GENERATORS AND RELATIONS FOR THE SYMMETRIC GROUP

MARKUS ROST

"Generators and relations for the symmetric group" [pdf].

We derive the well-known Coxeter-type presentation of the symmetric group (Corollary (3.7)). The method is pretty straightforward.

According to Coxeter-Moser 1980 (1957) [1, 6.2 The symmetric group, p. 63] the earliest presentation of this type is due to Moore 1897 (1896) [3]. The very elegant proof by Dickson 1899 [2] is discussed in section 5.

1. **Transpositions.** Let N be a finite set. The group of permutations of N (the symmetric group) is denoted by Σ_N. Moreover, let

$$N_{(2)} = \binom{N}{2} = \{\, x \subset N \mid |x| = 2 \,\}$$

denote the set of 2-element subsets of N.

For $x \in N_{(2)}$ let $\tau_x \in \Sigma_N$ be the transposition at x (swapping the 2 elements of x and leaving $a \in N \setminus x$ fixed). We use the abbreviated notations $\tau_{\{a,b\}} = \tau_{a,b} = \tau_{ab}$ for a, $b \in N$, $a \neq b$. Thus

$$\tau_{ab}(c) = \begin{cases} b & c = a \\ a & c = b \\ c & c \neq a, b \end{cases} \qquad (a \neq b)$$

It is plain that

(1.1) $$\tau_x^2 = 1 \qquad (x \in N_{(2)})$$

(1.2) $$g\tau_x g^{-1} = \tau_{g(x)} \qquad (g \in \Sigma_N,\ x \in N_{(2)})$$

where Σ_N acts on $N_{(2)}$ in the natural way: $g(\{a,b\}) = \{g(a), g(b)\}$.

Let x, $y \in N_{(2)}$, $x \neq y$. If x, $y \in N_{(2)}$ are disjoint, then $\tau_y(x) = x$ and τ_x, τ_y commute by (1.2). This yields (using (1.1))

(1.3) $$(\tau_x\tau_y)^2 = 1 \qquad (x, y \in N_{(2)},\ |x \cap y| = 0)$$

Otherwise τ_x, τ_y are two of the three transpositions of the 3-element set $x \cup y$ and $\tau_x\tau_y\tau_x = \tau_z$ is the third transposition at $z = (x \cup y) \setminus (x \cap y)$. Explicitly,

$$\tau_{ab}\tau_{bc}\tau_{ab} = \tau_{ac}$$

Upon interchanging $x \leftrightarrow y$ one gets

$$\tau_x\tau_y\tau_x = \tau_y\tau_x\tau_y$$

Hence (1.1) and (1.2) imply

(1.4) $$(\tau_x\tau_y)^3 = 1 \qquad (x, y \in N_{(2)},\ |x \cap y| = 1)$$

Date: August 7, 2026.

2. **Analyzing $\Sigma_{n-1} \subset \Sigma_n$.** Let $p \in N$ and let $M = N \setminus \{p\}$. The group Σ_M is a subgroup of Σ_N (the stabilizer of p) and

$$\Sigma_N/\Sigma_M \to N, \qquad g\Sigma_M \mapsto g(p)$$

is a bijection (since Σ_N acts transitively on N). If $\lambda\colon N \to \Sigma_N$ is a section to $g \mapsto g(p)$ (i.e., $\lambda(N)$ is a set of coset representatives), the map

$$N \times \Sigma_M \to \Sigma_N, \qquad (a, g) \mapsto \lambda(a)g$$

is a bijection. The product in Σ_N transfers to a multiplication table for $N \times \Sigma_M$ which leads to a presentation of Σ_N in terms of Σ_M and generators indexed by N. Taking

$$\lambda(a) = \begin{cases} 1 & a = p \\ \tau_{pa} & a \neq p \end{cases}$$

leads to the following Proposition (2.2).

(2.1) Lemma. *The group Σ_N is generated by transpositions.*

Proof: Follows by induction from the fact that one can choose transpositions as coset representatives for $\Sigma_N/\Sigma_M = N$. □

(2.2) Proposition. *Let $p \in N$, put $M = N \setminus \{p\}$ and let S be the group generated by Σ_M and symbols s_a $(a \in M)$ subject to the relations*[1]

$$s_a^2 = 1 \qquad (a \in M) \tag{2.3}$$

$$s_a s_b s_a = \tau_{ab} \qquad (\{a, b\} \in M_{(2)}) \tag{2.4}$$

$$g s_a g^{-1} = s_{g(a)} \qquad (g \in \Sigma_M,\, a \in M) \tag{2.5}$$

The homomorphism

$$\pi\colon S \to \Sigma_N$$

given by the inclusion $\Sigma_M \to \Sigma_N$ and by

$$s_a \mapsto \tau_{pa} \qquad (a \in M)$$

is an isomorphism.

Proof: π is well-defined because of (1.1), (1.2).

π is surjective since $\pi(S)$ contains the stabilizer of p and has the same p-orbit as Σ_N: $\pi(S)(p) = N = \Sigma_N(p)$.

Writing (2.3)–(2.5) in the form

$$s_a^2 = 1$$
$$s_b s_a = s_a \tau_{ab}$$
$$g s_a = s_{g(a)} g$$

one sees that the subset

$$\Omega = \{1, s_a\,(a \in M)\} \cdot \Sigma_M \subset S$$

is a subgroup. Since Ω generates S it follows that $S = \Omega$ and therefore

$$|S| = |\Omega| \leq (1 + |M|)|\Sigma_M| = |N||\Sigma_M| = |\Sigma_N|$$

The surjectivity of π implies that π is bijective. □

[1] S is the free product of Σ_M and the infinite cyclic groups $\langle s_a|\rangle$ modulo the smallest normal subgroup such that the relations hold.

3. **Rewriting relations.**

(3.1) Proposition. *Let N be a finite set, let $p,\ q \in N$ be distinct elements and put*

$$M = N \setminus \{p\}$$
$$L = M \setminus \{q\} = N \setminus \{p, q\}$$

Further let $r \in L$.

The group T generated by Σ_M and a symbol s subject to the relations

(3.2) $$s^2 = 1$$

(3.3) $$(s\tau_{qr})^3 = 1$$

(3.4) $$(s\tau_{ab})^2 = 1 \qquad (\{a, b\} \in L_{(2)})$$

is isomorphic to Σ_N via the inclusion $\Sigma_M \subset \Sigma_N$ and $s \mapsto \tau_{pq}$.

Clearly (3.3) holds then for all $r \in L$ since the isomorphism is independent of r. Proposition (3.1) says that (3.3) is required only for a single $r \in L$. The case $L = \emptyset$ is the case $|N| = 2$ for which $\Sigma_N = \langle s \,|\, s^2 = 1\rangle$.

Proof: By (2.3) and (2.5) one has in S:

(3.5) $$s_a = \tau_{qa} s_q \tau_{qa} \qquad (a \in L)$$

We use (3.5) to eliminate the s_a $(a \in L)$ from the generators of S, rewrite the relations (2.3)–(2.5) in terms of the only remaining generator s_q (besides Σ_M) and check that the resulting relations are equivalent to (3.2)–(3.4) for $s = s_q$. This means that $s \mapsto s_q$ gives an isomorphism $T \to S$. Then $T \to \Sigma_N$ is an isomorphism as well by Proposition (2.2).

Given $\{a, b\} \in M_{(2)}$, choose $g \in \Sigma_M$ with $g(a) = q$, $g(b) = r$. By (2.5), the family of relations (2.4) reduces to the single relation

$$s_q s_r s_q = \tau_{qr}$$

Rewriting this with (3.5) yields

$$s_q \tau_{qr} s_q \tau_{qr} s_q = \tau_{qr}$$

which is (3.3).

As for (2.5), note first that it sufficient to have (2.5) for g from a subset of generators of Σ_M. Since the elements τ_{bq} $(b \in L)$ generate Σ_M (by Lemma (2.1) and $\tau_{bq}\tau_{aq}\tau_{bq} = \tau_{ab}$ for $\{a, b\} \in L_{(2)}$), relations (2.5) reduce to

(3.6) $$\tau_{bq} s_a \tau_{bq} = s_{\tau_{bq}(a)} \qquad (a, b \in M,\ b \neq q)$$

For $a = q$ or $a = b$ relation (3.6) is just (3.5). For $a \neq q$, b one has $\tau_{bq}(a) = a$ and relation (3.6) rewrites via (3.5) as

$$\tau_{bq}\tau_{aq} s_q \tau_{aq} \tau_{bq} = \tau_{aq} s_q \tau_{aq}$$

Since $\tau_{aq}\tau_{bq}\tau_{aq} = \tau_{ab}$, this means $\tau_{ab} s_q \tau_{ab} = s_q$ which together with $s_q^2 = 1$ amounts to (3.4).

Finally, (2.3) reduces via (3.5) to (3.2). $\square$

(3.7) Corollary. *Let $N = \{1, \dots, n\}$. The group Σ_N has the presentation with generators t_i $(i = 1, \dots, n-1)$ corresponding to $\tau_{i,i+1}$ and defining relations*

(3.8) $$t_i^2 = 1 \qquad (i = 1, \dots, n-1)$$

(3.9) $$(t_i t_{i+1})^3 = 1 \qquad (i = 1, \dots, n-2)$$

(3.10) $$(t_i t_j)^2 = 1 \qquad (i, j = 1, \dots, n-1,\ |i-j| \geq 2)$$

Proof: By induction on n using Proposition (3.1) with $p = n$, $q = n-1$, $r = n-2$. The relations (3.8)–(3.10) involving $t_{n-1} = s$ correspond to relations (3.2)–(3.4). Here note that (3.4) is equivalent to $t_i t_{n-1} t_i^{-1} = t_{n-1}$ $(i = 1, \dots, n-3)$ since the τ_{ab} and, by induction, the t_i generate the same group $\Sigma_L = \Sigma_{n-2}$. □

4. **The complete presentation.** The obvious formulas (1.1), (1.2) can be combined to

(4.1) $$\tau_x \tau_y \tau_x = \tau_{\tau_x(y)} \qquad (x, y \in N_{(2)})$$

Indeed, $y = x$ yields $\tau_x^2 = 1$ and in (1.2) it suffices to take $g \in \Sigma_N$ from a set of generators (cf. Lemma (2.1)).

It is not difficult to see that (4.1) is a complete set of relations when taking as generators *all* reflections τ_x: The group Σ_N has the presentation with generators

$$t_x \qquad (x \in N_{(2)})$$

and relations

$$t_x t_y t_x = t_{\tau_x(y)} \qquad (x, y \in N_{(2)})$$

(use Proposition (2.2) and induction).

This presentation (let us call it the *complete presentation* of Σ_N) is highly redundant but has the advantage of being functorial with respect to arbitrary bijections $N \to N'$.

In contrast, the Coxeter-type presentation in Corollary (3.7) requires an order of N (up to order reversing) and transforms directly merely for order preserving or order reversing maps $N \to N'$ without "gaps".

By the way, the complete presentation shows already that the abelianized group $\Sigma_N/[\Sigma_N, \Sigma_N]$ is cyclic of order 2, providing a definition of $\mathrm{sgn}\colon \Sigma_N \to \{\pm 1\}$.

5. **Dickson's proof.** When preparing the bibliography entry for Moore [3], I found in the zbMATH list "Cited in 5 Reviews" the very elegant proof of Dickson 1899 [2].

In brief, Dickson argues as follows: Let $G(n)$ be the group given by the Coxeter-type presentation with generators $t_1, \dots, t_{n-1}$ (cf. Corollary (3.7)). Further let $G \subset G(n)$ be the subgroup generated by $t_1, \dots, t_{n-2}$. Dickson considers the following subsets of $G(n)$:

$$\begin{aligned} O_n &= G \\ O_{n-1} &= G t_{n-1} \\ O_{n-2} &= G t_{n-1} t_{n-2} \\ \cdots &\quad \cdots \\ O_1 &= G t_{n-1} t_{n-2} \cdots t_1 \end{aligned}$$

and shows that multiplication from the right with t_r interchanges O_r, O_{r+1} and leaves the other O_i fixed. The first statement is obvious from $t_r^2 = 1$. For $i > r+1$

the claim follows entirely from the commutation relations $t_j t_r = t_r t_j$ $(j > r+1)$. For $i < r$ one uses commutation relations and $t_r t_{r-1} t_r = t_{r-1} t_r t_{r-1}$. Thus $G(n)$ is the union of the O_i. Therefore $|G(n)| \leq n!$ by induction, whence $G(n) \to \Sigma_n$ is an isomorphism.

References

[1] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, fourth ed., Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 14, Springer-Verlag, Berlin-New York, 1980. MR 562913 zbMATH 0422.20001 1

[2] L. E. Dickson, *The abstract group isomorphic with the symmetric group on k letters*, Proc. Lond. Math. Soc. **31** (1899), 351–353 (English). MR 1576717 zbMATH 30.0141.03 1, 4

[3] E. H. Moore, *Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holohedrically isomorphic with the symmetric and the alternating substitution-groups on k letters*, Proc. Lond. Math. Soc. **28** (1897), 357–366 (English). MR 1576641 zbMATH 28.0121.03 1, 4

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

Email address: `rost at math.uni-bielefeld.de`

URL: `www.math.uni-bielefeld.de/~rost`