectors on $X$ with quadrics $Z$ resp. $Z'$ and isomorphisms

$$(X,p)\oplus Z\otimes L \xrightarrow{\sim} X \xrightarrow{\sim} (X,p')\oplus Z'\otimes L.$$

Let $F=k(Z)$. Note that $X_F$ is isotropic. Namely $Z_F$ is isotropic and any $F$-rational point on $Z_F$ corresponds to an element $u\in\operatorname{CH}_1(X_F)$ which gives in the split case a generator. Intersecting $u$ with a hyperplane section gives an $F$-rational point on $X_F$. By the previous step we have

$$\operatorname{Hom}\bigl(Z_F\otimes L,(X,p)_F\bigr)=0.$$

By a filtration argument one sees that $\operatorname{Hom}\bigl(Z\otimes L,(X,p')\bigr)$ is trivial. Similarly for $\operatorname{Hom}\bigl((X,p),Z'\otimes L\bigr)=0$. Hence the decomposition is unique. □

**Lemma 14.** *Let $p$ be a special projector. Then $p^t=p$.*

*Proof.* If $f\colon(X,\mathrm{id}_X-p)\to Z\otimes L$ is an isomorphism, then $f^t\colon Z\otimes L\to(X,\mathrm{id}_X-p^t)$ is an isomorphism. Hence $p^t$ is special und must be equal to $p$. □

**Lemma 15.** *Let $X$, $Z$ be quadrics with $\dim Z=\dim X-2$ and let $F$ be a field extension of $k$ such that $X$, $Z$ are split. Suppose that there exists a morphism*

$$f\colon Z\otimes L\to X$$

*such that*

$$(f_F)_*\colon\ \operatorname{CH}_{(p-1)}(Z_F)\to\operatorname{CH}_p(X_F)$$

*is an isomorphism for all $1<p<d$. Then there exists a special projector $p$ on $X$ and $f$ induces an isomorphism*

$$Z\otimes L\simeq(X,p).$$

*Proof.* The morphism $f^t\circ f$ is seen to be an isomorphism, by similar arguments as above. Then $\mathrm{id}_X-f\circ(f^t\circ f)^{-1}\circ f^t$ is a special projector. □

Let $p\in\operatorname{End}(X)$ be a special projector and $M=(X,p)$. We compute the endomorphism ring of $M$. Let $F$ be a splitting field of $X$ and let ($d=\dim X$)

$$\omega\colon\ \operatorname{End}(M)\to\operatorname{End}(M_F)=\operatorname{End}(L^{\otimes 0}\oplus L^{\otimes d})=\mathbb{Z}\oplus\mathbb{Z}$$

be the natural map. Let

$$\Gamma=2\mathbb{Z}\oplus 2\mathbb{Z}+(1,1)\mathbb{Z}\subset\mathbb{Z}\oplus\mathbb{Z}.$$

**Lemma 16.** *The map $\omega$ is injective. One has*

$$\operatorname{im}\omega = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } X \text{ is isotropic,} \\ \Gamma & \text{if } X \text{ is anisotropic.} \end{cases}$$

*Proof.* We have $(1,1) = \omega(\mathrm{id}_M) \in \operatorname{im}\omega$. Moreover $2\mathbb{Z}\oplus 2\mathbb{Z} \subset \operatorname{im}\omega$. Namely $X$ becomes isotropic over a quadratic extension $F/k$ and then

$$\omega \circ \operatorname{cor}_{F/k}\big(\operatorname{End}(M_F)\big) = 2\mathbb{Z}\oplus 2\mathbb{Z}.$$

If $(1,0) \in \operatorname{im}\omega$, there exists a zero cycle of degree 1 on $X$ and $X$ is isotropic. It remains to show that $\omega$ is injective.

Let $B = X$ and let $\pi\colon B\times X \to B$ be the projection to the second factor and let

$$E^2_{s,t} = A_s\big(B, A_t[\pi, K^M_*]\big) \Longrightarrow A_{s+t}(B\times X, K^M_*) \tag{6}$$

be the associated spectral sequence, see [3, Sect. 8]. This spectral sequence commutes with the projector $p$ acting on $X$. Hence we have a spectral sequence

$$E^2_{s,t} = A_s\big(B, p_*\big(A_t[\pi, K^M_*]\big)\big) \Longrightarrow p_*\big(A_{s+t}(B\times X, K^M_*)\big) \tag{7}$$

Since all fibers $X_b$ are isotropic, we have

$$p_*\big(A_t[\pi, K^M_*]\big) = \begin{cases} 0 & \text{if } 0 < t < d, \\ K^M_* & \text{if } t = 0,\ d. \end{cases}$$

It follows that there is a short exact sequence

$$\mathrm{CH}_0(B) \to \mathrm{CH}_d(B\otimes M) \to \mathrm{CH}_d(B) \to 0.$$

Hence $\mathrm{CH}_d(B\otimes M)$ is free of rank $\le 2$ and the same is true for the subgroup $\operatorname{End}(M)$. □

## 4. Construction of the motive

Let $a_n \in k^*$, $n \ge 1$ be a sequence of elements. We denote by

$$\varphi_n = \langle\!\langle a_1, \ldots, a_n\rangle\!\rangle = \otimes_{i=1}^n \langle 1, -a_i\rangle$$

the Pfister form corresponding to $a_1, \ldots, a_n$. Moreover $\varphi'_n$ denotes the prime subform of $\varphi_n$ defined by $\varphi_n = \langle 1\rangle \perp \varphi'_n$. Furthermore we put $\psi_n = \varphi_{n-1} \perp \langle -a_n\rangle$. Note that

$$\psi_n = \langle 1, -a_n\rangle \perp \varphi'_n.$$

We write $X_n = X_{\psi_n}$ and $Z_n = X_{\varphi'_n}$. Let $d_n = \dim X_n = 2^{n-1} - 1$.

**Theorem 17.** *On the quadric $X_n$ there exist a special projector $p_n$.*

*Let $M_n = (X_n, p_n)$. Then*

$$(X_n, \mathrm{id}_{X_n} - p_n) \simeq Z_{n-1} \otimes L \tag{8}$$

*and*

$$Z_n \simeq M_n \otimes \bigoplus_{i=0}^{d_n - 1} L^{\otimes i} \tag{9}$$

*Proof.* For $n = 1, 2$ one takes $p_n = \mathrm{id}_{X_n}$.

*Step* 1: We first assume that the theorem holds for $n$ and construct $p_{n+1}$ with property (8).

We may assume that $X_n$ is anisotropic.

We first choose a certain element

$$\Theta \in \mathrm{Hom}(M_n \otimes L^{\otimes d_n}, X_{n+1}) = \mathrm{CH}_{2d_n}(M_n \otimes X_{n+1}).$$

Let $F = k(X_n)$ and consider the sequence

$$\mathrm{CH}_{2d_n}(M_n \otimes X_{n+1}) \underset{p_n}{\overset{\text{incl.}}{\rightleftarrows}} \mathrm{CH}_{2d_n}(X_n \times X_{n+1}) \xrightarrow{j} \mathrm{CH}_{d_n}\big((X_{n+1})_F\big)$$

Here $j$ is the projection map. $j$ is surjective (by taking closure of cycles). Since $(X_{n+1})_F$ is split, the rightmost term is $\simeq \mathbb{Z}$ generated by the class $u$ of a maximal linear subspace.

We claim that $j(\ker p_n) = 0$. To check this, one may pass to a splitting field in which case the claim is easy to verify. Let $\Theta$ be an element which maps to the generator $u$.

Let $F$ be a splitting field of $\varphi_n$ and let

$$\begin{aligned}\rho\colon \mathrm{Hom}(M_n \otimes L^{\otimes d_n}, X_{n+1}) \to \mathrm{Hom}\big((M_n)_F \otimes L^{\otimes d_n}, (X_{n+1})_F\big) =& \\ = \mathrm{Hom}\big(L^{\otimes d_n} \otimes L^{\otimes 2d_n}, (X_{n+1})_F\big) = \mathbb{Z} \oplus \mathbb{Z}&\end{aligned}$$

be the restriction map. Note that

$$\rho(\Theta) \in (1, \mathbb{Z}).$$

**Lemma 18.** *If $X_n$ is anisotropic, then $2\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathrm{im}\,\rho \subset \Gamma$.*

*Proof.* The first inclusion is easily seen by passing to a quadratic splitting field and taking norms.

We replace $k$ by $k(X_{n+1})$. Then $X_{n+1}$ is isotropic, but $X_n$ is still anisotropic. One finds using the induction hypothesis:

$$\begin{aligned}\mathrm{Hom}(M_n \otimes L^{\otimes d_n}, X_{n+1}) &= \mathrm{Hom}(M_n \otimes L^{\otimes d_n}, L^{\otimes 0} \oplus L^{\otimes d_{n+1}} \oplus M_n \otimes \bigoplus_{i=1}^{d_n} L^{\otimes i}) \\ &= \mathrm{Hom}(M_n \otimes L^{\otimes d_n}, M_n \otimes \bigoplus_{i=1}^{d_n} L^{\otimes i})\end{aligned}$$

The map $\rho$ is then given by projection onto the summand for $i = d_n$ followed by passing to the split case. Hence it factors through the map $\omega\colon \mathrm{End}(M_n) \to \mathbb{Z} \oplus \mathbb{Z}$ and Lemma 16 yields the claim. $\square$

Since $\rho(\Theta) \notin 2\mathbb{Z} \oplus 2\mathbb{Z}$ we must have $\Gamma \subset \mathrm{im}\,\rho$ and we may assume that

$$\rho(\Theta) = (1, 1).$$

Let $h \in \mathrm{CH}^1(X_{n+1})$ be a hyperplane section and put

$$f = \Theta \cdot (1 + h + h^2 + \cdots + h^{d_n - 1}) \in$$

$$\mathrm{Hom}(M_n \otimes \bigoplus_{i=1}^{d_n} L^{\otimes i}, X_{n+1}) = \mathrm{Hom}(Z_n \otimes L, X_{n+1}).$$

One easily checks that $f$ satisfies the assumptions in Lemma 15. Hence Lemma 15 yields the projector $p_{n+1}$ together with property (8).

*Step* 2: We now assume the existence of $p_n$ with property (8) and show (9).

We may again assume that $X_n$ is anisotropic.

This time we look for an element

$$\Theta \in \operatorname{Hom}(M_n \otimes L^{\otimes d_n - 1}, Z_n) = \mathrm{CH}_{2d_n-1}(M_n \otimes Z_n).$$

Similar as above we have sequence (with $F = k(X_n)$)

$$\mathrm{CH}_{2d_n-1}(M_n \otimes Z_n) \underset{p_n}{\overset{\text{incl.}}{\rightleftarrows}} \mathrm{CH}_{2d_n-1}(X_n \times Z_n) \xrightarrow{j} \mathrm{CH}_{d_n-1}\big((Z_n)_F\big) = u\mathbb{Z}$$

and find $\Theta$ mapping to $u$.

By a spectral sequence argument (with base $Z_n$ and using the fact that $M_n \simeq L^{\otimes 0} \oplus L^{\otimes d_n}$ over any residue class field of $Z_n$) on gets an exact sequence

$$\mathrm{CH}_{d_n-1}(Z_n) \xrightarrow{M_n \times} \mathrm{CH}_{2d_n-1}(M_n \otimes Z_n) \xrightarrow{\ell} \mathrm{CH}_{2d_n-1}(Z_n) \to 0.$$

Suppose that $\ell(\Theta) \in \mathrm{CH}_{2d_n-1}(Z_n) = [Z_n]\mathbb{Z}$ is even. Then we can arrange $\ell(\Theta) = 0$ still keeping $j(\Theta) = u$. But then $\Theta$ would come from $\mathrm{CH}_{d_n-1}(Z_n)$ and the maximal linear subspace $u$ would be defined over $k$. This contradicts the fact that $X_n$ is anisotropic. We my therefore find $\Theta$ with

$$\ell(\Theta) = [Z_n], \qquad j(\Theta) = u.$$

Let $h \in \mathrm{CH}^1(Z_n)$ be a hyperplane section and put

$$f = \Theta \cdot (1 + h + h^2 + \cdots + h^{d_n-1}) \in \operatorname{Hom}(M_n \otimes \bigoplus_{i=0}^{d_n-1} L^{\otimes i}, Z_n).$$

Since $p_n^t = p_n$, the transpose $f^t$ is well defined.

It is easy to check that $f \circ f^t$ satisfies the assumption of Corollary 11 and hence $f \circ f^t$ is an isomorphism.

Similarly one proves that $f^t \circ f$ is an isomorphism (by extending Corollary 11 to motives of the form $M_n \otimes \mathbb{P}^m$). $\square$

**Proposition 19.** *One has*

$$X_{\varphi_n} \simeq M_n \otimes \mathbb{P}^{d_n}.$$

*Proof.* This follows as for (9). $\square$

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